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HYPERGEOMETRIC FORM OF $(1+x^2)^{\frac{ib}{2}}\exp(b\tan^{-1}x)$ AND ITS APPLICATIONS

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Abstract

In this article, we obtain hypergeometric forms (not available in the literature) of some composite functions like:

$$(1-y^2)^{\frac{d}{2}}\exp(d\tanh^{-1}y),\quad (1+x^2)^{\frac{g}{2}}\cos(g\tan^{-1}x),\quad (1+x^2)^{\frac{g}{2}}\sin(g\tan^{-1}x),$$
$$(1+x^2)^{\frac{ik}{2}}\cosh(k\tan^{-1}x),\quad (1+x^2)^{\frac{ik}{2}}\sinh(k\tan^{-1}x),\quad (1-y^2)^{\frac{g}{2}}\cosh(g\tanh^{-1}y),$$
$$(1-y^2)^{\frac{g}{2}}\sinh(g\tanh^{-1}y),\quad (1-y^2)^{\frac{ik}{2}}\cos(k\tanh^{-1}y),\quad (1-y^2)^{\frac{ik}{2}}\sin(k\tanh^{-1}y),$$

by using Leibniz theorem for successive differentiation and Maclaurin's series expansion. Some applications are also discussed.
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Keywords and Phrases: Hypergeometric function; Maclaurin series; Leibniz theorem.

1 Introduction and Preliminaries

In this paper, we shall use the following standard notations:

$\mathbb{N} :=\ \{1,2,3,\cdots\}\quad ;\mathbb{N}_0 := \mathbb{N}\bigcup\{0\}\,;\quad$ and $\quad \mathbb{Z}_0^- := \mathbb{Z}^-\bigcup\{0\} = \{0,-1,-2,-3,\cdots\}$.

The symbols $\mathbb{C},\mathbb{R},\mathbb{N},\mathbb{Z},\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.

The Pochhammer symbol $(\alpha)_p(\alpha,p\in\mathbb{C})$ is defined by ([15, p.22 Eq.(1), p.32, Q.N.(8) and Q.N.(9)],see also [17, p.23, Eq.(22) and Eq.(23)]).

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$ is accomplished by introducing any arbitrary number of numerator and denominator parameters [17, p.42, Eq.(1)].

Relations between hyperbolic and trigonometric functions are:

$$\cos(i\theta)=\cosh(\theta),\quad \sin(i\theta)=i\sinh(\theta), \tag{1.1}$$
$$\tan^{-1}(ix)=i\tanh^{-1}(x). \tag{1.2}$$

The Maclaurin's series is a particular case of Taylor's series expansion of a function about the origin, the Maclaurin series is given as:

$$\begin{aligned} y(x) &= (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \frac{x^5}{5!}(y_5)_0 + \cdots \\ &= \sum_{n=0}^{\infty}\frac{x^n}{n!}(y_n)_0 \qquad (1.3)\\ &= \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}(y_{2n})_0 + \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}(y_{2n+1})_0, \qquad (1.4)\end{aligned}$$

where, $(y_m)_0 = \left(\frac{d^m y}{dx^m}\right)_{x=0}$.

The general Leibniz rule, named after Gottfried Wilhelm Leibniz, generalizes the product rule (which is also known as "Leibniz's rule"), which states that if $U(x)$ and $T(x)$ are n-times differentiable functions, then the product $U(x).T(x)$ is also n-times differentiable and its nth derivative is given by:

$$D^n[U(x)\ T(x)] = ({}^nC_0)(D^n\ U)(D^0\ T) + ({}^nC_1)(D^{n-1}\ U)(D^1\ T) + ({}^nC_2)(D^{n-2}\ U)(D^2\ T) + \cdots +$$
$$+({}^nC_{n-1})(D^1\ U)(D^{n-1}\ T) + ({}^nC_n)(D^0\ U)(D^n\ T), \tag{1.5}$$
$$= \sum_{r=0}^{n} {}^nC_r(D^rT)(D^{n-r}U), \tag{1.6}$$
$$= \sum_{r=0}^{n} {}^nC_r(D^{n-r}T)(D^rU), \tag{1.7}$$

where $D = \frac{d}{dx}$.

Euler's formula is

$$\exp(i\theta) = \cos(\theta) + i\sin(\theta). \tag{1.8}$$

The present article is organized as follows. In section 3 we have given the proof of presented composite function. In section 4 we have discussed some applications using the relations between inverse trigonometric and inverse hyperbolic functions. The proof of the presented function is not available in the literature[1, 2, 3, 4, 6, 7, 9, 10, 5, 8] see also [11, 13, 12, 14, 16]. So we are interested to give the proof of hypergeometric form using Maclaurin series.

2 Hypergeometric Form of Composite Function

When the values of numerator, denominator parameters and arguments leading to the results which do not make sense are tacitly excluded, then the following hypergeometric form holds true:

$$(1+x^2)^{\frac{ib}{2}} \exp(b\tan^{-1}x) = {}_2F_1\left[\begin{array}{ccc} -\frac{ib}{2}, & \frac{-ib+1}{2}; & \\ & & -x^2 \\ & \frac{1}{2}; & \end{array}\right] + bx\,{}_2F_1\left[\begin{array}{ccc} \frac{1-ib}{2}, & \frac{2-ib}{2}; & \\ & & -x^2 \\ & \frac{3}{2}; & \end{array}\right]. \tag{2.1}$$

Note:In above hypergeometric function x and b can be purely real or purely imaginary or complex numbers.

3 Independent Proof of Hypergeometric Form

Proof of (2.1).

Let

$$y = (1+x^2)^{\frac{ib}{2}} \exp(b\tan^{-1}x). \tag{3.1}$$

Put $x = 0$ in equation (3.1), we get

$$(y)_0 = 1. \tag{3.2}$$

Differentiate equation (3.1) w.r.t. x and put $x = 0$, we get

$$(1+x^2)y_1 - (xi+1)yb = 0. \tag{3.3}$$
$$(y_1)_0 = b = i(-ib). \tag{3.4}$$

Differentiate equation (3.3) n-times w.r.t. x, and applying Leibniz theorem, we get

$$D^n\left\{(1+x^2)y_1\right\} - bD^n\left\{(xi+1)y\right\} = 0; \qquad n \geq 1.$$
$$(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} - b(xi+1)y_n - bin(1-b)y_{n-1} = 0; \quad n \geq 1. \tag{3.5}$$

Put $x = 0$ in equation(3.5) we get

$$(y_{n+1})_0 = -\left[n(n-1) - bin\right](y_{n-1})_0 + b(y_n)_0; \quad n \geq 1. \tag{3.6}$$

Put $n = 1, 2, 3, 4, 5, 6, 7, 8....$ in equation (3.6), we get

$$(y_2)_0 = b(b+i) = ib(1-ib), \tag{3.7}$$
$$(y_3)_0 = b(b+i)(b+2i) = i^2b(1-ib)(2-ib), \tag{3.8}$$
$$(y_4)_0 = b(b+i)(b+2i)(b+3i) = i^3b(1-ib)(2-ib)(3-ib), \tag{3.9}$$

$$(y_5)_0 = b(b+i)(b+2i)(b+3i)(b+4i) = i^4 b(1-ib)(2-ib)(3-ib)(4-ib), \tag{3.10}$$

$$(y_6)_0 = b(b+i)(b+2i)(b+3i)(b+4i)(b+5i) = i^5 b(1-ib)(2-ib)(3-ib)(4-ib)(5-ib) \tag{3.11}$$

$$(y_n)_0 = \prod_{k=1}^{n} \{b + (k-1)i\}, \tag{3.12}$$

$$(y_n)_0 = (i)^{n-1} b(1-ib)(2-ib)(3-ib)....(n-1-ib), \tag{3.13}$$

$$(y_n)_0 = i^n (-ib)_n. \tag{3.14}$$

We know by Maclaurin series expansion

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \frac{x^5}{5!}(y_5)_0 + ... \tag{3.15}$$

$$y = \sum_{n=0}^{\infty} \frac{(y_n)_0 \ x^n}{n!}, \tag{3.16}$$

$$y = \sum_{n=0}^{\infty} \frac{(-ib)_n (xi)^n}{n!}, \tag{3.17}$$

$$y = \sum_{n=0}^{\infty} \frac{(-ib)_{2n} (xi)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-ib)_{2n+1} (xi)^{2n+1}}{(2n+1)!}, \tag{3.18}$$

$$y = \sum_{n=0}^{\infty} \frac{\left(-\frac{ib}{2}\right)_n \left(-\frac{ib+1}{2}\right)_n (-x^2)^n}{\left(\frac{1}{2}\right)_n n!} + bx \sum_{n=0}^{\infty} \frac{\left(-\frac{ib+1}{2}\right)_n \left(-\frac{ib+2}{2}\right)_n (-x^2)^n}{\left(\frac{3}{2}\right)_n n!}. \tag{3.19}$$

Using definition of generalized hypergeometric function of one variable, we get the required result (2.1).

4 Applications

Suppose $x \in \mathbb{R}$ and b is purely imaginary in equation (2.1), then putting $x = iy$ and $b = -id$ in equation (2.1), where y is purely imaginary and d is purely real, we get

$$(1-y^2)^{\frac{d}{2}} \exp(d \tanh^{-1} y) = {}_2F_1 \left[\begin{matrix} -\frac{d}{2}, & \frac{1-d}{2}; & \\ & & y^2 \\ & \frac{1}{2}; & \end{matrix} \right] + dy \ {}_2F_1 \left[\begin{matrix} \frac{1-d}{2}, & \frac{2-d}{2}; & \\ & & y^2 \\ & \frac{3}{2}; & \end{matrix} \right]. \tag{4.1}$$

Putting $b = -ig$ in the equation (2.1), where g is purely real, we get

$$(1+x^2)^{\frac{g}{2}} \exp(-ig \tan^{-1} x) = {}_2F_1 \left[\begin{matrix} -\frac{g}{2}, & \frac{-g+1}{2}; & \\ & & -x^2 \\ & \frac{1}{2}; & \end{matrix} \right] - igx {}_2F_1 \left[\begin{matrix} \frac{1-g}{2}, & \frac{2-g}{2}; & \\ & & -x^2 \\ & \frac{3}{2}; & \end{matrix} \right]. \tag{4.2}$$

Applying Euler's formula on left hand side of equation (4.2), then on equating real and imaginary parts, we get

$$(1+x^2)^{\frac{g}{2}} \cos(g \tan^{-1} x) = {}_2F_1 \left[\begin{matrix} -\frac{g}{2}, & \frac{-g+1}{2}; & \\ & & -x^2 \\ & \frac{1}{2}; & \end{matrix} \right], \tag{4.3}$$

$$(1+x^2)^{\frac{g}{2}} \sin(g \tan^{-1} x) = gx \ {}_2F_1 \left[\begin{matrix} \frac{1-g}{2}, & \frac{2-g}{2}; & \\ & & -x^2 \\ & \frac{3}{2}; & \end{matrix} \right]. \tag{4.4}$$

Put $g = ik$ in equation (4.3) and (4.4), where k is purely imaginary, we get

$$(1+x^2)^{\frac{ik}{2}} \cosh(k \tan^{-1} x) = {}_2F_1 \left[\begin{matrix} -\frac{ik}{2}, & \frac{-ik+1}{2}; & \\ & & -x^2 \\ & \frac{1}{2}; & \end{matrix} \right], \tag{4.5}$$

$$(1+x^2)^{\frac{ik}{2}} \sinh(k \tan^{-1} x) = kx \ {}_2F_1 \left[\begin{matrix} \frac{1-ik}{2}, & \frac{2-ik}{2}; & \\ & & -x^2 \\ & \frac{3}{2}; & \end{matrix} \right]. \tag{4.6}$$

Putting $x = iy$ in equation (4.3) and (4.4) , where y is purely imaginary, we get

$$(1-y^2)^{\frac{g}{2}} \cosh(g \tanh^{-1} y) = {}_2F_1 \left[\begin{array}{c} -\frac{g}{2}, \quad \frac{-g+1}{2}; \\ \qquad\qquad\qquad y^2 \\ \frac{1}{2}; \end{array} \right], \tag{4.7}$$

$$(1-y^2)^{\frac{g}{2}} \sinh(g \tanh^{-1} y) = gy \ {}_2F_1 \left[\begin{array}{c} \frac{1-g}{2}, \quad \frac{2-g}{2}; \\ \qquad\qquad\qquad y^2 \\ \frac{3}{2}; \end{array} \right]. \tag{4.8}$$

Putting $x = iy$ and $g = ik$ in equation (4.3) and (4.4), where y and k are purely imaginary, we get

$$(1-y^2)^{\frac{ik}{2}} \cos(k \tanh^{-1} y) = {}_2F_1 \left[\begin{array}{c} -\frac{ik}{2}, \quad \frac{-ik+1}{2}; \\ \qquad\qquad\qquad y^2 \\ \frac{1}{2}; \end{array} \right], \tag{4.9}$$

$$(1-y^2)^{\frac{ik}{2}} \sin(k \tanh^{-1} y) = ky_2F_1 \left[\begin{array}{c} \frac{1-ik}{2}, \quad \frac{2-ik}{2}; \\ \qquad\qquad\qquad y^2 \\ \frac{3}{2}; \end{array} \right]. \tag{4.10}$$

5 Conclusion

In our present investigation, we have obtained hypergeometric forms of some composite functions using Maclaurin's series expansion and Leibniz theorem. We conclude our present investigation by observing that hypergeometric form of some other functions can be derived in an analogous manner. More over the results derived are significant. These are expected to find some potential applications in the fields of Applied Mathematics and Engineering Sciences.

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