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*(Dedicated to Professor D. S. Hooda on His 80th Birth Anniversary Celebrations)*

# GEOMETRY ON KAEHLERIAN WEYL-CONFORMAL AND WEYL-CONHARMONIC RECURRENT CURVATURE MANIFOLDS

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**Abstract**

Ozdemir and Yildirim (2005) has premeditated on conformally recurrent Kaehlerian weyl spaces. Also, Negi et al.(2019), has studied analytic HP-transformation in almost Kaehlerian spaces. In this paper, we have calculated geometry on Kaehlerian weyl-conformal and weyl-conharmonic recurrent curvature manifolds and some theorems are obtained.

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## 1. Introduction

The n-dimension differentiable manifold having a Riemann metric **g** with symmetric connection $\nabla$ and U is a 1-type field is called Weyl space $W_n(g, U)$ under the recalibration and transformed U gratifying the form Hlavaty[5], Calderbank and Pedersen [1]

$$\nabla_g = 2(U \bigotimes g), \tag{1.1}$$

$$\bar{g} = \lambda^2 g, \tag{1.2}$$

$$\bar{U} = U + dln\lambda, \tag{1.3}$$

where $\lambda$ is a scalar function on $W_n(g, U)$.

If point **P** defined on $W_n(g, U)$ is called a dependency of g of power **r** if it discloses a revolution of the type under the recalibration (1.2) of **g** given by Canfes and Ozdeger [3]. The expanded covariant derivative of dependency **P** of tensor $g_{ij}$ power **r** is defined in Norden [8]

$$\bar{P} = \lambda^r P, \tag{1.4}$$

$$\dot{\nabla} P = \nabla_k P - rU_k P. \tag{1.5}$$

Also, putting (1.1) in confined coordinates and using (1.5), then we find

$$\partial_k g_{ij} - g_{hj}\Gamma^h_{ik} - g_{ih}\Gamma^h_{jk} - 2U_k g_{ij} = 0, \partial_k = \frac{\partial}{\partial^k}, \dot{\nabla} g_{ij} = 0.$$

Here $\Gamma^i_{kl}$, $\{^i_{kl}\}$ are coefficients of Weyl and metric connection respectively defined

$$\Gamma^i_{kl} = \{^i_{kl}\} - g^{im}( g_{mk}U_l + g_{ml}U_k - g_{kl}U_m), \tag{1.6}$$

$$\{^i_{kl}\} = \frac{1}{2} g^{im}( \partial_k g_{ml} + \partial_l g_{km} - \partial_m g_{kl}). \tag{1.7}$$

The $n$-dimensional Kaehlerian Weyl manifolds $(KW_n)(n \geq 2m)$ with an almost complex structure $F^j_l$ fulfilling the tensors $F_{ij}$ and $F^{ij}$ are of power 2 and -2, respectively Demirbuker and Ozdemir [4]

$$F^j_i F^k_j = -\delta^k_i, \tag{1.8}$$

$$g_{ij}F^i_h F^j_k = g_{hk}, \tag{1.9}$$

$$\dot{\nabla} F^j_i = 0, \ (for\ all\ i, j, k), \tag{1.10}$$

$$F_{ij} = g_{jk}F^k_i = -F_{ji}, \tag{1.11}$$

$$F^{ij} = g^{ih}F^j_h = -F^{ji}, \tag{1.12}$$

The curvature tensor $R_{ijkl}$ and $R^{i}_{jkl}$ of $W_n(g, U)$ are following Hlavaty [5], Ozdemir and Yildirim [10]

$$R^{j}_{jkl} = \frac{\partial}{\partial x^l}\Gamma^{i}_{jk} - \frac{\partial}{\partial x^k}\Gamma^{i}_{jl} + \Gamma^{i}_{hl}\Gamma^{h}_{jk} - \Gamma^{i}_{hk}\Gamma^{h}_{jl}, \tag{1.13}$$

$$R_{ijkl} = g_{ih}R^{h}_{jkl}, R^{a}_{ija} = R_{ij} \text{ and } R = g^{ij}R_{ij}, \tag{1.14}$$

$$R_{[ij]} = n\nabla_{[i}U_{j]}, \tag{1.15}$$

$$H_{ij} = \frac{1}{2}R_{ijkl}F^{kl}, M_{ij} = g_{ki}R^{k}_{j}, R^{k}_{j} = R^{h}_{jkl}g^{kl}, \tag{1.16}$$

$$M_{ij} = \left(\frac{n-2}{n}\right)R_{ij} + \frac{2}{n}R_{ji} = R_{ij} + 2n(R_{ji} - R_{ij}), \tag{1.17}$$

$$H_{ij} = -M_{hj}F^{h}_{i} = M_{ih}F^{h}_{j}, \tag{1.18}$$

$$H_{hi}F^{h}_{j} = -H_{jh}F^{h}_{i} = M_{ji}, \tag{1.19}$$

$$H_{hi}F^{hi} = -M_{hi}g^{hi} = -R, \tag{1.20}$$

$$R_{ijkl} + R_{jikl} = 4\nabla_{[k}U_{l]}g_{ij}, \tag{1.21}$$

$$H_{ij} + H_{ji} = 0. \tag{1.22}$$

## 2. Geometry on Kaehlerian Weyl-Conformal recurrent curvature manifolds

Then-dimensional Weyl recurrent manifoldsof its curvature tensor $R_{lijk}$ satisfies the condition

$$\dot{\nabla}_r R_{lijk} = P_r R_{lijk} + Q_r(g_{lj}g_{ik} - g_{lk}g_{ij}), \tag{2.1}$$

where **P** and **Q** are two correspondingly non-zero 1-types of powers 0 and -2 given by Canfes [2]. Here putting $G_{lijk} = g_{lj}g_{ik} - g_{lk}g_{ij}$, then (2.1) becomes

$$\dot{\nabla}_r R_{lijk} = P_r R_{lijk} + Q_r G_{lijk}, \tag{2.2}$$

If the 1-type **Q** is zero, then it is Weyl recurrent manifolds given by Canfes and Ozdeger [3]

**Definition 2.1.** *The n-dimensional ($n \geq 2m$) Kaehlerian Weyl recurrent manifold is called a widespread Weyl recurrent manifold if its curvature tensor $R_{lijk}$ of power 2 fulfills the condition*

$$\dot{\nabla}_r R_{lijk} = P_r R_{lijk} + Q_r G_{lijk}, \tag{2.3}$$

where **P** and **Q** are correspondingly 1-type of powers 0 and -2.

Again, the conformal curvature tensor $C^{h}_{ijk}$ of $W_n(g, U)$ is given by Miron [6]

$$C^{h}_{ijk} = R^{h}_{ijk} + \delta^{h}_{k}L_{ij} - \delta^{h}_{j}L_{ik} + L^{h}_{k}g_{ij} - L^{h}_{j}g_{ik} - 2\delta^{h}_{i}L_{[jk]}, \tag{2.4}$$

where:

$$L_{ij} = -\frac{R_{ij}}{(n-2)} + \frac{2}{n(n-2)}R_{[ij]} + \frac{Rg_{ij}}{2(n-1)(n-2)}, \tag{2.5}$$

$$L^{h}_{k} = g^{lh}L_{lk}. \tag{2.6}$$

Considering (1.17) with (2.5) becomes

$$L_{ij} = -\frac{1}{n-2}M_{ij} + \frac{R_{ji} - R_{ij}}{n(n-2)} + \frac{Rg_{ij}}{2(n-1)(n-2)}, \tag{2.7}$$

and

$$L_{ij} = -\frac{1}{(n-2)}M_{ij} + \frac{1}{(n-4)(n-2)}(M_{ji} - M_{ij}) + \frac{1}{2(n-1)(n-2)}Rg_{ij}.(2.8) \tag{2.8}$$

Also, from (1.15), (1.17), (2.5) and (2.8), we obtain

$$L_{[ij]} = -\frac{1}{n}R_{[ij]} = -\nabla_{[i}U_{j]} = -\frac{1}{2(n-4)}(M_{ij} - M_{ji}). \tag{2.9}$$

**Definition 2.2.** *The n-dimensional ($n \geq 2m$) Kaehlerian Weyl recurrent manifold is called widespread conformal recurrent manifold if its conformal curvature tensor $C_{lijk}$ of power 2 fulfills the condition*

$$\dot{\nabla}_r C_{lijk} = P_r C_{lijk} + Q_r G_{lijk}. \tag{2.10}$$

where **Q** are correspondingly 1-type of powers 0 and -2 and $C_{lijk} = C^{h}_{ijk}g_{hl}$.

We can prove the following theorem relating to wide spread Kaehlerian Weyl-conformal recurrent curvature manifolds.

**Theorem 2.1.** *A Kaehlerian Weyl recurrent manifolds ($KW_n$) is widespread conformal recurrent manifold iff it is widespread recurrent.*

Proof. Assume $KW_n$ is widespread conformal recurrent manifold. Transvecting (2.4) by $g_{hl}$ we get

$$C_{lijk} = R_{lijk} + g_{kl}L_{ij} - g_{jl}L_{ik} + g_{ij}L_{lk} - g_{ik}L_{lj} - 2g_{il}L_{[jk]}. \tag{2.11}$$

By taking the expanded covariant derivative of (2.11) and using (2.10), (2.11), we obtain

$$\begin{aligned} &\dot{\nabla}_r R_{lijk} + g_{kl}\dot{\nabla}_r L_{ij} - g_{jl}\dot{\nabla}_r L_{ik} + g_{ij}\dot{\nabla}_r L_{lk} - g_{ik}\dot{\nabla}_r L_{lj} - 2g_{il}\dot{\nabla}_r L_{[jk]} \\ &= P_r\left[R_{lijk} + g_{kl}L_{ij} - g_{jl}L_{ik} + g_{ij}L_{lk} - g_{ik}L_{lj} - 2g_{il}L_{[jk]}\right] + Q_r G_{lijk}. \end{aligned} \tag{2.12}$$

Transvecting (2.12) by $F^{jk}$ and using (1.16), (1.17), (1.18) and (2.8), we derive

$$\begin{aligned} &\frac{(n-3)}{(n-2)}\dot{\nabla}_r H_{li} + \frac{1}{(n-2)}\dot{\nabla}_r H_{il} - \frac{1}{(n-1)(n-2)}F_{li}\dot{\nabla}_r R + \frac{1}{(n-4)}g_{il}F^{jk}\dot{\nabla}_r M_{jk} \\ &= P_r\left[\frac{(n-3)}{(n-2)}H_{li} + \frac{1}{(n-2)}H_{li} - \frac{1}{(n-1)(n-2)}RF_{li} + \frac{1}{(n-4)}g_{il}F^{jk}M_{jk}\right] + \frac{1}{2}Q_r G_{lijk}F^{jk}. \end{aligned} \tag{2.13}$$

Also, multiplying (2.13) by $F^{li}$ and adopting (1.20), we obtain

$$\dot{\nabla}_r R = P_r R + \frac{n(1-n)}{(n-2)}Q_r. \tag{2.14}$$

Again, multiplying (2.13) by $g^{ii}$ and using (1.18) establish

$$F^{jk}(\dot{\nabla}_r M_{jk}) = P_r F^{jk}M_{jk} + \frac{1}{2}Q_r G_{lijk}F^{jk}g^{li}. \tag{2.15}$$

Since $G_{lijk}F^{jk}g^{li} = 0$, therefore

$$F^{jk}\dot{\nabla}_r M_{jk} = P_r F^{jk}M_{jk}. \tag{2.16}$$

Applying (2.14) and (2.16) in (2.13), we get

$$\dot{\nabla}_r H_{li} = P_r H_{li} + \frac{(n-1)}{(n-2)}Q_r F_{li}. \tag{2.17}$$

Multiplying (2.17) by $F^i_j$, we obtain

$$\dot{\nabla}_r M_{lj} = P_r M_{lj} - \frac{(n-1)}{(n-2)}Q_r g_{lj}. \tag{2.18}$$

Employing (2.14) and (2.18) into (2.8), we derive

$$\dot{\nabla}_r L_{ij} = P_r L_{ij} + \frac{1}{2(n-2)}Q_r g_{ij}. \tag{2.19}$$

Using (2.19), (2.12) reduces to

$$\dot{\nabla}_r R_{lijk} = P_r R_{lijk} + \frac{(n-1)}{(n-2)}Q_r G_{lijk}. \tag{2.20}$$

Hence, the necessary part of the theorem is proved.

Conversely, assume that $KW_n$ is widespread recurrent with 1-types **P** and **Q**, then

$$\dot{\nabla}_r R_{lijk} = P_r R_{lijk} + Q_r G_{lijk}. \tag{2.21}$$

Multiplying (2.21) by $F^{jk}$ and using (1.16), we get

$$\dot{\nabla}_r H_{li} = P_r H_{li} + Q_r F_{li}, \tag{2.22}$$

Transvecting (2.22) by $F^{li}$, we find

$$\dot{\nabla}_r R = P_r R - nQ_r. \tag{2.23}$$

while for $H_{li} = M_{lh}Fi^h$ from (2.22), we get

$$\dot{\nabla}_r M_{lj} = P_r M_{lj} - Q_r g_{lj}. \tag{2.24}$$

Hence from (2.8), (2.22), (2.23), we obtain

$$\dot{\nabla}_r L_{ij} = P_r L_{ij} + \frac{1}{2(n-1)}Q_r g_{ij}, \tag{2.25}$$

$$\dot{\nabla}_r L_{[ij]} = P_r L_{[ij]}. \tag{2.26}$$

Taking expanded covariant derivative of (2.11) and using (2.25) and (2.26), we get

$$\dot{\nabla}_r C_{lijk} = P_r C_{lijk} + \frac{(n-2)}{(n-1)}Q_(r)G_{lijk}, \tag{2.27}$$

which implies that sufficient part of the theorem is proved.

## 3. Geometry on Kaehlerian Weyl-Conharmonic recurrent curvature manifolds

The conharmonic curvature tensor $K_{lijk}$ of $W_n(g, U)$ can be given by Ozen and Altay [9].

$$K_{lijk} = C_{lijk} + \frac{R}{(n-2)(n-1)} G_{lijk}, n > 2, \tag{3.1}$$

where $C_{lijk}$ is the conformal curvature tensor of Weyl space and $G_{lijk} = g_{lj}g_{ik} - g_{lk}g_{ij}$.

**Definition 3.1.** *The n-dimensional ($n \geq 2m$) Kaehlerian Weyl recurrent manifold is called widespread conharmonic recurrent manifold if its conharmonic curvature tensor $K_{lijk}$ of power 2 fulfills the condition*

$$\dot{\nabla}_r K_{lijk} = P_r K_{lijk} + Q_r G_{lijk}, \tag{3.2}$$

where **P** and **Q** are correspondingly non-zero 1-types of powers 0 and -2.

We can prove the following theorem relating to widespread Kaehlerian Weyl-conharmonic recurrent curvature manifolds.

**Theorem 3.1.** *A Kaehlerian Weyl recurrent manifold ($KW_n$) is widespread conharmonic recurrent manifold iff it is widespread recurrent.*

*Proof.* Asume $KW_n$ is widespread recurrent, then

$$\dot{\nabla}_r R_{lijk} = P_r R_{lijk} + Q_r G_{lijk}. \tag{3.3}$$

From Theorem 2.1 and (2.27), we have

$$\dot{\nabla}_r C_{lijk} = P_r C_{lijk} + \frac{(n-2)}{(n-1)} Q_r G_{lijk}. \tag{3.4}$$

Taking expanded covariant derivative of (3.1), we find

$$\dot{\nabla}_r K_{lijk} = \dot{\nabla}_r C_{lijk} + \frac{1}{(n-1)(n-2)} G_{lijk} \dot{\nabla}_r R. \tag{3.5}$$

Using (3.4) in (3.5), we obtain

$$\dot{\nabla}_r K_{lijk} = P_r C_{lijk} + \frac{(n-2)}{(n-1)} Q_r G_{lijk} + \frac{1}{(n-1)(n-2)} G_{lijk} \dot{\nabla}_r R. \tag{3.6}$$

Employing (2.23), (3.6) becomes

$$\dot{\nabla}_r K_{lijk} = P_r C_{lijk} + \frac{(n-2)}{(n-1)} Q_r G_{lijk} + \frac{1}{(n-1)(n-2)} G_{lijk}(P_r R - nQ_r). \tag{3.7}$$

Therefore, from (3.1) we get

$$\dot{\nabla}_r K_{lijk} = P_r K_{lijk} + \frac{(n-4)}{(n-1)} Q_r G_{lijk}. \tag{3.8}$$

Hence the necessary part of the theorem is proved.

Conversely, assume that

$$\dot{\nabla}_r K_{lijk} = P_r K_{lijk} + Q_r G_{lijk}, \tag{3.9}$$

therefore (3.5) becomes

$$P_r K_{lijk} + Q_r G_{lijk} = \dot{\nabla}_r C_{lijk} + \frac{1}{(n-1)(n-2)} G_{lijk} \dot{\nabla}_r R. \tag{3.10}$$

Using (3.1), we have

$$P_r \left( C_{lijk} + \frac{R}{(n-1)(n-2)} G_{lijk} \right) + Q_r G_{lijk} = \dot{\nabla}_r C_{lijk} + \frac{1}{(n-1)(n-2)} G_{lijk} \dot{\nabla}_r R. \tag{3.11}$$

Multiplying both sides of (3.11) by $F^{jk}$ and using (1.16), we get

$$\begin{aligned} 2P_r \left[ \frac{(n-3)}{(n-2)} H_{li} + \frac{1}{(n-2)} H_{il} - \frac{1}{(n-1)(n-2)} RF_{li} \right] + \frac{1}{(n-4)} g_{il} F^{jk} M_{jk} + P_r \frac{R}{(n-1)(n-2)} \\ G_{lijk} F^{jk} + Q_r G_{lijk} F^{jk} = 2 \left[ \frac{(n-3)}{(n-2)} \dot{\nabla}_r H_{li} + \frac{1}{(n-2)} \dot{\nabla}_r H_{il} - \frac{1}{(n-1)(n-2)} F_{li} \dot{\nabla}_r R \right] \\ + \frac{1}{(n-4)} g_{il} F^{jk} \dot{\nabla}_r M_{jk} + \frac{1}{(n-1)(n-2)} G_{lijk} F^{jk} \dot{\nabla}_r R. \end{aligned} \tag{3.12}$$

Since $G_{lijk} F^{li} F^{jk} = 2n$, by Transvecting (1.20) with $F^{li}$ and using (1.20), we obtain

$$\dot{\nabla}_r R = P_r R - \frac{n(n-2)}{(n-4)} Q_r, (n > 4). \tag{3.13}$$

Hence, by using (3.2) and (3.5), we get

$$\dot{\nabla}_r C_{lijk} = P_r C_{lijk} + \frac{(n-2)^2}{(n-1)(n-4)} Q_r G_{lijk}, (n > 4). \tag{3.14}$$

From Theorem 2.1, we obtain

$$\dot{\nabla}_r R_{lijk} = P_r R_{lijk} + \frac{(n-1)}{(n-4)} Q_r G_{lijk}, (n > 4). \tag{3.15}$$

the sufficient part of the Theorem 3.1 is proved.

## 4. Conclusion

We have established from above two Theorems 2.1 and 3.1, that a Kaehlerian Weyl recurrent manifolds is widespread conformal recurrent manifold and conharmonic recurrent manifold iff it is widespread recurrent respectively.

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