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CREATION OF SEQUENCES OF SINGULAR 3-TUPLES THROUGH ABEL AND CYCLOTOMIC POLYNOMIAL WITH COMMENSURABLE PROPERTY

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Abstract

In this paper, the sequence of 3-tuples named as singular 3-tuples $\{f(x), g(x), h(x)\}$, $\{g(x), h(x), i(x)\}$ etc concerning Abel's polynomial and Cyclotomic polynomial such that the arithmetic mean of any two polynomials increased by a monomial with integer coefficients provides square of a particular polynomial is enumerated. Furthermore, Python program for conformation of each of an evaluated singular 3-tuples is also exemplified.

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1 Introduction

"A set of s positive integers $\{r_1, r_2, \ldots, r_s\}$ is called a Diophantine s-tuples with the property $D(l), l \in Z-\{0\}$ if $r_m r_n + l$ is a perfect square for all $1 \leq m < n \leq s$." In [1, 2, 4], the writers originate limited number of Diophantine triples with precise properties. For an inclusive analysis of numerous problems on Diophantine 3-tuples with appropriate properties, see [5-11, 15-18]. In this communication, the process of finding sequence of singular 3-tuples $\{f(x), g(x), h(x)\}$, $\{g(x), h(x), i(x)\}$ etc consisting Abel's polynomial and Cyclotomic polynomial such that the arithmetic mean of any two polynomials in each 3-tuples enlarged by a monomial with integer coefficients remains square of a polynomial is demonstrated. Moreover, Python program for checking each of such singular 3-tuples with numerical values is illustrated.

2 Basic Definitions

2.1 Abel's Polynomial

The n^{th} term of Abel's Polynomial is defined by

$$p_{a,n}(x) = x(x-an)^{n-1}.$$

Therefore

$$P_{1,2}(x) = x^2 - 2x,\ P_{2,2}(x) = x^2 - 4x.$$

2.2 Cyclotomic Polynomial

The n^{th} term of Cyclotomic Polynomial is defined by

$$\varphi_n(x) = \begin{cases} \sum\limits_{k=0}^{n-1} x^k, & \text{if } n \text{ is a prime number} \\ \sum\limits_{k=0}^{p-1} (-x)^k, & \text{if } n = 2p \text{ where } p \text{ is an odd prime number.} \end{cases}$$

Then, $\varphi_3(x) = x^2 + x + 1$, $\varphi_6(x) = x^2 - x + 1$.

2.3 Singular 3-tuples

A set of 3-tuples $\{f(x), g(x), h(x)\}$ is called singular 3-tuples with property $D[r(x)]$ if the average of any two polynomials in the set added with $r(x)$ is a square of some other polynomial where $f(x), g(x), h(x), r(x)$ are polynomials with integer coefficients.

3 Process of Receiving Singular 3-tuples

The method of discovering two different types of singular 3-tuples in which the elements are Abel's polynomial and Cyclotomic polynomial such that average of any two polynomials added with certain monomial gives a square of a polynomial is expressed in Section 3.1 and Section 3.2.

3.1 Singular 3-tuples with Ables Polynomial

Choose $f(x) = P_{1,2}(16x) = 256x^2 - 32x$ and $g(x) = P_{2,2}(16x) = 256x^2 - 64x$ be such that average of $f(x)$ and $g(x)$ added with $80x + 1$ is a square of a polynomial.

Mathematically, the above hypothesis is expressed by

$$\left[\frac{f(x)+g(x)}{2}\right] + 80x + 1 = (16x+1)^2 = [\alpha(x)]^2 \text{ (say)}. \tag{3.1}$$

Let $h(x)$ be another polynomial collected with the resulting two conditions that

$$\left[\frac{g(x)+h(x)}{2}\right] + 80x + 1 = [\beta(x)]^2, \tag{3.2}$$

$$\left[\frac{f(x)+h(x)}{2}\right] + 80x + 1 = [\gamma(x)]^2. \tag{3.3}$$

Subtraction of (3.3) from (3.2) affords the succeeding equation

$$\left[\frac{g(x)-f(x)}{2}\right] = [\beta(x)]^2 - [\gamma(x)]^2. \tag{3.4}$$

For finding the third element $h(x)$ in an essential triple, let us select the proper conversions as specified below.

$$\beta(x) = A + 2 \text{ and } \gamma(x) = A. \tag{3.5}$$

Replacing the chosen values of $f(x)$, $g(x)$ and the alterations (3.5) in (3.4), the possibility of A and hence $\gamma(x)$ is attained by

$$\gamma(x) = A = 4x - 1.$$

Retaining $f(x)$ and the above derived value of $\gamma(x)$ in (3.3), it is scrutinized that

$$h(x) = -224x^2 - 112x.$$

Note that, $\{f(x), g(x), h(x)\} = \left\{256x^2 - 32x, 256x^2 - 64x, -224x^2 - 112x\right\}$ is a gorgeous singular 3-tuples with property $D(80x + 1)$.

Similarly opening with patterns of singular 2-tuples $\{g(x), h(x)\}$, $\{h(x), i(x)\}$ etc, it is possible to extend patterns of singular 3-tuples $\{g(x), h(x), i(x)\}$ $\{h(x), i(x), j(x)\}$ etc with an equivalent condition $D(80x+1)$. Here

$$i(x) = 7200x^4 + 1440x^3 + 56x^2 - 72x,$$
$$j(x) = 1620000x^8 + 648000x^7 + 190800x^6 + 43200x^5 + 2450x^4 - 20x^3 + 134x^2 - 68x.$$

Table 3.1 demonstrates singular 3-tuples for few values of x for easy understanding.

***Table* 3.1**

x	$D(80x+1)$	$\{f(x), g(x), h(x)\}$	$\{g(x), h(x)i(x)\}$	$\{h(x)i(x), j(x)\}$
1	$D(81)$	$\{224, 192, -336\}$	$\{192, -336, 8624\}$	$\{-336, 8624, 2504496\}$
2	$D(161)$	$\{960, 896, -1120\}$	$\{896, -1120, 126800\}$	$\{-1120, 126800, 511297040\}$
3	$D(241)$	$\{2208, 2112, -2352\}$	$\{2112, -2352, 622368\}$	$\{-2352, 622368, 12195785712\}$
4	D(321)	$\{3968, 3840, -4032\}$	$\{3840, -4032, 1935968\}$	$\{-4032, 1935968, 117611533392\}$
5	$D(401)$	$\{6240, 6080, -6160\}$	$\{6080, -6160, 4681040\}$	$\{-6160, 4681040, 686555281760\}$

3.2 Singular 3-tuples with Cyclotomic Polynomial

Let $\{k(x), l(x)\} = \{\varphi_3(16x), \varphi_6(16x)\} = \left\{256x^2 + 16x + 1,\ 256x^2 - 16x + 1\right\}$ be a pair comprising Cyclotomic Polynomials such that the average of these two polynomials augmented by the monomial $32x$ is a square of some other polynomial. Following the procedure as explained in Section 3.1, this pair is extended into singular triple $\{k(x), l(x), m(x)\}$ with property $D(32x)$. Here $m(x) = -224x^2 - 64x + 1$.

Similarly, if $\{l(x), m(x)\}, \{m(x), n(x)\}$ etc are pairs in which the elements are certain polynomials, then as in Section 3.1 each pair can be protracted into singular triples $\{l(x), m(x), n(x)\}, \{m(x), n(x), p(x)\}$ etc with the similar property $D(32x)$. Here

$$n(x) = 7200x^4 + 1440x^3 + 56x^2 - 24x + 1,$$

$$p(x) = 1620000x^8 + 648000x^7 + 190800x^6 + 43200x^5 + 2450x^4 - 20x^3 + 134x^2 - 20x + 1.$$

The ensuing Table 3.2 establishes the prescribed singular 3-tuples for limited values of x.

***Table* 3.2**

x	$D(32x)$	$\{k(x), l(x), m(x)\}$	$\{l(x), m(x), n(x)\}$	$\{m(x), n(x), p(x)\}$
1	$D(32)$	$\{273, 241, -287\}$	$\{241, -287, 8673\}$	$\{-287, 8673, 2504545\}$
2	$D(64)$	$\{1057, 993, -1023\}$	$\{993, -1023, 126897\}$	$\{-1023, 126897, 511297137\}$
3	$D(96)$	$\{2353, 2257, -2207\}$	$\{2257, -2207, 622513\}$	$\{-2207, 622513, 12195785857\}$
4	$D(128)$	$\{4161, 4033, -3839\}$	$\{4033, -3839, 1936161\}$	$\{-3839, 1936161, 117611533585\}$
5	$D(160)$	$\{6481, 6321, -5919\}$	$\{6321, -5919, 4681281\}$	$\{-5919, 4681281, 686555282001\}$

4 Python program

Python program is displayed below for endorsement of each of the singular 3-tuples with arithmetic values.

```
import math
Section=int(input('ENTER THE VALUE OF SECTION'))
if Section == 1:
  x=int(input('ENTER THE VALUE OF x = '))
  f=256 * x * x-32 * x
  g=256 * x * x-64 * x
  h=-224 * x * x-112 * x
  i=7200 * x * x * x * x+1440 * x * x * x+56 * x * x-72 * x
  j=1620000 * (x * x * x * x * x * x * x * x) + 648000 * (x * x * x * x * x * x * x)
    + 190800 * (x * x * x * x * x * x)+43200 * (x * x * x * x * x)
    + 2450 * (x * x * x * x)-20 * (x * x * x)+134 * (x * x)-68 * x
print('f(x)=', f,'g(x)=', g,'h(x)=', h,'i(x)=', i,'j(x)=', j)
  R=((f+g)/2)+80 * x+1
  Y=((g+h)/2)+80 * x+1
  Z=((h+f)/2)+80 * x+1
  M=((g+i)/2)+80 * x+1
  N=((h+i)/2)+80 * x+1
  P=((h+j)/2)+80 * x+1
  Q=((i+j)/2)+80 * x+1
root1=mathsqrt(R)
root2=mathsqrt(Y)
root3=mathsqrt(Z)
root4=mathsqrt(M)
root5=mathsqrt(N)
root6=mathsqrt(P)
root7=mathsqrt(Q)
if (int(root1+0.5) ** 2==R) and (int(root2+0.5) ** 2==Y) and (int(root3+0.5) ** 2==Z):
print('(f(x), g(x), h(x))=',(f, g, h), "is a Singular triple with D(80x+1)")
else:
print('(f(x), g(x), h(x))=',(f, g, h), "is not a Singular triple with D(80x+1)")
if (int(root2+0.5) ** 2==Y) and (int(root4+0.5) ** 2==M) and (int(root5+0.5) ** 2==N):
print('(g(x), h(x), i(x))=',(g, h, i), "is a Singular triple with D(80x+1)")
else:
print('(g(x), h(x), i(x))=',(g, h, i), "is not a Singular triple with D(80x+1)")
if (int(root5+0.5) ** 2==N) and (int(root6+0.5) * * 2==P) and (int(root7+0.5) ** 2==Q):
print('(h(x), i(x), j(x))=',(h, i, j), "is a Singular triple with D(80x+1)")
else:
print('(h(x), i(x), j(x))=',(h, i, j), "is not a Singular triple with D(80x+1)")
elif Section == 2:
X=int(input('ENTER THE VALUE OF X = '))
k=256 * X * X+16 * X+1
l=256 * X * X-16 * X+1
```

```
m=-224 * X * X-64 * X+1
n=7200 * X * X * X * X+1440 * X * X * X+56 * X * X-24 * X+1
p=1620000 * (X * X * X * X * X * X * X * X) + 648000 * (X * X * X * X * X * X * X)
  + 190800 * (X * X * X * X * X * X)+43200 * (X * X * X * X * X)
  + 2450 * (X * X * X * X)-20 * (X * X * X)+134 * (X * X)-20 * X+1
print('k(x)=', k,'l(x)=', l,'m(x)=', m,'n(x)=', n,'p(x)=', p)
A=((k+l)/2)+32 * X
B=((l+m)/2)+32 * X
C=((m+k)/2)+32 * X
D=((l+n)/2)+32 * X
E=((m+n)/2)+32 * X
F=((m+p)/2)+32 * X
G=((n+p)/2)+32 * X
root1=mathsqrt(A)
root2=mathsqrt(B)
root3=mathsqrt(C)
root4=mathsqrt(D)
root5=mathsqrt(E)
root6=mathsqrt(F)
root7=mathsqrt(G)
if (int(root1+0.5) ** 2==A) and (int(root2+0.5) ** 2==B) and (int(root3+0.5) ** 2==C):
print('(k(x), l(x), m(x))=',(k,l,m), "is a Singular triple with D(32x)")
else:
print('(k(x), l(x)m(x))=',(k,l,m), "is not a Singular triple with D(32x)")
if (int(root2+0.5) ** 2==B) and (int(root4+0.5) ** 2==D) and (int(root5+0.5) ** 2==E):
print('(l(x), m(x), n(x))=',(l,m,n), "is a Singular triple with D(32x)")
else:
print('(l(x), m(x), n(x))=',(l,m,n), "is not a Singular triple with D(32x)")
if (int(root5+0.5) ** 2==E) and (int(root6+0.5) ** 2==F)
  and (int(root7+0.5) ** 2==G):
print('(m(x), n(x), p(x))=',(m, n, p), "is a Singular triple with D(32x)")
else:
print('(m(x), n(x), p(x))=',(m, n, p), "is not a Singular triple with D(32x)")
```

Output of Some Examples

```
ENTER THE VALUE OF SECTION 1
ENTER THE VALUE OF x = 1
f(x)= 224 g(x)= 192 h(x)=-336 i(x)= 8624 j(x)= 2504496
(f(x), g(x), h(x))= (224, 192,-336) is a Singular triple with D(80x+1)
(g(x), h(x), i(x))= (192, -336, 8624) is a Singular triple with D(80x+1)
(h(x), i(x), j(x))= (-336, 8624, 2504496) is a Singular triple with D(80x+1)

ENTER THE VALUE OF SECTION 1
ENTER THE VALUE OF x = 2
f(x)= 960 g(x)= 896 h(x)=-1120 i(x)= 126800 j(x)= 511297040
(f(x), g(x), h(x))= (960, 896,-1120) is a Singular triple with D(80x+1)
(g(x), h(x), i(x))= (896, -1120, 126800) is a Singular triple with D(80x+1)
(h(x), i(x), j(x))= (-1120,126800,511297040) is a Singular triple with D(80x+1)

ENTER THE VALUE OF SECTION 2
ENTER THE VALUE OF X = 1
k(x)= 273 l(x)= 241 m(x)=-287 n(x)= 8673 p(x)= 2504545
(k(x), l(x), m(x))= (273, 241,-287) is a Singular triple with D(32x)
(l(x), m(x), n(x))= (241, -287, 8673) is a Singulartriple with D(32x)
(m(x), n(x), p(x))= (-287, 8673, 2504545) is a Singular triple with D(32x)

ENTER THE VALUE OF SECTION 2
ENTER THE VALUE OF X = 2
k(x)= 1057 l(x)= 993 m(x)=-1023 n(x)= 126897 p(x)= 511297137
(k(x), l(x), m(x))= (1057, 993, -1023) is a Singular triple with D(32x)
(l(x), m(x), n(x))= (993,-1023, 126897) is a Singular triple with D(32x)
(m(x), n(x), p(x))= (-1023, 126897, 511297137) is a Singular{triple with D(32x)
```

5 Conclusion

In this paper, the development of finding order of singular 3-tuples $\{f(x), g(x), h(x)\}$, $\{g(x), h(x), i(x)\}$ etc entailing Abel's polynomial and Cyclotomic polynomial in which the arithmetic mean of any two polynomials in each 3-tuples added by a monomial with integer coefficients leftovers square of a polynomial is recognized. Additionally, Python program for inspection of each of such singular 3-tuples with numerical values is presented. To conclude this, one can pursuit varieties of 3-tuples nourishing innumerable exciting features.

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