CREATION OF SEQUENCES OF SINGULAR 3-TUPLES THROUGH ABEL AND CYCLOTOMIC POLYNOMIAL WITH COMMENSURABLE PROPERTY

R. Vanaja\(^1\) and V. Pandichelvi\(^2\)

\(^1\)Department of Mathematics, AIMAN College of Arts & Science for Women, Tiruchirappalli, Tamil Nadu, India-620021

(Affiliated to Bharathidasan University)

\(^2\)Post Graduate & Research Department of Mathematics, Urumu Dhanalakshmi College, Tiruchirappalli, Tamil Nadu, India-620019

(Affiliated to Bharathidasan University)

Email: vanajvicky09@gmail.com, mvpmahesh2017@gmail.com

(Received: June 06, 2022; In format: July 28, 2022; Revised: August 24, 2022; Accepted: April 08, 2023)

DOI: https://doi.org/10.58250/jnanabha.2023.53112

Abstract

In this paper, the sequence of 3-tuples named as singular 3-tuples \(\{f(x), g(x), h(x)\}\), \(\{g(x), h(x), i(x)\}\) etc concerning Abel’s polynomial and Cyclotomic polynomial such that the arithmetic mean of any two polynomials increased by a monomial with integer coefficients provides square of a particular polynomial is enumerated. Furthermore, Python program for conformation of each of an evaluated singular 3-tuples is also exemplified.

2020 Mathematical Sciences Classification: 11B83

Keywords and Phrases: Singular 3-tuples, Abel’s polynomial, Cyclotomic polynomial.

1 Introduction

“A set of \(s\) positive integers \(\{r_1, r_2, \ldots, r_s\}\) is called a Diophantine \(s\)-tuples with the property \(D(l), l \in \mathbb{Z} - \{0\}\) if \(r_mr_n + l\) is a perfect square for all \(1 \leq m < n \leq s\)” In [1, 2, 4], the writers originate limited number of Diophantine triples with precise properties. For an inclusive analysis of numerous problems on Diophantine 3-tuples with appropriate properties, see [5-11, 15-18]. In this communication, the process of finding sequence of singular 3-tuples \(\{f(x), g(x), h(x)\}\), \(\{g(x), h(x), i(x)\}\) etc consisting Abel’s polynomial and Cyclotomic polynomial such that the arithmetic mean of any two polynomials in each 3-tuples enlarged by a monomial with integer coefficients remains square of a polynomial is demonstrated. Moreover, Python program for checking each of such singular 3-tuples with numerical values is illustrated.

2 Basic Definitions

2.1 Abel’s Polynomial

The \(n^{th}\) term of Abel’s Polynomial is defined by

\[
p_{a,n}(x) = x(x-an)^{n-1}.
\]

Therefore

\[
P_{1,2}(x) = x^2 - 2x, \quad P_{2,2}(x) = x^2 - 4x.
\]

2.2 Cyclotomic Polynomial

The \(n^{th}\) term of Cyclotomic Polynomial is defined by

\[
\varphi_n(x) = \begin{cases} 
\sum_{k=0}^{n-1} x^k, & \text{if } n \text{ is a prime number} \\
\sum_{k=0}^{p-1} (-x)^k, & \text{if } n = 2p \text{ where } p \text{ is an odd prime number.}
\end{cases}
\]

Then, \(\varphi_3(x) = x^2 + x + 1\), \(\varphi_4(x) = x^2 - x + 1\).

2.3 Singular 3-tuples

A set of 3-tuples \(\{f(x), g(x), h(x)\}\) is called singular 3-tuples with property \(D[r(x)]\) if the average of any two polynomials in the set added with \(r(x)\) is a square of some other polynomial where \(f(x), g(x), h(x), r(x)\) are polynomials with integer coefficients.
3 Process of Receiving Singular 3-tuples

The method of discovering two different types of singular 3-tuples in which the elements are Abel’s polynomial and Cyclotomic polynomial such that average of any two polynomials added with certain monomial gives a square of a polynomial is expressed in Section 3.1 and Section 3.2.

3.1 Singular 3-tuples with Ables Polynomial

Choose \( f(x) = P_{1,2}(16x) = 256x^2 - 32x \) and \( g(x) = P_{2,2}(16x) = 256x^2 - 64x \) be such that average of \( f(x) \) and \( g(x) \) added with \( 80x + 1 \) is a square of a polynomial.

Mathematically, the above hypothesis is expressed by

\[
\frac{f(x) + g(x)}{2} + 80x + 1 = (16x + 1)^2 = [\alpha(x)]^2 \text{ (say).} \tag{3.1}
\]

Let \( h(x) \) be another polynomial collected with the resulting two conditions that

\[
\frac{g(x) + h(x)}{2} + 80x + 1 = [\beta(x)]^2, \tag{3.2}
\]

\[
\frac{f(x) + h(x)}{2} + 80x + 1 = [\gamma(x)]^2. \tag{3.3}
\]

Subtraction of (3.3) from (3.2) affords the succeeding equation

\[
\frac{g(x) - f(x)}{2} = [\beta(x)]^2 - [\gamma(x)]^2. \tag{3.4}
\]

For finding the third element \( h(x) \) in an essential triple, let us select the proper conversions as specified below.

\[
\beta(x) = A + 2 \quad \text{and} \quad \gamma(x) = A. \tag{3.5}
\]

Replacing the chosen values of \( f(x), g(x) \) and the alterations (3.5) in (3.4), the possibility of \( A \) and hence \( \gamma(x) \) is attained by

\[
\gamma(x) = A = 4x - 1.
\]

Retaining \( f(x) \) and the above derived value of \( \gamma(x) \) in (3.3), it is scrutinized that

\[
h(x) = -224x^2 - 112x.
\]

Note that, \( \{f(x), g(x), h(x)\} = \{256x^2 - 32x, 256x^2 - 64x, -224x^2 - 112x\} \) is a gorgeous singular 3-tuples with property \( D(80x + 1) \).

Similarly opening with patterns of singular 2-tuples \( \{g(x), h(x)\}, \{h(x), i(x)\} \) etc, it is possible to extend patterns of singular 3-tuples \( \{g(x), h(x), i(x)\}, \{h(x), i(x), j(x)\} \) etc with an equivalent condition \( D(80x + 1) \).

Here

\[
i(x) = 7200x^4 + 1440x^3 + 56x^2 - 72x;
\]

\[
j(x) = 1620000x^8 + 648000x^7 + 190800x^6 + 43200x^5 + 2450x^4 - 20x^3 + 134x^2 - 68x.
\]

Table 3.1 demonstrates singular 3-tuples for few values of \( x \) for easy understanding.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( D(80x + 1) )</th>
<th>( {f(x), g(x), h(x)} )</th>
<th>( {g(x), h(x), i(x)} )</th>
<th>( {h(x), i(x), j(x)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( D(81) )</td>
<td>{224, 192, -336}</td>
<td>{192, -336, 8624}</td>
<td>{-336, 8624, 2504496}</td>
</tr>
<tr>
<td>2</td>
<td>( D(161) )</td>
<td>{960, 896, -1120}</td>
<td>{896, -1120, 126800}</td>
<td>{-1120, 126800, 511297040}</td>
</tr>
<tr>
<td>3</td>
<td>( D(241) )</td>
<td>{2208, 2112, -2352}</td>
<td>{2112, -2352, 622368}</td>
<td>{-2352, 622368, 12195785712}</td>
</tr>
<tr>
<td>4</td>
<td>( D(321) )</td>
<td>{3968, 3840, -4032}</td>
<td>{3840, -4032, 1935968}</td>
<td>{-4032, 1935968, 11761153392}</td>
</tr>
<tr>
<td>5</td>
<td>( D(401) )</td>
<td>{6240, 6080, -6160}</td>
<td>{6080, -6160, 4681040}</td>
<td>{-6160, 4681040, 686555281760}</td>
</tr>
</tbody>
</table>

3.2 Singular 3-tuples with Cyclotomic Polynomial

Let \( \{k(x), l(x)\} = \{\varphi_3(16x), \varphi_6(16x)\} = \{256x^2 + 16x + 1, 256x^2 - 16x + 1\} \) be a pair comprising Cyclotomic Polynomials such that the average of these two polynomials augmented by the monomial 32x is a square of some other polynomial. Following the procedure as explained in Section 3.1, this pair is extended into singular triple \( \{k(x), l(x), m(x)\} \) with property \( D(32x) \). Here \( m(x) = -224x^2 - 64x + 1 \).
Similarly, if \( \{l(x), m(x)\} \), \( \{m(x), n(x)\} \) etc are pairs in which the elements are certain polynomials, then as in Section 3.1 each pair can be protracted into singular triples \( \{l(x), m(x), n(x)\} \), \( \{m(x), n(x), p(x)\} \) etc with the similar property \( D(32x) \). Here
\[
\begin{align*}
n(x) &= 7200x^4 + 1440x^3 + 56x^2 - 24x + 1, \\
p(x) &= 162000x^8 + 648000x^7 + 190800x^6 + 43200x^5 + 2450x^4 - 20x^3 + 13x^2 - 20x + 1.
\end{align*}
\]

The ensuing Table 3.2 establishes the prescribed singular 3-tuples for limited values of \( x \).

<table>
<thead>
<tr>
<th>Section</th>
<th>( {l(x), m(x), n(x)} )</th>
<th>( {l(x), m(x), n(x)} )</th>
<th>( {m(x), n(x), p(x)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{273, 241, -287}</td>
<td>{241, -287, 8673}</td>
<td>{-287, 8673, 2504545}</td>
</tr>
<tr>
<td>2</td>
<td>{1057, 993, -1023}</td>
<td>{993, -1023, 126897}</td>
<td>{-1023, 126897, 511297137}</td>
</tr>
<tr>
<td>3</td>
<td>{2353, 2257, -2207}</td>
<td>{2257, -2207, 622513}</td>
<td>{-2207, 622513, 1219575857}</td>
</tr>
<tr>
<td>4</td>
<td>{4161, 4033, -3839}</td>
<td>{4033, -3839, 1936161}</td>
<td>{-3839, 1936161, 11761153585}</td>
</tr>
<tr>
<td>5</td>
<td>{6481, 6321, -5919}</td>
<td>{6321, -5919, 4681281}</td>
<td>{-5919, 4681281, 686555282001}</td>
</tr>
</tbody>
</table>

4 Python program

Python program is displayed below for endorsement of each of the singular 3-tuples with arithmetic values.

```python
import math
Section=int(input('ENTER THE VALUE OF SECTION'))
if Section == 1:
x=int(input('ENTER THE VALUE OF x = '))
f=256 * x * x-32 * x
f=256 * x * x-64 * x
h=-224 * x * x-112 * x
g=256 * x * x-32 * x
i=7200 * x * x * x * x+1440 * x * x * x+56 * x * x-72 * x
j=648000 * (x * x * x * x + x * x * x * x) + 648000 * (x * x * x * x + x * x * x * x)
if (int(root5+0.5) ** 2==N) and (int(root6+0.5) ** 2==P) and (int(root7+0.5) ** 2==Q):
print('(g(x), h(x), i(x))=',(g, h, i), "is not a Singular triple with D(80x+1)"
else:
print('(g(x), h(x), i(x))=',(g, h, i), "is a Singular triple with D(80x+1)"
if (int(root2+0.5) ** 2==Y) and (int(root4+0.5) ** 2==M) and (int(root5+0.5) ** 2==N):
print('(f(x), g(x), h(x))=',(f, g, h), "is not a Singular triple with D(80x+1)"
else:
print('(f(x), g(x), h(x))=',(f, g, h), "is a Singular triple with D(80x+1)"
```

The Python program is shown above for endorsement of each of the singular 3-tuples with arithmetic values.
Output of Some Examples

1. **ENTER THE VALUE OF SECTION 1**
2. **ENTER THE VALUE OF X = 1**
3. f(x) = 224  g(x) = 192  h(x) = -336  i(x) = 8624  j(x) = 2504496
4. (f(x), g(x), h(x)) = (224, 192, -336) is a Singular triple with D(80x+1)
5. (g(x), h(x), i(x)) = (192, -336, 8624) is a Singular triple with D(80x+1)
6. (h(x), i(x), j(x)) = (-336, 8624, 2504496) is a Singular triple with D(80x+1)

7. **ENTER THE VALUE OF SECTION 1**
8. **ENTER THE VALUE OF X = 2**
9. f(x) = 960  g(x) = 896  h(x) = -1120  i(x) = 126800  j(x) = 511297040
10. (f(x), g(x), h(x)) = (960, 896, -1120) is a Singular triple with D(80x+1)
11. (g(x), h(x), i(x)) = (896, -1120, 126800) is a Singular triple with D(80x+1)
12. (h(x), i(x), j(x)) = (-1120, 126800, 511297040) is a Singular triple with D(80x+1)

13. **ENTER THE VALUE OF SECTION 2**
14. **ENTER THE VALUE OF X = 1**
15. k(x) = 273  l(x) = 241  m(x) = -287  n(x) = 8673  p(x) = 2504545
16. (k(x), l(x), m(x)) = (273, 241, -287) is a Singular triple with D(32x)
17. (l(x), m(x), n(x)) = (241, -287, 8673) is a Singular triple with D(32x)
18. (m(x), n(x), p(x)) = (-287, 8673, 2504545) is a Singular triple with D(32x)

19. **ENTER THE VALUE OF SECTION 2**
20. **ENTER THE VALUE OF X = 2**
21. k(x) = 1057  l(x) = 993  m(x) = -1023  n(x) = 126897  p(x) = 511297137
22. (k(x), l(x), m(x)) = (1057, 993, -1023) is a Singular triple with D(32x)
23. (l(x), m(x), n(x)) = (993, -1023, 126897) is a Singular triple with D(32x)
24. (m(x), n(x), p(x)) = (-1023, 126897, 511297137) is a Singular triple with D(32x)
5 Conclusion
In this paper, the development of finding order of singular 3-tuples \( \{ f(x), g(x), h(x) \} \), \( \{ g(x), h(x), i(x) \} \) etc entailing Abel’s polynomial and Cyclotomic polynomial in which the arithmetic mean of any two polynomials in each 3-tuples added by a monomial with integer coefficients leftovers square of a polynomial is recognized. Additionally, Python program for inspection of each of such singular 3-tuples with numerical values is presented. To conclude this, one can pursue varieties of 3-tuples nourishing innumerable exciting features.

Acknowledgement. Authors are grateful to the Editor and Reviewer for their fruitful suggestions to improve the paper.

References