GEOMETRY ON KAELERIAN WEYL-CONFORMAL AND WEYL-CONHARMONIC RECURRENT CURVATURE MANIFOLDS
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Abstract
Ozdemir and Yildirim (2005) has premeditated on conformally recurrent Kaehlerian weyl spaces. Also, Negi et al. (2019), has studied analytic HP-transformation in almost Kaehlerian spaces. In this paper, we have calculated geometry on Kaehlerian weyl-conformal and weyl-conharmonic recurrent curvature manifolds and some theorems are obtained.

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1. Introduction
The n-dimension differentiable manifold having a Riemann metric \( g \) with symmetric connection \( \nabla \) and \( U \) is a 1-type field is called Weyl space \( W_n(g, U) \) under the recalibration and transformed \( U \) gratifying the form Hlavaty [5], Calderbank and Pedersen [1]

\[ \nabla_g = 2(U \otimes g), \]
\[ \bar{g} = \lambda^2 g, \]
\[ \bar{U} = U + d \ln \lambda, \]

where \( \lambda \) is a scalar function on \( W_n(g, U) \).

If point \( P \) defined on \( W_n(g, U) \) is called a dependency of \( g \) of power \( r \) if it discloses a revolution of the type under the recalibration (1.2) of \( g \) given by Canfes and Ozdeger [3]. The expanded covariant derivative of dependency \( P \) of tensor \( g_{ij} \) power \( r \) is defined in Norden [8]

\[ \bar{P} = \lambda' P, \]
\[ \bar{\nabla} P = \nabla_k P - rU_k P. \]

Also, putting (1.1) in confined coordinates and using (1.5), then we find

\[ \partial_k g_{ij} = g_{hj} \Gamma^k_{ih} - g_{ih} \Gamma^k_{j} - 2U_k g_{ij} = 0, \partial_k = \frac{\partial}{\partial \bar{x}^k} \bar{\nabla} g_{ij} = 0. \]

Here \( \Gamma^i_{kl}, \\{\zeta^i_{kl}\} \) are coefficients of Weyl and metric connection respectively defined

\[ \Gamma^i_{kl} = \{\zeta^i_{kl}\} = \delta^{im} (g_{mk} U_l + g_{ml} U_k - g_{kl} U_m), \]
\[ \{\zeta^i_{kl}\} = \frac{1}{2} \delta^{im} (\partial_k g_{ml} + \partial_l g_{km} - \partial_m g_{kl}). \]

The \( n \)-dimensional Kaehlerian Weyl manifolds \( (KW_n) \) with an almost complex structure \( F^i_j \) fulfilling the tensors \( F_{ij} \) and \( F^{ij} \) are of power 2 and -2, respectively Demirbuker and Ozdemir [4]

\[ F^i_j F^j_k = -\delta^i_k, \]
\[ g_{ij} F^i_k F^j_k = g_{kk}, \]
\[ \nabla F^i_i = 0, \text{ (for all } i, j, k), \]
\[ F_{ij} = g_{jk} F^j_i = -F_{ji}, \]
\[ F^{ij} = g^{ih} F^i_h = -F^{ji}. \]
The curvature tensor $R_{ijkl}$ and $R^g_{ijkl}$ of $W_n(g, U)$ are following Hlavaty [5], Ozdemir and Yildirim [10]

\[ R^j_{ijkl} = \frac{\partial}{\partial x^k} \Gamma^j_{ik} - \frac{\partial}{\partial x^l} \Gamma^j_{li} + \Gamma^j_{ik} \Gamma^i_{lj} - \Gamma^j_{il} \Gamma^i_{kj}, \]  
(1.13)

\[ R_{ijkl} = g_{bh} R^b_{ijkl}, R^h_{ijkl} = R_{ijkl} \text{ and } R = g^{ij} R_{ij}, \]  
(1.14)

\[ R_{ij(l)} = n \nabla_i U_j, \]  
(1.15)

\[ H_{ij} = \frac{1}{2} R_{ijkl} F^{kl}, M_{ij} = g_{bh} \delta^i_j, R^h_{ijkl} = R_{ijkl} \delta^g_{bh}, \]  
(1.16)

\[ M_{ij} = \left( \frac{n-2}{n} \right) R_{ij} + \frac{2}{n} R_{ji} = R_{ij} + 2n(R_{ji} - R_{ij}), \]  
(1.17)

\[ H_{ij} = -M_{hi} F^h_j = M_{ji}, \]  
(1.18)

\[ H_{hi} F^j_i = -H_{jh} F^h_i = M_{ji}, \]  
(1.19)

\[ H_{hi} F^h_i = -M_{hi} F^h_i = -R, \]  
(1.20)

\[ R_{ijkl} + R_{ijkl} = 4 \nabla_i U_j g_{ij}, \]  
(1.21)

\[ H_{ij} + H_{ji} = 0. \]  
(1.22)

2. Geometry on Kaehlerian Weyl-Conformal recurrent curvature manifolds

Then-dimensional Weyl recurrent manifolds of its curvature tensor $R_{ij}, k$ satisfies the condition

\[ \nabla_i R_{jk} = P_i R_{jk} + Q_i (g_{ij} g_{lk} - g_{ik} g_{lj}), \]  
(2.1)

where $P$ and $Q$ are two correspondingly non-zero 1-types of powers 0 and -2 given by Canfes [2]. Here putting

\[ G_{ik} = g_{ik} g_{lk} - g_{ik} g_{lj}, \]  
then (2.1) becomes

\[ \nabla_i R_{jk} = P_i R_{jk} + Q_i G_{ik}. \]  
(2.2)

If the 1-type $Q$ is zero, then it is Weyl recurrent manifolds given by Canfes and Ozdegir [3]

**Definition 2.1.** The n-dimensional ($n \geq 2m$) Kaehlerian Weyl recurrent manifold is called a widespread Weyl recurrent manifold if its curvature tensor $R_{ij}, k$ of power 2 fulfills the condition

\[ \nabla_i R_{jk} = P_i R_{jk} + Q_i G_{ik}, \]  
(2.3)

where $P$ and $Q$ are correspondingly 1-type of powers 0 and -2.

Again, the conformal curvature tensor $C^h_{ijkl}$ of $W_n(g, U)$ is given by Miron [6]

\[ C^h_{ijkl} = R^h_{ijkl} + \delta^h_j L_{ij} - \delta^h_i L_{jk} + \delta^h_k g_{ij} - L^h_{ij} g_{ik} - 2 h_i L_{[ij]}, \]  
(2.4)

where:

\[ L_{ij} = -\frac{R_{ij}}{n(n-2)} + \frac{2}{n(n-2)} R_{ijkl} + \frac{R_{ij}}{2(n-1)(n-2)} R_{ijkl}, \]  
(2.5)

\[ L^h_k = g^h_k L_k. \]  
(2.6)

Considering (1.17) with (2.5) becomes

\[ L_{ij} = -\frac{1}{n-2} L_{ij} + \frac{R_{ij}}{n(n-2)} + \frac{R_{ijkl}}{2(n-1)(n-2)}, \]  
(2.7)

and

\[ L_{ij} = -\frac{1}{n-2} M_{ij} + \frac{1}{n(n-2)(n-4)} (M_{ij} - M_{ij}) + \frac{1}{2(n-1)(n-2)} R_{ijkl}. \]  
(2.8)

Also, from (1.15), (1.17), (2.5) and (2.8), we obtain

\[ L_{ij} = -\frac{1}{n-2} R_{ij} = -\nabla_i U_j = -\frac{1}{2(n-4)} (M_{ij} - M_{ij}). \]  
(2.9)

**Definition 2.2.** The n-dimensional ($n \geq 2m$) Kaehlerian Weyl recurrent manifold is called widespread conformal recurrent manifold if its conformal curvature tensor $C^h_{ijkl}$ of power 2 fulfills the condition

\[ \nabla_i C^h_{ijkl} = P_i C^h_{ijkl} + Q_i G_{ijkl}. \]  
(2.10)

where $Q$ are correspondingly 1-type of powers 0 and -2 and $C_{ijkl} = C^h_{ijkl} g_{hi}$. We can prove the following theorem relating to widespread Kaehlerian Weyl-conformal recurrent curvature manifolds.

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Theorem 2.1. A Kaehlerian Weyl recurrent manifolds \( (KW_n) \) is widespread conformal recurrent manifold iff it is widespread recurrent.

Proof. Assume \( KW_n \) is widespread conformal recurrent manifold. Transvecting (2.4) by \( g_{hi} \) we get

\[
C_{ijk} = R_{ijk} + g_{ik}L_{lj} - g_{jl}L_{ki} + g_{ij}L_{lk} - g_{kl}L_{ij} - 2g_{il}L_{kj}.
\]

(2.11)

By taking the expanded covariant derivative of (2.11) and using (2.10), (2.11), we obtain

\[
\nabla_j R_{ijk} + g_{ik} \nabla_j L_{lj} - g_{jl} \nabla_i L_{ki} + g_{ij} \nabla_k L_{lk} - g_{kl} \nabla_i L_{ij} - 2g_{il} \nabla_j L_{kj} = P_r \left( R_{ijk} + g_{ik}L_{lj} - g_{jl}L_{ki} + g_{ij}L_{lk} - g_{kl}L_{ij} - 2g_{il}L_{kj} \right) + Q_r G_{ijk}.
\]

Transvecting (2.12) by \( F^{jk} \) and using (1.16), (1.17), (1.18) and (2.8), we derive

\[
\frac{(n-3)}{(n-2)} \nabla_i H_{li} + \frac{1}{(n-2)} \nabla_i H_{li} - \frac{1}{(n-1)(n-2)} F_l \nabla_i R + \frac{1}{(n-4)} g_{li} F^{jk} \nabla_j M_{jk} = P_i \left( \frac{(n-3)}{(n-2)} H_{li} + \frac{1}{(n-2)} H_{li} - \frac{1}{(n-1)(n-2)} RF_{li} + \frac{1}{(n-4)} g_{li} F^{jk} M_{jk} \right) + \frac{1}{2} Q_r G_{ijk} F^{jk}.
\]

(2.13)

Also, multiplying (2.13) by \( F^{ij} \) and adopting (1.20), we obtain

\[
\nabla_i R = P_i R + n \frac{(1-n)}{(n-2)} Q_r.
\]

(2.14)

Again, multiplying (2.13) by \( g^{ji} \) and using (1.18) establish

\[
F^{jk} \nabla_j M_{jk} = P_r F^{jk} M_{jk} + \frac{1}{2} Q_r G_{ijk} F^{jk} g^{li}.
\]

(2.15)

Since \( G_{ijk} F^{jk} g^{li} = 0 \), therefore

\[
F^{jk} \nabla_j M_{jk} = P_r F^{jk} M_{jk}.
\]

(2.16)

Applying (2.14) and (2.16) in (2.13), we get

\[
\nabla_i H_{li} = P_i H_{li} + \frac{(n-1)}{(n-2)} Q_r F_{li}.
\]

(2.17)

Multiplying (2.17) by \( F^j \), we obtain

\[
\nabla_i M_{lj} = P_i M_{lj} - \frac{(n-1)}{(n-2)} Q_r g_{lj}.
\]

(2.18)

Employing (2.14) and (2.18) into (2.8), we derive

\[
\nabla_i L_{lj} = P_i L_{lj} + \frac{1}{2(n-2)} Q_r g_{lj}.
\]

(2.19)

Using (2.19), (2.12) reduces to

\[
\nabla_i R_{ijk} = P_i R_{ijk} + \frac{(n-1)}{(n-2)} Q_r G_{ijk}.
\]

(2.20)

Hence, the necessary part of the theorem is proved.

Conversely, assume that \( KW_n \) is widespread recurrent with 1-types \( P \) and \( Q \), then

\[
\nabla_i R_{ijk} = P_i R_{ijk} + Q_r G_{ijk}.
\]

(2.21)

Multiplying (2.21) by \( F^{jk} \) and using (1.16), we get

\[
\nabla_i H_{li} = P_i H_{li} + Q_r F_{li}.
\]

(2.22)

Transvecting (2.22) by \( F^{li} \), we find

\[
\nabla_i R = P_i R - n Q_r.
\]

(2.23)

while for \( H_{li} = M_{li} F^{li} \) from (2.22), we get

\[
\nabla_i M_{lj} = P_i M_{lj} - Q_r g_{lj}.
\]

(2.24)

Hence from (2.8), (2.22), (2.23), we obtain

\[
\nabla_i L_{lj} = P_i L_{lj} + \frac{1}{2(n-1)} Q_r g_{lj},
\]

(2.25)

\[
\nabla_i L_{lji} = P_i L_{lji}.
\]

(2.26)

Taking expanded covariant derivative of (2.11) and using (2.25) and (2.26), we get

\[
\nabla_i C_{ijk} = P_i C_{ijk} + \frac{(n-2)}{(n-1)} Q_r G_{ijk},
\]

(2.27)

which implies that sufficient part of the theorem is proved.
3. Geometry on Kaehlerian Weyl-Conharmonic recurrent curvature manifolds

The conharmonic curvature tensor $K_{ljik}$ of $W_{n}(g, U)$ can be given by Ozen and Altay [9].

$$K_{ljik} = C_{ljik} + \frac{R}{(n - 1)(n - 2)} G_{ljik}, n > 2,$$  \hspace{1cm} (3.1)

where $C_{ljik}$ is the conformal curvature tensor of Weyl space and $G_{ljik} = g_{ljik} - g_{li} g_{kj} g_{ij}$.

**Definition 3.1.** The $n$-dimensional ($n \geq 2m$) Kaehlerian Weyl recurrent manifold is called widespread conharmonic recurrent manifold if its conharmonic curvature tensor $K_{ljik}$ of power 2 fulfills the condition

$$\nabla_{r} K_{ljik} = P_{r} K_{ljik} + Q_{r} G_{ljik},$$  \hspace{1cm} (3.2)

where $P$ and $Q$ are correspondingly non-zero 1-types of powers 0 and -2.

We can prove the following theorem relating to widespread Kaehlerian Weyl-conharmonic recurrent curvature manifolds.

**Theorem 3.1.** A Kaehlerian Weyl recurrent manifold ($KW_{n}$) is widespread conharmonic recurrent manifold iff it is widespread recurrent.

**Proof.** Asume $KW_{n}$ is widespread recurrent, then

$$\nabla_{r} R_{ljik} = P_{r} R_{ljik} + Q_{r} G_{ljik}.$$  \hspace{1cm} (3.3)

From Theorem 2.1 and (2.27), we have

$$\nabla_{r} C_{ljik} = P_{r} C_{ljik} + \frac{(n - 2)}{(n - 1)} Q_{r} G_{ljik}.$$  \hspace{1cm} (3.4)

Taking expanded covariant derivative of (3.1), we find

$$\nabla_{r} K_{ljik} = \nabla_{r} C_{ljik} + \frac{1}{(n - 1)(n - 2)} G_{ljik} \nabla_{r} R.$$  \hspace{1cm} (3.5)

Using (3.4) in (3.5), we obtain

$$\nabla_{r} K_{ljik} = P_{r} C_{ljik} + \frac{(n - 2)}{(n - 1)} Q_{r} G_{ljik} + \frac{1}{(n - 1)(n - 2)} G_{ljik} \nabla_{r} R.$$  \hspace{1cm} (3.6)

Employing (2.23), (3.6) becomes

$$\nabla_{r} K_{ljik} = P_{r} C_{ljik} + \frac{(n - 2)}{(n - 1)} Q_{r} G_{ljik} + \frac{1}{(n - 1)(n - 2)} G_{ljik} (P_{r} R - n Q_{r}).$$  \hspace{1cm} (3.7)

Therefore, from (3.1) we get

$$\nabla_{r} K_{ljik} = P_{r} K_{ljik} + \frac{(n - 4)}{(n - 1)} Q_{r} G_{ljik}.$$  \hspace{1cm} (3.8)

Hence the necessary part of the theorem is proved.

Conversely, assume that

$$\nabla_{r} K_{ljik} = P_{r} K_{ljik} + Q_{r} G_{ljik},$$  \hspace{1cm} (3.9)

therefore (3.5) becomes

$$P_{r} K_{ljik} + Q_{r} G_{ljik} = \nabla_{r} C_{ljik} + \frac{1}{(n - 1)(n - 2)} G_{ljik} \nabla_{r} R.$$  \hspace{1cm} (3.10)

Using (3.1), we have

$$P_{r} \left( C_{ljik} + \frac{R}{(n - 1)(n - 2)} G_{ljik} \right) + Q_{r} G_{ljik} = \nabla_{r} C_{ljik} + \frac{1}{(n - 1)(n - 2)} G_{ljik} \nabla_{r} R.$$  \hspace{1cm} (3.11)

Multiplying both sides of (3.11) by $F^{lj}$ and using (1.16), we get

$$2 P_{r} \left( \frac{(n - 3)}{(n - 2)} H_{li} + \frac{1}{(n - 1)(n - 2)} R F_{li} \right) + \frac{1}{(n - 1)(n - 2)} g_{il} F_{jk} M_{jk} + P_{r} \frac{R}{(n - 1)(n - 2)}$$

$$G_{ljik} F^{jk} + Q_{r} G_{ljik} F^{jk} = 2 \left( \frac{(n - 3)}{(n - 2)} \nabla_{r} H_{li} + \frac{1}{(n - 1)(n - 2)} \nabla_{r} H_{il} - \frac{1}{(n - 1)(n - 2)} F_{il} \nabla_{r} R \right)$$

$$+ \frac{1}{(n - 1)(n - 2)} g_{il} F_{jk} \nabla_{r} M_{jk} + \frac{1}{(n - 1)(n - 2)} G_{ljik} F^{jk} \nabla_{r} R.$$  \hspace{1cm} (3.12)

Since $G_{ljik} F^{lk} F^{jk} = 2n$, by Transvecting (1.20) with $F^{li}$ and using (1.20), we obtain

$$\nabla_{r} R = P_{r} R - \frac{n(n - 2)}{(n - 4)} Q_{r}, (n > 4).$$  \hspace{1cm} (3.13)

Hence, by using (3.2) and (3.5), we get

$$\nabla_{r} C_{ljik} = P_{r} C_{ljik} + \frac{(n - 2)^{2}}{(n - 1)(n - 4)} Q_{r} G_{ljik}, (n > 4).$$  \hspace{1cm} (3.14)

From Theorem 2.1, we obtain

$$\nabla_{r} R_{ljik} = P_{r} R_{ljik} + \frac{(n - 1)}{(n - 4)} Q_{r} G_{ljik}, (n > 4).$$  \hspace{1cm} (3.15)

the sufficient part of the Theorem 3.1 is proved.
4. Conclusion
We have established from above two Theorems 2.1 and 3.1, that a Kaehlerian Weyl recurrent manifolds is widespread conformal recurrent manifold and conharmonic recurrent manifold if and only if it is widespread recurrent respectively.

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