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(Dedicated to Professor G. C. Sharma on His 85^{th} Birth Anniversary Celebrations)

DETECTION OF RELATIVELY PRIME INTEGER SOLUTIONS FOR TWO DISPARATE FORMS OF MORDELL CURVES

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Abstract

In this paper, two unrelated natures of Mordell curves $y^3 = x^2 + k$ where k is a multinomial of degree four and six are scrutinized for many sets of relatively prime integer solutions. The geometrical description of the curves for each set of solutions are also exhibited with an assistance of *MATLAB* tools.

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Keywords and Phrases: Mordell equation, relatively prime, integer solutions.

1 Introduction

A Diophantine problem is one in which the solutions are mandatory to be integers. If a Diophantine equation has a supplementary variable or variables occurring as exponents, it is known as an exponential Diophantine equation [3, 4, 5, 6]. In [7], the author considered the Diophantine equation $Y^3 = X^2 + C$ and found out the numerical solutions for $C = 9, 36, -16$. In [1, 2, 8] authors discovered consecutive integer solutions to the Diophantine equation $y^3 = x^2 + k$.

In this communication, two dissimilar forms of Mordell type curves $y^3 = x^2 + k$ where k is a polynomial of degree four and six are examined for various sets of relatively prime integer solutions. The geometrical representation of the curves for each set of solutions are also displayed with the help of *MATLAB* tools.

2 Evaluation of relatively prime integer solutions to Mordell type equations

It is well recognized that the Mordell equation is

$$a^2 = b^3 + r, \tag{2.1}$$

where r is a constant. If $S'(r)$ is the number of relatively prime integral solutions $(a, b) \in Z(u)$ of (2.1) where Z is the ring of integer, then $\limsup_{r\to\infty} S'(r) \geq 1$. Here the solutions to (2.1) are denoted by $M_j = (a_j, b_j)$, $j = 1, 2, 3$ etc.

In this paper two different kinds of Mordell type equations are considered in (2.1) and (2.2) for sleuthing relatively prime integer solutions.

2.1 Equation of the form $a^2 = b^3 + r$ where $r = 64u^4 + 64u^3 + 16u^2 + 1$

Consider the equation of type (2.1) as

$$a_j^2 = b_j^3 + 64u^4 + 64u^3 + 16u^2 + 1,\ u \in N. \tag{2.2}$$

The probable three sets of relatively prime integer solutions to (2.2) which are identified by

$$M_1 : a_1 = 8u^2 + 1,\ b_1 = -4u,$$
$$M_2 : a_2 = 8u^2 + 4u,\ b_2 = -1,$$
$$M_3 : a_3 = 8u^2 + 8u + 3,\ b_3 = 4u + 2.$$

Since (2.2) may have more than three solutions, $\limsup_{r\to\infty} S'(r) \geq 3$.

The following *MATLAB* program supports to treasure the numerical values for all variables.

```
clear all;close all;clc;
disp('the Equation is a^2 = b^3+(64*u^4)+(64*u^3)+(16*u^2)+1)');
u = -50:50;
k = (64*u.^4)+(64*u.^3)+(16*u.^2)+1;
a1 = 8*u.^2+1;b1 = -4*u;
a2 =(8*u.^2)+4*u;
[r,c]=size(a2);
b2 = -1*ones(r,c);a3 = (8*u.^2)+(8*u)+3;
b3 = (4*u)+2;
fprintf('The first solution \n');fprintf('a1 = %d\n',a1);
fprintf('b1 = %d\n',b1);fprintf('The second  solution \n');
fprintf('a2 = %d\n',a2);fprintf('b2 = %d\n',b2);
fprintf('The third solution\n');fprintf('a3 = %d\n',a3);
fprintf('b3 = %d\n',b3);
z1=[b1;a1;k];
surf(z1)
colormap(cool)
title('8*u^2+1,-4*u,(64*u^4)+(64*u^3)+(16*u^2)+1)')
figure
z2=[b2;a2;k];
surf(z2)
colormap(cool)
title('(8*u^2)+4*u,-1,(64*u^4)+(64*u^3)+(16*u^2)+1)')
figure
z3=[b3;a3;k];
surf(z3)
colormap(cool)
title('(8*u^2)+(8*u)+3,(4*u)+2,(64*u^4)+(64*u^3)+(16*u^2)+1)')
```

For easy verification the arithmetic values of $(a_j b_j)$ for few natural numbers u are tabulated in Table 2.1.

***Table* 2.1**

u	r	$M_1 = (a_1, b_1)$	$M_2 = (a_2, b_2)$	$M_3 = a_3, b_3)$
1	145	$(9, -4)$	$(12, -1)$	$(19, 6)$
2	1601	$(33. - 8)$	$(40, -1)$	$(51, 10)$
3	7057	$(73, -12)$	$(84, -1)$	$(99, 14)$
4	20737	$(129, -16)$	$(144, -1)$	$(163, 18)$
5	48401	$(201, -20)$	$(220, -1)$	$(243, 22)$

The subsequent table (Table 2.2) displays the left-hand and right-hand side values of (2.2) for all the above three sets of solutions.

***Table* 2.2**

u	M_1		M_2		M_3	
	a_1^2	$b_1^3 + r$	a_2^2	$b_2^3 + r$	a_3^2	$b_3^3 + r$
1	81	81	144	144	361	361
2	1089	1089	1600	1600	2601	2601
3	5329	5329	7056	7056	9801	9801
4	16641	16641	20736	20736	26569	26569
5	40401	40401	48400	48400	59049	59049

For all other values of u the values of (a_j, b_j) can be calculated by using the above *MATLAB* algorithm.

The geometrical representation of (2.2) for the above three sets of solutions are visualized in Figures 2.1, 2.2 and 2.3.

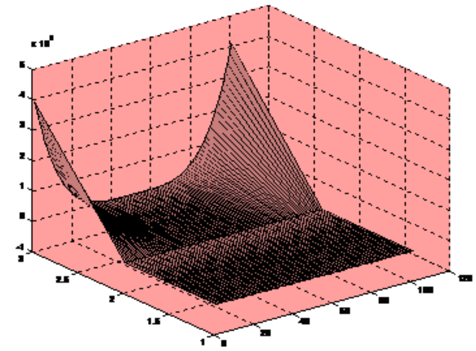

***Figure* 2.1:** Visualization of (2.2) for the solution M_1

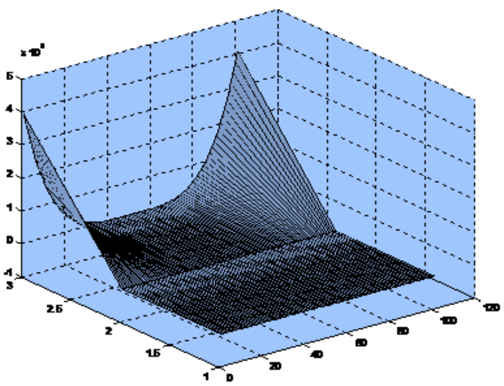

***Figure* 2.2:** Visualization of (2.2) for the solution M_2

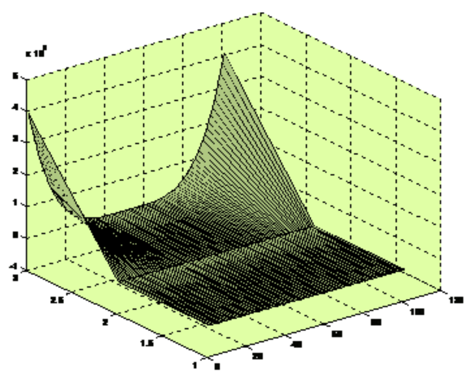

***Figure* 2.3:** Visualization of (2.2) for the solution M_3

2.2 Equation of the form $a^2 = b^3 + r$ where $r = \left(16(n+1)^6u^6 + 1\right)$

Consider an additional Mordell kind equation as

$$a_j^2 = b_j^3 + \left(16(n+1)^6u^6 + 1\right)u, \ n \in N. \tag{2.3}$$

It is experiential that (2.3) is fulfilled by six pair of values of (a_j, b_j). They are denoted by $\pm M_j, \ j = 1, 2, 3$ where $M_j = (a_j, b_j)$ and $-M_j = (a_j - b_j)$.

Among six set of solutions, the first three set of values of $(a_j, b_j), j = 1, 2, 3$ are pointed out by the following equations:

$$M_1 : a_1 = 4(n+1)^3u^3 + 1, \ b_1 = 2(n+1)u.$$

$$M_2 : a_2 = 4(n+1)^3u^3, \ b_2 = -1.$$

$$M_3 : a_3 = 4(n+1)^3 u^3 - 1,\ b_3 = -2(n+1)u.$$

Let us find the remaining three pairs (a_j, b_j), $j = 4,5,6$. If (a_1, b_1) and (a_2, b_2) are two distinct points on E, their sum (a', b') is given by

$$a' = \left(\frac{b_2 - b_1}{a_2 - a_1}\right)^2 - a_1 - a_2,$$
$$b' = \left(\frac{b_2 - b_1}{a_2 - a_1}\right)(a' - a_1) + b_1.$$

Since $-M_j = (a_j, -b_j)$ the co-ordinates of a_i, b_j for $M_j \pm M_k, 1 \le j < k \le 3$ such that only three out of these six points sustaining (2.3) are scrutinized that

$$\begin{aligned}
&M_4 = M_1 - M_2 :\\
&a_4 = -(64(n+1)^6u^6 - 96(n+1)^5u^5 + 96(n+1)^4u^4\\
&\qquad - 68(n+1)^3u^3 + 36(n+1)^2u^2 - 12(n+1)u + 3),\\
&b_4 = 16(n+1)^4u^4 - 16(n+1)^3u^3 + 12(n+1)^2u^2 - 18(n+1)u + 2,\\
&M_5 = M_2 - M_3 :\\
&a_5 = -(64(n+1)^6u^6 + 96\,(n+1)^5\,u^5 + 96(n+1)^4u^4 + 68(n+1)^3u^3\\
&\qquad + 36(n+1)^2u^2 + 12(n+1)u + 3),\\
&b_5 = 16(n+1)^4u^4 + 16(n+1)^3u^3 + 12(n+1)^2u^2 + 18(n+1)u + 2,\\
&M_6 = M_1 - M_3 :\\
&a_6 = -\left(8(n+1)^6u^6 + 1\right),\ b_6 = 4(n+1)^6u^6.
\end{aligned}$$

To check all the above coordinates (a_j, b_j) are coprime for all integer u, Euclid's algorithm may be applied. Example for $M_1 - M_2$,

$$\begin{aligned}
&(16(n+1)^4u^4 - 16(n+1)^3u^3 + 12(n+1)^2u^2 - 18(n+1)u + 2,\\
&\quad\Big(-64(n+1)^6u^6 + 96(n+1)^5u^5 - 96(n+1)^4u^4\\
&\quad + 68(n+1)^3u^3 - 36(n+1)^2u^2 + 12(n+1)u - 3\Big)\\
&= (8(n+1)^3u^3 + 8(n+1)^2u^2 + 2(n+1)u + 1,\\
&\qquad 6(n+1)^4u^4 + 16(n+1)^3u^3 + 12(n+1)^2u^2 + 18(n+1)u + 2)\\
&= (8(n+1)^2u^2 + 4(n+1) + 2, 4(n+1) + 1)\\
&= (4(n+1), 1)\\
&= 1.
\end{aligned}$$

Thus for $M_1 - M_2$, $\gcd(a_4, b_4) = 1$ and subsequently the pair $(a_4 b_4)$ is relatively prime to each other.

Similarly, it is evidenced that all other pairs (a_j, b_j) for the remaining five sets of solutions are relatively prime by utilizing Euclid's algorithm. Since, it can be able to find more than six groups of solutions in co-prime integers $\limsup_{r\to\infty} S'(r) \ge 6$.

3 *MATLAB* program for finding (a_j, b_j)

MATLAB program for finding (a_j, b_j) for distinct values of u are illustrated below:

```
clear all;close all;clc;
disp('the Equation is a^2 = b^3+(16(n+1)^6*u^6+1)');
n = 1;u = -50:50;k =(16*(n+1)^6*u.^6+1)
disp('the following 6 solution');
a1 = 4*(n+1)^3*u.^3+1;b1 = 2*(n+1)*u;
a2 = 4*(n+1)^3*u.^3;[r,c]=size(a2);
b2 = -1*ones(r,c);a3 = 4*(n+1)^3*u.^3-1;b3 = -2*(n+1)*u;
a4 = (-64*(n+1)^6*u.^6 )-(96*(n+1)^5*u.^5 )+(96*(n+1)^4*u.^4 )
-(68*(n+1)^3*u.^3)+(36*(n+1)^2*u.^2)-(12*(n+1)*u)+3;
```

```
b4 = (16*(n+1)^4*u.^4)-(16*(n+1)^3*u.^3)+(12*(n+1)^2*u.^2)
-(6*(n+1)*u)+2;
a5 = -((64*(n+1)^6*u.^6)+(96*(n+1)^5*u.^5)+(96*(n+1)^4*u.^4)
+(68*(n+1)^3*u.^3)+(36*(n+1)^2*u.^2)+(12*(n+1)*u)+3);
b5 = (16*(n+1)^4*u.^4)+(16*(n+1)^3*u.^3)+(12*(n+1)^2*u.^2)
+(6*(n+1)*u)+2;
a6 = -(8*(n+1)^6*u.^6+1);b6 = 4*(n+1)^4*u.^4;
fprintf('the first solution \n');fprintf('a1 = %d\n^',a1);
fprintf('b1 = %d\n',b1);fprintf('the second  solution \n');
fprintf('a2 = %d\n^',a2);fprintf('b2 = %d\n',b2);
fprintf('the third solution\n');fprintf('a3 = %d\n^',a3);
fprintf('b3 = %d\n',b3);fprintf('the fourth solution \n');
fprintf('a4= %d\n',a4);fprintf('b4 = %d\n',b4);
fprintf('the fifth solution \n');fprintf('a5 = %d\n^',a5);
fprintf('b5 = %d\n',b5);fprintf('the Sixth solution \n');
fprintf('a6 = %d\n^',a6);fprintf('b6 = %d\n',b6);
z1=[b1;a1;k];surf(z1)
colormap(cool)
title('4*(n+1)^3*u^3+1,2*(n+1)*u,(16(n+1)^6*u^6+1)')
figure
z2=[b2;a2;k];surf(z2)
colormap(cool)
title('4*(n+1)^3*u^3,-1,(16(n+1)^6*u^6+1)')
figure
z3=[b3;a3;k];
surf(z3)
colormap(cool)
title('4*(n+1)^3*u^3+1,-2*(n+1)*u,(16(n+1)^6*u^6+1)')
figure
z4=[b4;a4;k];surf(z4)
colormap(cool)
title('(-64*(n+1)^6*u^6 )-(96*(n+1)^5*u^5 )+(96*(n+1)^4*u^4 )
-(68*(n+1)^3*u^3)+(36*(n+1)^2*u^2)-(12*(n+1)*u)+3,
(16*(n+1)^4*u.^4)-(16*(n+1)^3*u.^3)+(12*(n+1)^2*u^2)
-(6*(n+1)*u)+2,(16(n+1)^6*u^6+1)')
figure
z5=[b5;a5;k];
surf(z5)
colormap(cool)
title('-(64*(n+1)^6*u^6)+(96*(n+1)^5*u^5)+(96*(n+1)^4*u^4)
+(68*(n+1)^3*u^3)+(36*(n+1)^2*u^2)+(12*(n+1)*u)+3),
(16*(n+1)^4*u^4)+(16*(n+1)^3*u^3)+(12*(n+1)^2*u^2)
+(6*(n+1)*u)+2,(16(n+1)^6*u^6+1)')
figure
z6=[b6;a6;k];
surf(z6)
colormap(cool)
title('-(8*(n+1)^6*u^6+1),4*(n+1)^4*u^4,(16(n+1)^6*u^6+1)')
```

Illustration: A

If $n = 1$, then the corresponding representation of (2.3) is

$$a_j^2 = b_j^3 + \left(1024u^6 + 1\right). \tag{3.1}$$

The six pair of solutions to (2.3) are viewed by

$M_1 : a_1 = 32u^3 + 1,\ b_1 = 4u.$

$M_2 : a_2 = 32u^3,\ b_2 = -1.$

$M_3 : a_3 = 32u^3 - 1,\ b_3 = -4u.$

$M_4 = M_1 - M_2 : a_4 = -\left(4096u^6 - 3072u^5 + 1536u^4 - 544u^3 + 144u^2 - 24u + 3\right).$

$b_4 = 256u^4 - 128u^3 + 48u^2 - 12u + 2.$

$M_5 = M_2 - M_3 : a_5 = -\left(4096u^6 + 3072u^5 + 1536u^4 + 544u^3 + 144u^2 + 24u + 3\right).$

$b_5 = 256u^4 + 128u^3 + 48u^2 + 12u + 2.$

$M_6 = M_1 - M_3 : a_6 = -\left(512u^6 + 1\right),\ b_6 = 64u^4.$

Table 3.1 shows the six values of (a_j, b_j) equivalent to $n = 1$.

***Table* 3.1**

u	r	M_1	M_2	M_3
1	1025	(33,4)	$(32, -1)$	$(31, -4)$
2	65537	(257,8)	$(256, -1)$	$(255, -8)$
3	746497	(865,12)	$(864, -1)$	$(863, -12)$
4	4194305	(2049,16)	$(2048, -1)$	$(2047, -16)$
5	16000001	(4001,20)	$(4000, -1)$	$(3999, -20)$
		M_4	M_5	M_6
		$(-2139, 166)$	$(-9419, 446)$	$(-513, 64)$
		$(-184595, 3242)$	$(-390003, 5338)$	$(-32769, 1024)$
		$(-2350443, 17678)$	$(-3872955, 24662)$	$(-373249, 5184)$
		$(-13992099, 58066)$	$(-20353379, 74546)$	$(-2097153, 16384)$
		$(-55295483, 145142)$	$(-74631723, 177262)$	$(-8000001, 40000)$

Table 3.2 displays the left-hand and right-hand side values of (2.3) for all the above six sets of solutions equivalent to $n = 1$.

***Table* 3.2**

u	M_1		M_2		M_3	
	a_1^2	$b_1^3 + r$	a_2^2	$b_2^3 + r$	a_3^2	$b_3^3 + r$
1	1089	1089	1024	1024	961	961
2	66049	66049	65536	65536	65025	65025
3	748225	748225	746496	746496	744769	744769
4	4198401	4198401	4190209	4190209	4190209	4190209
	M_4		M_5		M_6	
	a_4^2	$b_4^3 + r$	a_5^2	$b_5^3 + r$	a_6^2	$b_6^3 + r$
	4575321	4575321	88717561	88717561	263169	263169
	3.407×10^{10}	3.407×10^{10}	1.521×10^{11}	1.521×10^{11}	107380736	107380736
	5.524×10^{12}	5.524×10^{12}	1.499×10^{13}	1.499×10^{13}	1.393×10^{11}	1.393×10^{11}
	1.957×10^{14}	1.957×10^{14}	4.142×10^{14}	4.142×10^{14}	4.398×10^{12}	4.398×10^{12}

Illustration: B

If $n = 2$, then the required equation is

$$a_j^2 = b_j^3 + \left(11664u^6 + 1\right). \tag{3.2}$$

The equivalent solutions to (3.2) are given by

$M_1 : a_1 = 108u^3 + 1,\ b_1 = 6u.$

$M_2 : a_2 = 108u^3,\ b_2 = -1.$

$M_3 : a_3 = 108u^3 - 1,\ b_3 = -6u.$

$M_4 = M_1 - M_2 :$

$$a_4 = -\left(46656u^6 - 23328u^5 + 7776u^4 - 1836u^3 + 324u^2 - 36u + 3\right).$$
$$b_4 = 1296u^4 - 432u^3 + 108u^2 - 18u + 2.$$
$$M_5 = M_2 - M_3:$$
$$a_5 = -\left(46656u^6 + 23328u^5 + 7776u^4 + 1836u^3 + 324u^2 + 36u + 3\right).$$
$$b_5 = 1296u^4 + 432u^3 + 108u^2 + 18u + 2.$$
$$M_6 = M_1 - M_3 : a_6 = -\left(5832u^6 + 1\right).$$
$$b_6 = 324u^4.$$

Table 3.3 shows the six values of (a_j, b_j) equivalent $n = 2$.

***Table* 3.3**

u	r	M_1	M_2	M_3
1	11665	(109,6)	$(108, -1)$	$(107, -6)$
2	746497	(865,12)	$(864, -1)$	$(863, -12)$
3	8503057	(2917,18)	$(2916, -1)$	$(2915, -18)$
4	47775745	(6913,24)	$(6912, -1)$	$(6911, -24)$
		M_4	M_5	M_6
		$(-29559, 956)$	$(-79959, 1856)$	$(-5833, 324)$
		$(-2350443, 17678)$	$(-3872955, 24662)$	$(-373249, 5184)$
		$(-28926615, 94232)$	$(-40363383, 117668)$	$(-4251529, 26244)$
		$(-169093299, 365786)$	$(-217104339, 361226)$	$(-23887873, 82944)$

Table 3.4 displays the left-hand and right-hand side values of (2.3) for all the above six sets of solutions equivalent to $n = 2$.

***Table* 3.4**

u	M_1		M_2		M_3	
	a_1^2	$b_1^3 + r$	a_2^2	$b_2^3 + r$	a_3^2	$b_3^3 + r$
1	11881	11881	11664	11664	11449	11449
2	748225	748225	746496	746496	744769	744769
3	8508889	8508889	8503056	8503056	8497225	8497225
4	4.8×10^4	4.8×10^4	4.8×10^4	4.8×10^4	1.82×10^8	1.82×10^8
	M_4		M_5		M_6	
	a_4^2	$b_4^3 + r$	a_5^2	$b_5^3 + r$	a_6^2	$b_6^3 + r$
	873734481	873734481	6393441681	6393441681	34023889	34023889
	5.524×10^{12}	5.524×10^{12}	1.499×10^{13}	1.499×10^{13}	1.393×10^{11}	1.393×10^{11}
	8.367×10^{14}	8.367×10^{14}	1.629×10^{15}	1.629×10^{15}	1.807×10^{13}	1.807×10^{13}
	2.859×10^{16}	2.859×10^{16}	4.713×10^{16}	4.713×10^{16}	5.706×10^{14}	5.706×10^{14}

The three-dimensional shape of the original equation (2.3) for each of the above six set of solutions are envisaged below.

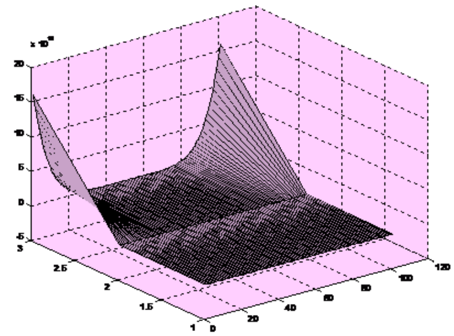

Figure **3.1:** Visualization of (2.3) for the solution M_1

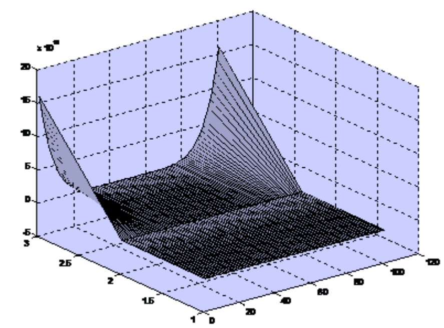

Figure **3.2:** Visualization of (2.3) for the solution M_2

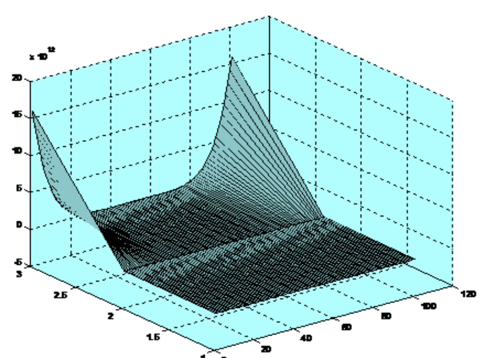

Figure **3.3:** Visualization of (2.3) for the solution M_3

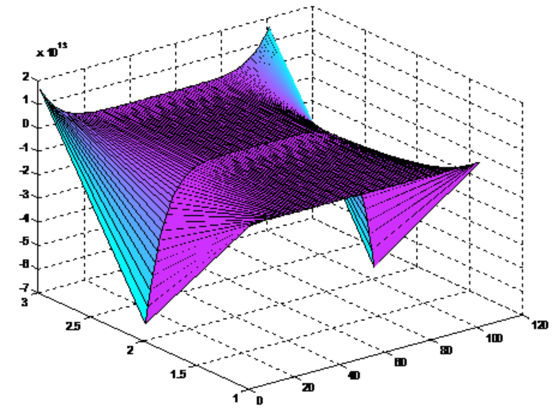

***Figure* 3.4:** Visualization of (2.3) for the solution M_4

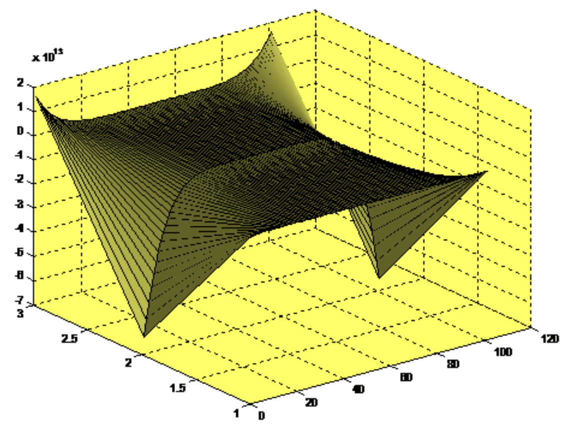

***Figure* 3.5:** Visualization of (2.3) for the solution M_5

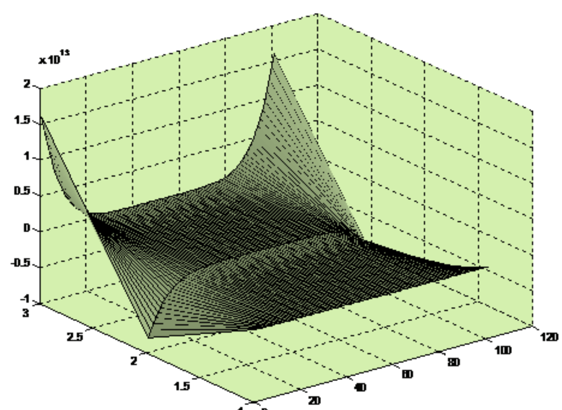

***Figure* 3.6:** Visualization of (2.3) for the solution M_6

4 Conclusion

Two discrete Mordell kinds equations $y^3 = x^2 + k$ where k is a polynomial of degree four and six are scrutinized for finite sets of relatively prime integer solutions. The pictures of the curves for each pair of solutions are also unveiled with the support of *MATLAB* program. In this manner, one can search varieties of such types of equations for consecutive odd and even integers.

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