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(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

A GEOMETRIC PROGRAMMING APPROACH TO CONVEX MULTI-OBJECTIVE PROGRAMMING PROBLEMS

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Abstract

Over the past few years, convex optimization has played a vital role in the study of complex engineering problems in different fields. Geometric programming is one of the available techniques particularly used for solving nonconvex programming problems. But in this article, a suitable attempt has been made to solve a real-life model on convex multi-objective using geometric programming technique with help of the ϵ-constraint method and result is compared with the solutions obtained by fuzzy technique. Finally, a conclusion is presented by analyzing the solutions to a numerical problem.

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Keywords and Phrases: Convex optimization; Multi-objective optimization; Geometric programming; ϵ-constraint method; Fuzzy.

1 Introduction

There is hardly ever a situation that arises where one can expect only one goal at the same time. For example, while purchasing something we are expecting a high-quality product at a low price. In the same manner, most of the technical problems involve more than one goal to maximize quality versus minimizing the cost. This ambiguity proceeds to the field of multi-objective optimization. There exists an infinite number of optimal solutions to multi-objective problems because of conflicting objectives. The group of these agreement solutions is called the pareto set[8] and solutions are called pareto solutions. But the question arises how to combine different objectives to yield optimal solutions for our modeled problems. In this article geometric programming technique has been discussed for solving different engineering applications, which was developed by Duffin, Peterson and Zener [9]. Nowadays, it can be used in various fields like circuit design [4, 5], production, and constructing models for market planning [2, 11]. Many important problems in engineering need to solve non-convex multi-objective optimization problems to achieve the proper results. But in this article, we have tried to discuss convex multi-objective optimization problems. The optimization problem in which objective functions, as well as, constraints are convexly is known as the convex problems. Recently convex optimization methods are widely used in the design and analysis of communication systems and signal processing algorithms because in convex problems local optimum is considered as global optimum. Luo et al.[12], in their recent paper have shown how convex optimization is useful for communications and signal processing. Different applications in the field of automatic control systems, electronic circuit design, data analysis, statistics, and finance has been discovered since its development. The basic advantages of the convex optimization problem for solving a problem very reliably and efficiently using interior-point methods or other special methods have been shown by Boyd et al.[6]. The connectedness properties of quasi-convex problems using cone-efficient set of the solution have been shown by Zhou[16]. An and Liu[1] have proven different necessary and sufficient conditions for getting weakly Pareto solutions and weakly efficient solutions of convex multi-objective programming problems. For deriving the solutions of multi-objective convex problems using both equality and inequality constraints, Shang et al.[15] have discussed the homotopy method which does not require any starting point to be the feasible point.

The paper is structured as follows: beginning with the introduction, the basic concept of convex optimization has been discussed in sec 2. The modeling of multi-objective convex geometric problems discussed in sec 3 and corresponding solution procedure by ϵ-constraint method discussed in sec 4. The rule of convergence of solutions by ϵ-constraint method and a suitable example based on our discussion given in sec 5 and sec 6 respectively. Finally, conclusion drawn is presented in Sec 7.

2 Basic concepts of convex set, convex function and convex programming

The study of fundamental concepts of convexity and its use in the construction of mathematical models[7, 14] related to various physical problems are key for everyone.

Convex sets:

A set $S \subset R^n$ is said to be convex if for any two points $x, y \in S$, their convex linear combination also lies in S. Mathematically, it is represented as.

λ x + (1-λ)y$\in$ S for all $\lambda \in$ [0, 1] and $x, y \in$ S

Since the line segment joining any two distinct points is no longer on the unit sphere, the unit sphere is not convex whereas the unit ball is a convex set. Generally, a convex set is a solid object having no holes and always curved outward. An important property regarding convex sets is that the intersection of more than one number of convex sets is again convex.

Convex functions

A function $f(x) : R^n \to R$ is called convex if for any two points $x, y \in R^n$ the following condition must be satisfied

f(λ x + (1-λ)y)$\leq$ λ $f(x)$ + (1-λ)$f(y)$ for all $\lambda \in$ [0, 1].

Geometrically, a function is called convex if the line joining x and y lies above the graph of f that is called an epigraph.

Theorem 2.1. Let f is a function which is defined and differentiable on $domf$. Then f is called convex if and only if $f(y) \geq f(x) + \nabla f(x)^T (y - x)$ for all $x, y \in domf$.

It will be strictly convex if and only if $domf$ is convex for every $x, y \in domf$and $x \neq y$. Then we have $f(y) > f(x) + \nabla f(x)^T (y - x)$.

Theorem 2.2. *Let f be a twice differentiable function on its convex domain $domf$. Then f is called convex if and only if the hessian of the function should be positive semi definite: $\nabla^2 f(x) \geq 0$ for all$x \in domf$. However, if $\nabla^2 f(x) > 0$ for all $x \in domf$, then f will be strictly convex.*

Basic Properties of convex functions

· f is called strictly convex if the strict inequality holds and $x \neq y$.
· if f is concave then $-f$ will be convex.
· if f is convex then its epigraph epi f is also a convex set.
· If f is a convex function over a convex set S, then the local minimum will be the global minimum.

Convex Optimization

The optimization problem of the form

$$minf(x) \tag{2.1}$$

subject to

$$f_i(x) \leq 0, \quad i = 1, 2, ..., m, \tag{2.2}$$

$$g_j(x) = 0, \quad j = 1, 2, 3, ..., n, \qquad x \in S, \tag{2.3}$$

is called convex optimization problem if inequality constraints are convex and equality constraints are affine where S is a convex set. The optimization problem will be non-convex if one of the conditions is violated. Convex optimization problems have three important properties that make them fundamentally more powerful than any generic non-convex optimization problems.

· Local optimum is necessarily a global optimum;
· Detection of exact infeasibility ;
· Very large problems can be handled by efficient numerical solution methods.

Theorem 2.3. *(Local optima implies global optima)Let Q be a convex optimization problem and let $x^* \in S$ be a point such that $f(x^*) \leq f(y)$ for all feasible y with $||x^* - y|| \leq \rho$. Then $f(x^*) \leq f(y)$ for all feasible y.*

proof: The proof is by contradiction. Assume that there is some feasible t such that $f(t) < f(x^*)$. Then take $y = \alpha x^* + (1 - \alpha)t$ for $\alpha \in (0, 1)$ close to 1. We claim this point is feasible. The affine constraints are satisfied due to linearity, since both $x^*$and t are feasible. As for the inequality constraints, by convexity we get

$g_i(\alpha x^* + (1 - \alpha)t) \leq \alpha g_i(x^*) + (1 - \alpha)g_i(t) \leq 0$.

Hence y is feasible. However, the objective value is strictly smaller than $f(x^*)$, since

$f(\alpha x^* + (1 - \alpha)t) \leq \alpha f(x^*) + (1 - \alpha)f(t) < f(x^*)$.

For α close to one, we will get $||x^* - y|| \leq \rho$, which is a contradiction.

3 Multi-objective convex geometric programming problem *(MOCGPP)*

Optimization of multiple objective functions subject to given constraints is known as multi-objective optimization. In this process a solution which is optimal with respect to one objective function may not be same for remaining objectives. As a result, we can not find only one global optimal solution. Therefore optimizing a problem means, find a solution such that it should acceptable to all the decision makers. A multi-objective geometric problem will be convex if all objective functions as well as constraints are convex. It can be defined mathematically as follows:

To determine $x = (x_1, x_2, ..., x_n)^T$ in order to

$$\min : f_k(x) = \sum_{i=1}^{\alpha^k} \beta_{0i}^k \prod_{j=1}^{n} x_j^{c_{0ij}^k}, \quad k = 1, 2, ..., p. \tag{3.1}$$

Subject to

$$g_t(x) = \sum_{i=1}^{\alpha_t} \beta_{it} \prod_{j=1}^{n} x_j^{c_{itj}} \leq 1, \quad t = 1, 2, ..., m, \tag{3.2}$$

$$x_j > 0, \quad j = 1, 2, ..., n, \tag{3.3}$$

where all objective functions and constraints are convex and
$\beta_{0i}^k \geq 0 \ \forall \ k$ and i,
$\beta_{it} \geq 0 \ \forall \ i$ and t,
c_{0tj}^k and c_{itj} are real numbers $\forall \ i, j, k, t$,
α^k = no. of terms in the k^{th} objective function $f_k(x)$,
α_t= no. of terms in the t^{th} constraint.

4 The ϵ-constraintmethod

This method was developed first by Haimes et al.[10] for generating pareto optimal solutions for the multi-objective optimization problems. In this method at a time, only one of the objective functions solved expressing other objective functions as constraints. This method can be stated as:

$$\min : \ f_0^l(x), \quad where \quad l \in \{1, 2, ..., k\}, \tag{4.1}$$

subject to

$$f_0^p(x) \leq \epsilon_p, \quad p = 1, 2, ..., k, \quad p \neq l, \tag{4.2}$$

$$g_i(x) \leq 1, \quad i = 1, 2, ..., m. \tag{4.3}$$

We define

$$L_p \leq \epsilon_p \leq U_p, \quad p = 1, 2, ..., k, \quad p \neq l.$$

where

$$L_p = \min_{\forall x \in X} \quad f_0^p(x), \quad p = 1, 2, ..., k.$$

and

$$U_p = \max_{\forall x \in X} \quad f_0^p(x), \quad p = 1, 2, ..., k,$$

$$x \in X, \quad X \quad being \ \ the \ \ feasible \ \ region \ .$$

Compromise solutions of the problems can be generated considering the values of ϵ_p in the interval $[L_p, U_p]$ for $p = 1, 2, ..., k$.

5 Test of convergence of solutions by ϵ-constraint method

Sometimes, it is necessary to check the convergence of optimal solutions in multi-objective problems. Regarding this, Ojha and Biswal[13] in their recent paper have shown how the pareto solutions converges using ϵ-constraint method. Below given steps are some of the notes representing converges of the solutions.

Step 1. Determine the bounds of the objectives $(f_0^{(l)}(x), l = 1, 2, ..., k)$ using ideal points $X^{(1)}, X^{(2)}, ..., X^{(k)}$ as obtained by geometric programming technique. Let L_l and U_l are the least and best values of $f_0^{(l)}$ i.e $L_l \leq f_0^{(l)}(x) \leq U_l, \quad l = 1, 2, ..., k$.

Step 2. Consider ϵ_l be a point in the interval $[L_l, U_l]$ such that $L_l \leq \epsilon_l \leq U_l, \quad l = 1, 2, ..., k$.

Step 3. If we assign different values to ϵ_l in the interval $[L_l, U_l]$, it will initiate a set of pareto optimal

solution.
Step 4. Weigh the differences between the pareto solutions with the solution obtained by fuzzy method.
Step 5. If the solution obtained in step 3 is same as that obtained in step 4, then stop and accept the solution of the problem.

6 Numerical example

Let's consider and solve the following example on the basis of our above discussion.
Example 6.1.
Find x_1, x_2 and x_3 to

$$\min f_1(x) = 2x_1^{-1}x_2^{-1} + 20x_2 + 12x_3^{-1}, \tag{6.1}$$

$$\min f_2(x) = 4x_1^{-1}x_2^{-1}x_3^{-1} + 20x_2^2 + 10x_3^{-1}, \tag{6.2}$$

subject to

$$2x_1^2x_2^{-1/2} + 2x_2^{-1}x_3^2 \leq 1, \tag{6.3}$$

$$where \quad x_1, x_2, x_3 \geq 0. \tag{6.4}$$

Verification of convexity of objective functions

It can be shown that a function $f(x_1, x_2, ...x_n)$ is a convex function if and only if the matrix of second order derivatives or Hessian matrix is positive semi-definite and principal minor determinants of this matrix are all non negative.

for example, if $H(x) = [\frac{\partial^2 f}{\partial_{x_i}.\partial_{x_j}}] = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, where i=1, 2, 3 and j=1, 2, 3

Let us verify the convexity of 1st objective function $f_1(x) = 2x_1^{-1}x_2^{-1} + 20x_2 + 12x_3^{-1}$
As per the definition, Hessian matrix for the given problem will be

$$H(x) = \begin{pmatrix} 4x_1^{-3}x_2^{-1} & 2x_1^{-2}x_2^{-2} & 0 \\ 2x_1^{-2}x_2^{-2} & 4x_1^{-1}x_2^{-3} & 0 \\ 0 & 0 & 24x_3^{-3} \end{pmatrix}.$$

Corresponding to the above hessian matrix, the determinant of the principal minors $D_1 = 4x_1^{-3}x_2^{-1}, D_2 = 12x_1^{-4}x_2^{-4}$ and $D_3 = 288x_1^{-4}x_2^{-4}x_3^{-3}$ are positive for the design variables x_1, x_2 and x_3. So it is a convex function.

Similarly for the 2nd objective function $f_2(x) = 4x_1^{-1}x_2^{-1}x_3^{-1} + 20x_2^2 + 10x_3^{-1}$, hessian matrix will be

$$H(x) = \begin{pmatrix} 8x_1^{-3}x_2^{-1}x_3^{-1} & 4x_1^{-2}x_2^{-2}x_3^{-1} & 4x_1^{-2}x_2^{-1}x_3^{-2} \\ 4x_1^{-2}x_2^{-2}x_3^{-1} & 84x_1^{-1}x_2^{-3}x_3^{-1} + 40 & 4x_1^{-1}x_2^{-2}x_3^{-2} \\ 4x_1^{-2}x_2^{-1}x_3^{-2} & 4x_1^{-1}x_2^{-2}x_3^{-2} & 8x_1^{-1}x_2^{-1}x_3^{-3} + 20x_3 \end{pmatrix}.$$

Corresponding to the above hessian matrix, the determinant of the principal minors $D_1 = 8x_1^{-3}x_2^{-1}x_3^{-1}, D_2 = 64x_1^{-4}x_2^{-4}x_3^{-2} + 320x_1^{-3}x_2^{-1}x_3^{-1} - 16x_1^{-4}x_2^{-4}x_3^{-2}$ and $D_3 = 2564x_1^{-5}x_2^{-5}x_3^{-5} + 19208x_1^{-4}x_2^{-2}x_3^{-4} + 6400x_1^{-3}x_2^{-1}x_3^{-4} - 192x_1^{-4}x_2^{-4}x_3^{-5}$ are positive for the design variables x_1, x_2 and x_3. So it is a convex function.

For the given constraint $2x_1^2x_2^{-1/2} + 2x_2^{-1}x_3^2 \leq 1$, the hessian matrix will be

$$H(x) = \begin{pmatrix} 4x_2^{-1/2} & -2x_1x_2^{-3/2} & 0 \\ -2x_1x_2^{-3/2} & 3/2x_1^2x_2^{-5/2} + 4x_2^{-3}x_3^2 & -4x_2^{-2}x_3 \\ 0 & -4x_2^{-2}x_3 & 4x_2^{-1} \end{pmatrix}.$$

Now this hessian matrix can be checked for convexity of constraints. We found it is also convex function. Therefore the given optimization problem is a convex optimization problem.

Here we have divided this problem into two sub problems as primal(i) and primal(ii)in order to find its optimal solutions.

Primal (i) Corresponding dual and their solutions
Primal(i).
Find x_1, x_2 and x_3 to

$$\min f_1(x) = 2x_1^{-1}x_2^{-1} + 20x_2 + 12x_3^{-1}, \tag{6.5}$$

subject to

$$2x_1^2x_2^{-1/2} + 2x_2^{-1}x_3^2 \leq 1, \tag{6.6}$$

$$where \;\; x_1,\; x_2,\; x_3 \geq 0. \tag{6.7}$$

Dual:

The Dual of the above primal will be as follows:

$$\max_t : V(t) = \left(\frac{2}{t_{01}}\right)^{t_{01}} \left(\frac{20}{t_{02}}\right)^{t_{02}} \left(\frac{12}{t_{03}}\right)^{t_{03}} \left(\frac{2}{t_{11}}\right)^{t_{11}} \left(\frac{2}{t_{12}}\right)^{t_{12}} (t_{11}+t_{12})^{(t_{11}+t_{12})}. \tag{6.8}$$

Subject to

$$\begin{aligned} t_{01}+t_{02}+t_{03} &= 1, \\ -t_{01}+2t_{11} &= 0, \\ -t_{01}+t_{02}-\frac{1}{2}t_{11}-t_{12} &= 0, \\ -t_{03}+2t_{12} &= 0, \\ t_{01},\; t_{02},\; t_{03},\; t_{11},\; t_{12} &\geq 0, \end{aligned}$$

Solution of primal(i) is $f_1(x) = 45.10214$ for $x_1 = 0.3390946,\ x_2 = 0.9131737$ and $x_3 = 0.5888183$ where as its corresponding dual is $f_1^* = 45.10214$ for $t_{01} = 0.1432052,\ t_{02} = 0.4049359,\ t_{03} = 0.4518589,$ '$t_{11} = 0.0716025,\ t_{12} = 0.2259295$.

Primal(ii) Corresponding dual and their solutions

Primal(ii).

Find x_1, x_2 and x_3 to minimize,

$$\min f_2(x) = 4x_1^{-1}x_2^{-1}x_3^{-1} + 20x_2^2 + 10x_3^{-1}, \tag{6.9}$$

subject to

$$2x_1^2x_2^{-1/2} + 2x_2^{-1}x_3^2 \leq 1, \tag{6.10}$$

$$where \;\; x_1,\; x_2,\; x_3 \geq 0. \tag{6.11}$$

Dual.

$$\max_t : V(t) = \left(\frac{4}{t_{01}}\right)^{t_{01}} \left(\frac{20}{t_{02}}\right)^{t_{02}} \left(\frac{10}{t_{03}}\right)^{t_{03}} \left(\frac{2}{t_{11}}\right)^{t_{11}} \left(\frac{2}{t_{12}}\right)^{t_{12}} (t_{11}+t_{12})^{(t_{11}+t_{12})}. \tag{6.12}$$

Subject to

$$\begin{aligned} t_{01}+t_{02}+t_{03} &= 1, \\ -t_{01}+2t_{11} &= 0, \\ -t_{01}+2t_{02}-\frac{1}{2}t_{11}-t_{12} &= 0, \\ -t_{01}-t_{03}+2t_{12} &= 0, \\ t_{01}, t_{02}, t_{03}, t_{11}, t_{12} &\geq 0. \end{aligned}$$

Solution of primal $f_2(x) = 54.28115$ for $x_1 = 0.40671,\ x_2 = 0.9880575$ and $x_3 = 0.5741136$ where as its dual will be $f_2^* = 54.28115$ for $t_{01} = 0.3194082,\ t_{02} = 0.3597041,\ t_{03} = 0.3208877,\ t_{11} = 0.1597041,\ t_{12} = 0.3201480$.

Replacing the value of f_1 in f_2 and f_2 in f_1, we find both lower and upper bound of the functions:

$L_1 = 45.10214 \leq f_1 \leq 45.63988 = U_1$,

and $L_2 = 54.28115 \leq f_2 \leq 55.64130 = U_2$.

Defining ϵ_1 and ϵ_2 based on the values of f_1 and f_2, we have

$45.10214 \leq \epsilon_1 \leq 45.63988$ and $54.28115 \leq \epsilon_2 \leq 55.64130$.

We can observe, as the value of ϵ_1 and ϵ_2 changes within their range, the value of objective functions f_1 and f_2 also changes and are converging towards their suitable compromise values.

Primal(i) and its solution by ϵ-constraint method

Find $x_1,\ x_2$ and x_3 to

$$\min f_1(x) = 2x_1^{-1}x_2^{-1} + 20x_2 + 12x_3^{-1}, \tag{6.13}$$

subject to

$$4x_1^{-1}x_2^{-1}x_3^{-1} + 20x_2^2 + 10x_3^{-1} \leq \epsilon_2, \tag{6.14}$$

$$2x_1^2x_2^{-1/2} + 2x_2^{-1}x_3^2 \leq 1, \tag{6.15}$$

$$where \;\; x_1,\; x_2,\; x_3 > 0. \tag{6.16}$$

Different values of the Primal(i) will be obtained by changing ϵ_2 between 54.28115 to 55.63988 given in Table 6.1.

***Table* 6.1:** (Optimal solution of Primal(i))

ϵ_2	x_1	x_2	x_3	primalf_1
54.3	0.3968714	0.9834149	0.5792341	45.50971
54.5	0.3744542	0.9677492	0.5881633	45.27651
54.7	0.3634936	0.9560997	0.5906395	45.19375
54.9	0.3557495	0.9455062	0.5913476	45.14871
55.1	0.3497639	0.9355950	0.5911580	45.12281
55.3	0.3449175	0.9262459	0.5904456	45.10875
55.5	0.3408725	0.9173958	0.5894118	45.10279
55.6	0.3390946	0.9131737	0.5888183	45.10214

Dual

The Dual of the above primal will be:

$$\max_t : V(t) = \left(\frac{1}{t_{01}}\right)^{t_{01}} \left(\frac{1}{4t_{02}}\right)^{t_{02}} \left(\frac{3}{4t_{11}}\right)^{t_{11}} \left(\frac{3}{8t_{12}}\right)^{t_{12}} (t_{11}+t_{12})^{(t_{11}+t_{12})}$$
$$\left(\frac{2}{\epsilon_2 t_{21}}\right)^{t_{21}} \left(\frac{2}{\epsilon_2 t_{22}}\right)^{t_{22}} (t_{21}+t_{22})^{(t_{21}+t_{22})}, \tag{6.17}$$

subject to

$$\begin{aligned} t_{01} + t_{02} &= 1, \\ -2t_{01} + 2t_{11} - t_{21} + t_{22} &= 0, \\ 2t_{02} - 2t_{11} + t_{12} - t_{21} + t_{22} &= 0, \\ -t_{02} + 2t_{12} - t_{21} &= 0, \\ t_{01},\; t_{02},\; t_{11},\; t_{12},\; t_{21},\; t_{22} &\geq 0. \end{aligned} \tag{6.18}$$

As the value of ϵ_2 will change between 54.28115 to 55.64130, the changes occur in the dual value is given in the Table 6.2.

***Table* 6.2:** (Dual Solution)

ϵ_2	t_{01}	t_{02}	t_{03}	t_{11}	t_{12}	t_{13}	t_{21}	t_{22}	$Dual f_1$
54.3	0.1126002	0.432175	0.455221	1.23822	1.35357	1.208159	0.675410	1.45080	45.50971
54.5	0.1218983	0.427481	0.450619	0.240704	0.240232	0.218063	0.181301	0.454694	45.27651
54.7	0.1273361	0.423111	0.449552	0.127157	0.119299	0.110479	0.127246	0.343594	45.19375
54.9	0.1316970	0.418840	0.449462	0.075176	0.668398	0.063216	0.103437	0.293928	45.14871
55.1	0.1354475	0.414688	0.449864	0.043940	0.037172	0.035917	0.089675	0.264842	45.12281
55.3	0.1387802	0.410672	0.450547	0.022454	0.018169	0.0017934	0.080617	0.245468	45.10875
55.5	0.1418002	0.406802	0.451397	0.006534	0.005608	0.0005108	0.074167	0.231528	45.10279
55.6	0.1432051	0.404935	0.451859	0.22940×10^{-7}	0.17625×10^{-7}	0.18190×10^{-7}	0.071600	0.225929	45.10214

From table 6.2, we found the optimal value of dual, that is 45.10214 for $t_{01} = 0.143051$, $t_{02} = 0.4049359$, $t_{03} = 0.4518591$, $t_{11} = 0.22940 \times 10^{-7}$, $t_{12} = 0.17625 \times 10^{-7}$, $t_{13} = 0.18190 \times 10^{-7}$, $t_{21} = 0.071600$, $t_{22} = 0.225929$

Primal(ii) and its solution by ϵ-constraint method
Find x_1, x_2 and x_3 to

$$\min f_2(x) = 4x_1^{-1}x_2^{-1}x_3^{-1} + 20x_2^2 + 10x_3^{-1}, \tag{6.19}$$

subject to

$$2x_1^{-1}x_2^{-1} + 20x_2 + 12x_3^{-1} \leq, \epsilon_1 \tag{6.20}$$

$$2x_1^2x_2^{-1/2} + 2x_2^{-1}x_3^2 \leq 1, \tag{6.21}$$

$$where \;\; x_1, \; x_2, \; x_3 \geq 0. \tag{6.22}$$

Solutions of the Primal(ii) obtained by changing the values of ϵ_1 between 45.10214 to 45.63988 given in Table 6.3.

***Table* 6.3:** (Optimal solution of Primal(ii))

ϵ_1	x_1	x_2	x_3	primalf_2
45.11	0.345469	0.927377	0.590554	55.2752
45.3	0.377145	0.970123	0.587336	54.46271
45.5	0.396079	0.982994	0.579618	54.30320
45.6	0.403834	0.986803	0.575673	54.28274
45.61	0.404565	0.987130	0.575281	54.28203
45.63	0.406007	0.987758	0.574499	54.28125
45.639	0.406647	0.988031	0.574148	54.28115

Dual
The Dual of the above Primal will be

$$\max_t : V(t) = \left(\frac{1}{t_{01}}\right)^{t_{01}} \left(\frac{1}{4t_{02}}\right)^{t_{02}} \left(\frac{3}{4t_{11}}\right)^{t_{11}} \left(\frac{3}{8t_{12}}\right)^{t_{12}} (t_{11}+t_{12})^{(t_{11}+t_{12})}$$
$$\left(\frac{2}{\epsilon_2 t_{21}}\right)^{t_{21}} \left(\frac{2}{\epsilon_2 t_{22}}\right)^{t_{22}} (t_{21}+t_{22})^{(t_{21}+t_{22})}. \tag{6.23}$$

Subject to

$$\begin{aligned} t_{01} + t_{02} &= 1 \\ -2t_{01} + 2t_{11} - t_{21} + t_{22} &= 0, \\ 2t_{02} - 2t_{11} + t_{12} - t_{21} + t_{22} &= 0, \\ -t_{02} + 2t_{12} - t_{21} &= 0, \\ t_{01}, \; t_{02}, \; t_{11}, \; t_{12}, \; t_{21}, \; t_{22} &\geq 0. \end{aligned} \tag{6.24}$$

As the value of ϵ_1 will change between 45.10214 to 45.63988,the changes occur in dual value is given in Table 6.4.

***Table* 6.4:** (Dual Solution)

ϵ_1	t_{01}	t_{02}	t_{03}	t_{11}	t_{12}	t_{13}	t_{21}	t_{22}	$Dual f_2$
45.11	0.327074	0.350879	0.322046	0.0075578	0.028352	0.031227	0.167315	0.340174	54.29273
45.3	0.322068	0.356626	0.321305	0.0025116	0.0097014	0.010488	0.162289	0.326931	54.28255
45.5	0.319408	0.359704	0.320887	0.116862×10^{-7}	0.453369×10^{-7}	0.487989×10^{-7}	0.159704	0.320148	54.28115
45.6	0.319408	0.359704	0.320887	0.116030×10^{-7}	0.453361×10^{-7}	0.491941×10^{-7}	0.159704	0.320148	54.28115

From table 6.4, it can be found that the optimal value of the dual is 54.28115 for $t_{01} = 0.319408, \; t_{02} = 0.359704, \; t_{03} = 0.320887, \; t_{11} = 0.116030 \times 10^{-7}, \; t_{12} = 0.453361 \times 10^{-7}, \; t_{13} = 0.491941 \times 10^{-7}, \; t_{21} = 0.159704, \; t_{22} = 0.320148$

Solution by fuzzy method

Case-1.Solution of f_1

The crisp model of f_1 using fuzzy method can be stated as:

$max : \theta$

subject to

$$2x_1^{-1}x_2^{-1} + 20x_2 + 12x_3^{-1} + (45.63988 - 45.10214)\theta \le 45.63988,$$
$$2x_1^2x_2^{-1/2} + 2x_2^{-1}x_3^2 \le 1,$$
$$\theta > 0,\ x_i > 0\ for\ \ i = 1, 2, 3. \tag{6.25}$$

The optimal value of $f_1 = 45.10214$ for $\theta = 1.00008$, $x_1 = 0.3390946$, $x_2 = 0.9131737$, $x_3 = 0.5888183$.

Case-2.Solution of f_2

The crisp model of f_2 using fuzzy method is defined as follows:

$max : \theta$

s.to

$$4x_1^{-1}x_2^{-1}x_3^{-1} + 20x_2^2 + 10x_3^{-1} + (55.64130 - 54.28115)\theta \le 55.64130,$$
$$2x_1^2x_2^{-1/2} + 2x_2^{-1}x_3^2 \le 1,$$
$$\theta > 0,\ x_i > 0\ for\ \ i = 1, 2, 3. \tag{6.26}$$

The optimal value of $f_2 = 54.28115$ for $\theta = 1.0000$, $x_1 = 0.4067100$, $x_2 = 0.9880575$ and $x_3 = 0.5741136$.

Result Analysis

Usually, the geometric programming problems are non-convex in nature. In this paper, the problem taken for our research purposes is a convex problem. The main aim of taking convex problem is that global minima of a problem will be global optima if the considered test problem is convex.

The above work out shows how the solutions converging to $f_1 = 45.10214$ for $x_1 = 0.3390946$, $x_2 = 0.9131737$, $x_3 = 0.588183$ and $f_2 = 54.281159$ for $x_1 = 0.4067100$, $x_2 = 0.9880575$, $x_3 = 0.5741136$ by obtained by ϵ-constraint method which is exactly same as obtained by fuzzy method. However, the decision makers have multiple choices in ϵ-constraint method But there is only one choice in fuzzy method.

7 Conclusion

It is very interesting to search a suitable solution for the multi-objective problems. But only one difficulty arises because of conflicting of objectives. Due to non-convexity nature,sometimes it is difficult to find a best compromise solutions for multi-objective problems. Here we are not interested to explain whether a generic multi-objective optimization problem is efficiently solvable or not. However, we are interested,how to solve the problem efficiently. As far as the solutions of the problem is concerned, there exists optimization problems in which both objective and the constraints are convex. Under the given conditions a convex optimization problem can be solve up to to a given accuracy. In contrast, a non-convex problems is difficult to solve. The computational effort required to solve such problems by the best known numerical methods grows fast with the dimensions of the problems and therefore it is difficult to study an intrinsic nature of non-convex problems. Because of this, we have considered a multi-objective convex geometric problem to study its behaviour in order to find best compromise solutions.

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