EQUATIONAL CLASS-LIKE PROPERTIES OF 0-DISTRIBUTIVE LATTICES
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(Received : January 20, 2022; Revised : May 09, 2022; Accepted : August 31, 2022)
DOI: https://doi.org/10.58250/jnanabha.2022.52208

Abstract
In generalizing the notion of pseudo complemented lattice, Varlet [8] introduced the notion of 0-distributive lattices. In this paper, we prove that the class of 0-distributive lattices is not an equational class, but it is an equational class-like in the sense that while an equational class is closed under the operations of subalgebras, direct products and homomorphic images, the class of 0-distributive lattices is closed under the first two operations and as far as the third one is concerned, the homomorphism should be a monomorphism. We also prove that if CS$(L)$ is 0-semimodular then so is $L$.

2020 Mathematical Sciences Classification: 06A06, 06A07, 06B20
Keywords and Phrases: 0-distributive lattices, 0-modular, Equational classes, Sublattices, Direct Products, Homomorphic images.

1. Introduction
An equational class of universal algebras is class of universal algebras which satisfies a set of identities. Equivalently, an equational class of universal algebras is a family of universal algebras which is closed under the operations of taking subalgebras, homomorphic images and direct product of members.

The following result is appeared in [2].

Theorem 1.1. Let $K$ be a class of lattices. A Class $K$ of lattices is equational class or a variety iff $K$ is closed under the formation of homomorphic images, sublattices and direct products.

In the variety of lattices, the classes of modular lattices and distributive lattices are equational, while complete lattices and complemented lattices are not.

The concept of 0-distributive lattices was first introduced by Verlet [8]. Several authors have made contributions in different aspects of 0-distributive lattices. For example, one can refer to Subbarayan and Vethamanickam [7] and Balasubramani and Venkatanarasimhan [1], etc.

2. Class of 0-distributive lattices
In this section, we examine whether the class of 0-distributive is equational. We prove that it is closed under sublattices, direct products but a homomorphic image of a 0-distributive lattice is 0-distributive, only if the homomorphism is a monomorphism. For all undefined terms we refer to [3].

Definition 2.1. A lattice $L$ with 0 is said to be 0-distributive if $a \land b = 0$ and $a \land c = 0$ imply $a \land (b \lor c) = 0$, for any $a, b, c$ in $L$.

Lemma 2.1. A sublattice of a 0-distributive lattice is 0-distributive.
Proof. Since $x \land y = 0, x \land z = 0$ in the sublattice imply $x \land (y \lor z) = 0$ in the sublattice, the sublattice is 0-distributive.

Lemma 2.2. A product of 0-distributive lattices is 0-distributive.
Proof. If $\{L_i/i \in I\}$ is family of 0-distributive lattices and if $X = x_i/i \in I, Y = y_i/i \in I$ and $Z = z_i/i \in I$ are any three elements of $\pi_{i \in I}L_i$, then $X \land Y = 0$ and $X \land Z = 0$ in $\pi_{i \in I}L_i$ imply that $x_i \land y_i = 0$ and $x_i \land z_i = 0$ for all $i \in I$ imply that $x_i \land (y_i \lor z_i) = 0$ for all $i \in I$, which implies that $X \land (Y \lor Z) = 0$ in $\{L_i/i \in I\}$

Lemma 2.3. Homomorphic image of a 0-distributive lattice is 0-distributive, only if the homomorphism is one-one.
Proof. Since, if $L$ is a 0-distributive lattice and $L_1$ its homomorphic image, then let $f : L \rightarrow L_1$ be an one-one, onto homomorphism and let $x_1, y_1, z_1 \in L_1$ such that $x_1 \land y_1 = 0, x_1 \land z_1 = 0$ then there exists $x, y, z \in L$ such that $f(x) = x_1, f(y) = y_1, f(z) = z_1$.

Therefore, $f(x) \land f(y) = 0$ and $f(x) \land f(z) = 0$.

Hence $f(x \land y) = f(x \land z) = 0$.

Therefore, $f(x \land y) = f(0)$ and $f(x \land z) = f(0)$ which implies that $x \land y = 0$ and $x \land z = 0 \in L$, as $f$ is one-one.

So, $x \land (y \lor z) = 0$ in $L$, as $L$ is 0-distributive.

Hence $f[x \land (y \lor z)] = f(0)$.

That is, $f(x) \land [f(y) \lor f(z)] = f(0) = 0$.

Hence $x_1 \land (y_1 \lor z_1) = 0$.

So, $L_1$ is 0-distributive.

Theorem 2.1. A Class of 0-distributive lattices is closed under the operations of taking sublattices, direct products and monomorphic images.

Proof. It follows from Lemmas 2.1, 2.2 and 2.3.

3. Class of $CS(L)$

In the theory of lattice of convex sublattices, another approach was developed by Lavanya and Bhatta in their paper [4]. They define a new partial ordering relation on $CS(L)$. They proved that both $L$ and $CS(L)$ are in the same equational class with respect to this new partial ordering. They have shown that $L$ and $CS(L)$ satisfy the same identities with respect to this new partial ordering. This motivated us to look into the connection between $L$ and $CS(L)$ for Eulerian lattices which are a class of lattices not defined by identities.

This new partial ordering was made of use by Ramanamurty [5]. He proved that for a lattice $L$, $CS(L)$ is semimodular then so is $L$.

The results of Lavanya and Bhatta [4] motivated us to work with the new partial ordering. In this section, we show by a counter example that with respect to this partial ordering relation $CS(L)$ need not be Eulerian, for an Eulerian lattice $L$ and prove that if $CS(L)$ is 0-semimodular then so is $L$. The next definition appeared in [4].

Definition 3.1. The binary relation $\leq$ on $CS(L)$, defined by, for $A, B \in CS(L), A \leq B$ if and only if "for every $a \in A$ there exists a $b \in B$ such that $a \leq b$ and for every $b \in B$ there exists an $a \in A$ such that $b \geq a$".

We provide the basic definitions and examples of Eulerian lattices that are needed to study of $CS(L)$, if $L$ is Eulerian.

Definition 3.2. Let $P$ be a finite poset with a unique minimum and a unique maximum element. The poset $P$ is said to be graded if all the maximal chains in $P$ have the same length.

Definition 3.3. A function $r : P \rightarrow \{0, 1, ..., n\}$ is said to be the rank function on $P$ if $r(x) = 0$ if $x$ is a minimal element of $P$ and $r(y) = r(x) + 1$ if $y$ covers $x$ in $P$. If $r(x) = i$ then we say that $x$ has rank $i$.

Definition 3.4. The M"obius function $\mu$ on a poset $P$ is an integer-valued function $\mu : P \times P \rightarrow \mathbb{Z}$ satisfying the following conditions:

$\mu(x, y) = \begin{cases} 1 & \text{if } x = y \\ - \sum_{x \leq z < y} \mu(x, z) & \text{if } x \leq y \\ 0 & \text{if } x \not\leq y. \end{cases}$

Definition 3.5. A finite graded poset $P$ is said to be Eulerian if its M"obius function assumes the value $\mu(x, y) = (-1)^{r(x, y)}$ for all $x \leq y$ in $P$, where $r(x, y) = r(y) - r(x)$ and $r$ is the rank function on $P$.

Every Boolean algebra of rank $n$ is Eulerian and the lattice $C_n$ is Eulerian which is given in figure 3.1. For the concept of Eulerian poset refer to [6, 7, 9].
Every Boolean algebra of rank $n$ is Eulerian and the lattice $C_4$ is Eulerian which is given in figure 1. For the concept of Eulerian poset refer to [6],[7],[9].

Lemma 3.1. If $a, b \in L$ then we show that $a < b$ in $L$ if and only if $(a] < (b]$ in $CS(L)$.

Lemma 3.2. If $a$ is atom of $L$ if and only if $(a]$ is an atom of $CS(L)$.

Definition 3.6. If $L$ is said to be 0-semimodular if whenever $a$ is an atom of $L$ and $x \in L$ such that $a \wedge x = 0$ then $x \vee a$ covers $x$.

Now, we show that $CS(L)$ need not be Eulerian even though $L$ is Eulerian by the following counter-example. The lattice $C_4$ given in Figure 3.1 is an Eulerian lattice. Its lattice of convex sublattices ($CS(C_4), \leq$) is given in Figure 3.2. Here, $CS(C_4)$ need not be Eulerian. It contains a 3-element interval $[a, b]$ whose M"obius function is $-1 \neq (-1)^{\rho(b)-\rho(a)}$.

The following two lemmas were proved by P.V.Ramanamurty in his paper [5].

Lemma 3.3. If $CS(L)$ is 0-semimodular then so is $L$.

Proof. If $a$ is an atom of $L$ and $x \in L$ such that $a \wedge x = 0$

Then $(a]$ is an atom of $CS(L)$, by the Lemma 3.2.

Now $a \wedge x = 0$ implies that $(a] \wedge (x] = 0$ in $CS(L)$.

So, $(a] \vee (x]$ covers $(x]$ in $CS(L)$, by hypothesis.

That is, $(a \vee x]$ covers $(x]$ in $CS(L)$.

This implies that, $x < a \vee x$, by the Lemma 3.2.

Hence the lemma.
4. Conclusion
The class of 0-distributive lattices is not an equational. The problem of equational class of weaker class of 0-distributive lattices, like, Pseudo-0-distributive and super-0-distributive lattices are equational is still open.

Acknowledgement. We are very much grateful to the Editor and Reviewer for their fruitful suggestions to bring the paper in the present form.

References