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(Dedicated to Professor D. S. Hooda on His 80^{th} Birth Anniversary Celebrations)

EQUATIONAL CLASS-LIKE PROPERTIES OF 0-DISTRIBUTIVE LATTICES

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Abstract

In generalizing the notion of pseudo complemented lattice, Varlet [8] introduced the notion of 0-distributive lattices. In this paper, we prove that the class of 0-distributive lattices is not an equational class, but it is an equational class-like in the sense that while an equational class is closed under the operations of subalgebras, direct products and homomorphic images, the class of 0-distributive lattices is closed under the first two operations and as far as the third one is concerned, the homomorphism should be a monomorphism. We also prove that if $CS(L)$ is 0-semimodular then so is L.

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1. Introduction

An equational class of universal algebras is class of universal algebras which satisfies a set of identities. Equivalently, an equational class of universal algebras is a family of universal algebras which is closed under the operations of taking subalgebras, homomorphic images and direct product of members.

The following result is appeared in [2].

Theorem 1.1. *Let K be a class of lattices. A Class K of lattices is equational class or a variety iff K is closed under the formation of homomorphic images, sublattices and direct products.*

In the variety of lattices, the classes of modular lattices and distributive lattices are equational, while complete lattices and complemented lattices are not.

The concept of 0-distributive lattices was first introduced by Verlet [8]. Several authors have made contributions in different aspects of 0-distributive lattices. For example, one can refer to Subbarayan and Vethamanickam [7] and Balasubramani and Venkatanarasimhan [1], etc.

2. Class of 0-distributive lattices

In this section, we examine whether the class of 0-distributive is equational. We prove that it is closed under sublattices, direct products but a homomorphic image of a 0-distributive lattice is 0-distributive, only if the homomorphism is a monomorphism. For all undefined terms we refer to [3].

Definition 2.1. *A lattice L with 0 is said to be 0-distributive if $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$, for any a, b, c in L.*

Lemma 2.1. *A sublattice of a 0-distributive lattice is 0-distributive.*

Proof. Since $x \wedge y = 0$, $x \wedge z = 0$ in the sublattice imply $x \wedge (y \vee z) = 0$ in the sublattice, the sublattice is 0-distributive.

Lemma 2.2. *A product of 0-distributive lattices is 0-distributive.*

Proof. If $\{L_i / i \in I\}$ is family of 0-distributive lattices and if $X = x_i / i \in I, Y = y_i / i \in I$ and $Z = z_i / i \in I$ are any three elements of $\pi_{i \in I} L_i$, then $X \wedge Y = 0$ and $X \wedge Z = 0$ in $\pi_{i \in I} L_i$ imply that $x_i \wedge y_i = 0$ and $x_i \wedge z_i = 0$ for all $i \in I$ imply that $x_i \wedge (y_i \vee z_i) = 0$ for all $i \in I$, which implies that $X \wedge (Y \vee Z) = 0$ in $\{L_i / i \in I\}$

Lemma 2.3. *Homomorphic image of a 0-distributive lattice is 0-distributive, only if the homomorphism is one-one.*

Proof. Since, if L is a 0-distributive lattice and L_1 its homomorphic image, then let $f : L \rightarrow L_1$ be an one-one, onto homomorphism and let $x_1, \ y_1, \ z_1 \in L_1$ such that $x_1 \wedge y_1 = 0, x_1 \wedge z_1 = 0$ then there exists $x, \ y, \ z \in L$ such that $f(x) = x_1, \ f(y) = y_1, \ f(z) = z_1$.
Therefore, $f(x) \wedge f(y) = 0$ and $f(x) \wedge f(z) = 0$.
Hence $f(x \wedge y) = f(x \wedge z) = 0$.
Therefore, $f(x \wedge y) = f(0)$ and $f(x \wedge z) = f(0)$ which implies that $x \wedge y = 0$ and $x \wedge z = 0 \in L$, as f is one-one.
So, $x \wedge (y \vee z) = 0$ in L, as L is 0-distributive.
Hence $f[x \wedge (y \vee z)] = f(0)$.
That is, $f(x) \wedge [f(y) \vee f(z)] = f(0) = 0$.
Hence $x_1 \wedge (y_1 \vee z_1) = 0$.
So, L_1 is 0-distributive.

Theorem 2.1. *A Class of* 0*-distributive lattices is closed under the operations of taking sublattices, direct products and monomorphic images.*

Proof. It follows from Lemmas 2.1, 2.2 and 2.3.

3. Class of *CS(L)*

In the theory of lattice of convex sublattices, another approach was developed by Lavanya and Bhatta in their paper [4]. They define a new partial ordering relation on $CS(L)$. They proved that both L and $CS(L)$ are in the same equational class with respect to this new partial ordering.They have shown that L and $CS(L)$ satisfy the same identities with respect to this new partial ordering. This motivated us to look into the connection between L and $CS(L)$ for Eulerian lattices which are a class of lattices not defined by identities.

This new partial ordering was made of use by Ramanamurty [5]. He proved that for a lattice L, $CS(L)$ is semimodular then so is L.

The results of Lavanya and Bhatta [4] motivated us to work with the new partial ordering. In this section, we show by a counter example that with respect to this partial ordering relation $CS(L)$ need not be Eulerian, for an Euelrian lattice L and prove that if $CS(L)$ is 0-semimodular then so is L. The next definition appeared in [4].

Definition 3.1. *The binary relation $\leq$ on $CS(L)$, defined by, for $A, \ B$ in $CS(L)$, $A \leq B$ if and only if "for every $a \in A$ there exists a $b \in B$ such that $a \leq b$ and for every $b \in B$ there exists an $a \in A$ such that $b \geq a$".*

We provide the basic definitions and examples of Eulerian lattices that are needed to study of $CS(L)$, if L is Eulerian.

Definition 3.2. *Let P be a finite poset with a unique minimum and a unique maximum element. The poset P is said to be graded if all the maximal chains in P have the same length.*

Definition 3.3. *A function $r : P \rightarrow \{0, 1, ..., n\}$ is said to be the rank function on P if $r(x) = 0$ if x is a minimal element of P and $r(y) = r(x) + 1$ if y covers x in P. If $r(x) = i$ then we say that x has rank i.*

Definition 3.4. *The Möbius function μ on a poset P is an integer-valued function $\mu : P \times P \rightarrow Z$ satisfying the following conditions:*

$$\mu(x, y) = \begin{cases} 1 & \text{if } x = y \\ -\sum_{x \leq z < y} \mu(x, z) & \text{if } x \leq y \\ 0 & \text{if } x \nleq y. \end{cases}$$

Definition 3.5. *A finite graded poset P is said to be Eulerian if its Möbius function assumes the value $\mu(x, y) = (-1)^{l(x,y)}$ for all $x \leq y$ in P, where $l(x, y) = r(y) - r(x)$ and r is the rank function on P.*

Every Boolean algebra of rank n is Eulerian and the lattice C_4 is Eulerian which is given in figure 3.1. For the concept of Eulerian poset refer to [6, 7, 9].

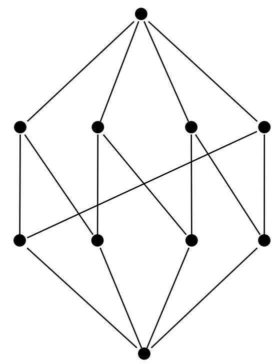

***Figure* 3.1**

Now, we show that $CS(L)$ need not be Eulerian even though L is Eulerian by the following counter-example. The lattice C_4 given in Figure 3.1 is an Eulerian lattice. Its lattice of convex sublattices $(CS(C_4), \leq)$ is given in Figure 3.2. Here, $CS(C_4)$ need not be Eulerian. It contains a 3-element interval $[a, b]$ whose Möbius function is $-1 \neq (-1)^{\rho(b)-\rho(a)}$.

The following two lemmas were proved by P.V.Ramanamurty in his paper [5].

Lemma 3.1. *If $a, b \in L$ then we show that $a \prec b$ in L if and only if $(a] \prec (b]$ in $CS(L)$.*

Lemma 3.2. *If a is atom of L if and only if $(a]$ is an atom of $CS(L)$.*

Definition 3.6. *If L is said to be 0-semimodular if whenever a is an atom of L and $x \in L$ such that $a \wedge x = 0$ then $x \vee a$ covers x.*

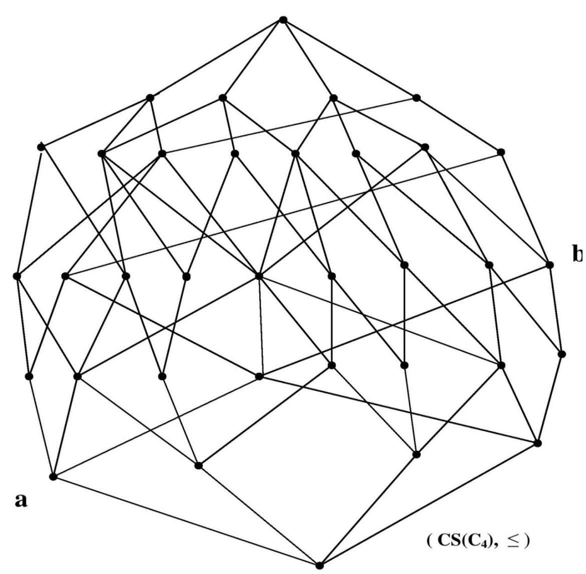

***Figure* 3.2**

Lemma 3.3. *If $CS(L)$ is 0-semimodular then so is L.*

Proof. If a is an atom of L and $x \in L$ such that $a \wedge x = 0$

Then $(a]$ is an atom of $CS(L)$, by the Lemma 3.2.

Now $a \wedge x = 0$ implies that $(a] \wedge (x] = 0$ in $CS(L)$.

So, $(a] \vee (x]$ covers $(x]$ in $CS(L)$, by hypothesis.

That is, $(a \vee x]$ covers $(x]$ in $CS(L)$.

This implies that, $x \prec a \vee x$, by the Lemma 3.2.

Hence the lemma.

4. Conclusion

The class of 0-distributive lattices is not an equational. The problem of equational class of weaker class of 0-distributive lattices, like, Pseudo-0-distributive and super-0-distributive lattices are equational is still open.

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