COMMON FIXED POINT THEOREMS FOR GENERALIZED MULTI-VALUED CONTRACTIONS IN b-METRIC AND DISLOCATED b-METRIC SPACES

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Abstract

In this paper, we extend the contraction to find some common fixed point theorems of multivalued contraction in b-metric and dislocated b-metric spaces. Also, we give some examples to vindicate our results. Moreover, the obtained results extend and improve some well known results of the literature.

Keywords and Phrases: Multi-valued mapping, common fixed point, b-metric space, dislocated b-metric space

1. Introduction and Preliminaries

Let \((X, d)\) be a metric space, then a mapping \(T : X \to X\) is said to be contraction if there exists a positive real number \(r < 1\) such that \(d(Tx, Ty) \leq rd(x, y)\) for all \(x, y \in X\). In the literature of fixed point theory, Banach contraction principle (BCP) plays an important role which states that every contraction on a complete metric space \((X, d)\) has a unique fixed point, i.e., there is a point \(z \in X\) such that \(Tz = z\). It has a wide range of applications in physical sciences, computer sciences and Engineering Sciences. BCP is further improved, extended and generalized by many researchers in the fixed point theory by weakening the contractive condition and structure of the metric space \([1, 18]\). In 1969, Nadler\([17]\) introduced the notion of multi-valued contractive mapping in a complete metric space, and established a fixed point theorem for multivalued mapping which is a generalization of BCP. In fact, Nadler proved the following result:

**Theorem 1.1** ([17]). Let \((X, d)\) be a complete metric space and \(T : X \to \text{CB}(X)\) be a mapping. Assume there exists \(r \in [0, 1)\) such that \(H(Tx, Ty) \leq rd(x, y)\) for all \(x, y \in X\). Then, there exists \(z \in X\) such that \(z \in Tz\).

Thereafter, Nadler’s fixed point theorem, many authors have given results for multi-valued and hybrid contractive mappings \([8, 9, 10, 11, 19, 4]\) and references therein.

In generalization process of metric structure, the concept of b-metric space was introduced by Czerwik\([7]\). Since then many authors used notion of b-metric to obtain various fixed point theorems. Moreover, several results of metric spaces are presented for b-metric spaces, see \([3, 13, 12, 16]\) and references therein. Similarly, some interesting results in dislocated b-metric space are also proved by researchers, see \([14, 20]\).

Now we give some definitions and important theorems of literature which will be useful for our main results.

**Definition 1.1** ([5]). Let \(X\) be a non-empty set and \(s \geq 1\) be a given real number. A function \(d : X \times X \to [0, \infty)\) is called a b-metric iff for all \(x, y, z \in X\), the following conditions are satisfied:

\[\begin{align*}
(i) \quad d(x, y) &= 0 \text{ if and only if } x = y; \\
(ii) \quad d(x, y) &= d(y, x); \\
(iii) \quad d(x, z) &\leq s[d(x, y) + d(y, z)],
\end{align*}\]

the triplet \((X, d, s)\) is known as a b-metric space.

**Definition 1.2** ([5]). Let \((X, d, s)\) be a b-metric space. A sequence \(x_n\) in \(X\) is said to be:

1. Cauchy if and only if \(d(x_n, x_m) \to 0\) as \(n, m \to \infty\);
2. convergent iff there exists \(x \in X\) such that \(d(x_n, x) \to 0\) as \(n \to \infty\) (or if for all \(\epsilon > 0\), there exists \(k \in \mathbb{N}\) we have such that \(d(x_n, x) < \epsilon\) for all \(n \geq k\)), and we write \(\lim_{n \to \infty} x_n = x\);
3. complete if every Cauchy sequence in \(X\) is convergent.
Lemma 1.1. Let \((d, s, s)\) be a b-metric space. A subset \(Y\) of \(X\) is said to be:

1. closed if for each sequence \(x_n\) in \(Y\), which converges to an element \(x\), we have \(x \in Y\);
2. compact if for every sequence of element in \(Y\) there exists a subsequence that converges to an element in \(Y\);
3. bounded if \(\sup\{d(x, y) : x, y \in Y\} < \infty\).

The extension of Banach contraction principle in b-metric spaces as follows:

Theorem 1.2 ([5]). Let \((X, d)\) be a complete b-metric space with constant \(s \geq 1\) such that b-metric is a continuous functional. Let \(T : X \rightarrow X\) be a contraction having contraction constant \(k \in [0, 1)\) such that \(ks < 1\). Then \(T\) has a unique fixed point.

Definition 1.3 ([2]). Let \(X\) be a nonempty set and \(s \geq 1\) be a given real number. A function \(d_b : X \times X \rightarrow [0, \infty)\) is called a dislocated b-metric if for all \(x, y, z \in X\), the following conditions are satisfied:

1. \(d_b(x, y) = 0 \Rightarrow x = y;\)
2. \(d_b(x, y) = d_b(y, x);\)
3. \(d_b(x, z) \leq s[d_b(x, y) + d_b(y, z)].\)

the triplet \((X, d_b, s)\) is known as dislocated b-metric space. If \(s = 1\), the dislocated b-metric space \((X, d_b, s)\) is called a dislocated metric space.

Definition 1.4 ([2]). Let \((X, d_b)\) be a dislocated b-metric space. A sequence \(\{x_n\}\) in \(X\) is said to be:

1. Cauchy if and only if \(d_b(x_n, x_m) \rightarrow 0\) as \(n, m \rightarrow \infty;\)
2. convergent if there exist \(x \in X\) such that \(d_b(x_n, x) \rightarrow 0\) as \(n \rightarrow \infty\) and we write \(\lim_{n \rightarrow \infty} x_n = x;\)
3. complete if every Cauchy sequence in \(X\) is convergent.

Example 1.1. Let \(X = \mathbb{Q}^+ \cup \{0\}\) and let \(d_b : X \times X \rightarrow X\) defined by \(d_b(x, y) = (x + y)^2\) for all \(x, y \in X\). Then \(d_b\) is dislocated b-metric on \(X\) with \(s = 2\).

Let \((X, d)\) be a metric (resp. b-metric, dislocated b-metric) space and \(CB(X)\) the collection of all non-empty closed and bounded subsets of \(X\). The Hausdorff metric \(H(\text{resp.} H, H_b)\) on \(CB(X)\) induced by the metric \(d\) is given by

\[
H(A, B) = \max\left\{\sup_{p \in A} d(p, B), \sup_{q \in B} d(q, A)\right\}
\]

for \(A, B \in CB(X)\), where \(d(x, A) = \inf_{y \in A} d(x, y)\). Moreover, the distance between sets \(A, B \in CB(X)\) is defined as

\[
d(A, B) = \inf\{d(p, q) : p \in A, q \in B\}.
\]

We recall the following properties from [3, 6] and the references therein.

Lemma 1.1. Let \((X, d, s)\) be a b-metric space. For any \(A, B, C \in CB(X)\) and any \(x, y \in X\), the following statements are true:

1. \(d(x, B) \leq d(x, q), \text{ for any } q \in B;\)
2. \(\sup_{p \in A} d(p, B) \leq H(A, B);\)
3. \(H(A, B) = 0 \text{ iff } A = B;\)
4. \(d(x, B) \leq H(A, B), \text{ for any } x \in A;\)
5. \(H(A, B) = H(B, A);\)
6. \(H(A, C) \leq s[H(A, B) + H(B, C)]\)
7. \(d(x, A) \leq s[d(x, y) + d(y, A)];\)

Following the Nadler’s fixed point theorem, Czerwik [6] established the following theorem for multivalued mapping.

Theorem 1.3 ([6]). Let \((X, d, s)\) be a complete b-metric space and let \(T : X \rightarrow CB(X)\) be a multi-valued mapping such that \(T\) satisfies the inequality \(H(Tx, Ty) \leq ad(x, y)\) for all \(x, y \in X\), where \(0 < a < \frac{1}{s}\). Then \(T\) has a fixed point.

Subsequently a number of fixed point theorems have been obtained by the researchers for multivalued mappings in different settings of spaces see [12, 19, 18, 4] and references therein. In this paper, we extend and generalize the result of [12] for multivalued mappings satisfying generalized contractive type condition in complete b-metric and dislocated b-metric spaces.
2. Main Results

Lemma 2.1 ([7]). Let \((X, d, s)\) be a b-metric (or dislocated b-metric) space and \(A, B \in CB(X)\). Then, for each \(\epsilon > 0\) and for all \(p \in A\), there exists \(q \in B\) such that \(d(p, q) \leq H(A, B) + \epsilon\).

Theorem 2.1. Let \((X, d, s)\) be a complete b-metric space and let \(S, T : X \to CB(X)\) be two multivalued mappings satisfying for all \(x, y \in X\),

\[
H(Tx, Sy) \leq a d(x, Ty) + b \left[d(x, Sx) + d(Tx, Ty) + c d(Tx, Sy)\right],
\]

(2.1)

where \(a, b, c \geq 0\) with \(a + 2bs + bc < 1\) and \(b + bc < 1/s^2\). Then \(S\) and \(T\) have a unique common fixed point in \(X\).

Proof. Fix \(x \in X\). Without loss of generality, we choose \(\epsilon_0\) such that \(0 < \epsilon_0 < 1 - bs - bc\). Then, for \(x \in X\), define \(x_0 = x\) and let \(x_1 \in Tx_0\). By Lemma 2.1, we may choose \(x_2 \in Sx_0\) such that

\[
d(x_1, x_2) \leq H(Tx_0, Sx_0) + \epsilon_0
\]

\[
\leq ad(x_0, Tx_0) + b \left[d(x_0, Sx_0) + d(Tx_0, Tx_2) + c d(Tx_0, Sx_0)\right] + \epsilon_0
\]

\[
\leq ad(x_0, x_1) + bcd(x_1, x_2) + \epsilon_0
\]

\[
\Rightarrow d(x_1, x_2)(1 - bs - bc) \leq ad(x_0, x_1) + bsd(x_1, x_1) + \epsilon_0
\]

\[
\leq (a + bs)d(x_0, x_1) + \epsilon_0
\]

\[
\Rightarrow d(x_1, x_2) \leq \frac{(a + bs)}{(1 - bs - bc)} d(x_0, x_1) + \frac{\epsilon_0}{(1 - bs - bc)}.
\]

Thus,

\[
d(x_1, x_2) \leq \frac{(a + bs)}{(1 - bs - bc)} d(x_0, x_1) + \epsilon,
\]

where \(\epsilon = \frac{\epsilon_0}{(1 - bs - bc)}\).

Similarly, there exists \(x_3 \in T_{x_2}\) such that

\[
d(x_2, x_3) \leq H(Tx_2, Sx_0) + \frac{\epsilon_0^2}{(1 - bs - bc)}
\]

\[
\leq ad(x_2, Tx_0) + b \left[d(x_2, Sx_0) + d(Tx_2, Tx_2) + c d(Tx_2, Sx_0)\right] + \frac{\epsilon_0^2}{(1 - bs - bc)}
\]

\[
\leq ad(x_2, x_1) + bd(x_1, x_3) + bcd(x_1, x_2) + \frac{\epsilon_0^2}{(1 - bs - bc)}
\]

\[
\Rightarrow d(x_2, x_3)(1 - bs - bc) \leq ad(x_2, x_1) + bsd(x_1, x_2) + \frac{\epsilon_0^2}{(1 - bs - bc)}
\]

\[
\Rightarrow d(x_2, x_3) \leq \frac{(a + bs)}{(1 - bs - bc)} d(x_1, x_2) + \frac{\epsilon_0^2}{(1 - bs - bc)}
\]

Continuing in this way, we obtain by induction a sequence \(\{x_n\}_{n \in \mathbb{N}}\) such that \(x_{2n} \in Sx_{2n-2}\), \(x_{2n+1} \in T_{x_{2n}}\), such that

\[
d(x_{2n+1}, x_{2n+2}) \leq H(Tx_{2n}, Sx_{2n}) + \frac{\epsilon_0^{2n+1}}{(1 - bs - bc)^{2n+1}}
\]

\[
d(x_{2n}, x_{2n+1}) \leq H(Sx_{2n-2}, Tx_{2n}) + \frac{\epsilon_0^{2n}}{(1 - bs - bc)^{2n-1}}
\]

Now,

\[
d(x_{2n}, x_{2n+1}) \leq H(Sx_{2n-2}, T_{x_{2n}}) + \frac{\epsilon_0^{2n}}{(1 - bs - bc)^{2n-1}}
\]

\[
\leq ad(x_{2n}, Tx_{2n}) + b \left[d(x_{2n}, Sx_{2n-2}) + d(Tx_{2n}, Tx_{2n-2}) + c d(Tx_{2n}, Sx_{2n-2})\right]
\]

\[
+ \frac{\epsilon_0^{2n}}{(1 - bs - bc)^{2n-1}}
\]
\[ d(x_{2n}, x_{2n+1}) \leq ad(x_{2n}, x_{2n-1}) + b[d(x_{2n}, x_{2n}) + d(x_{2n+1}, x_{2n-1}) + cd(x_{2n+1}, x_{2n})] + \frac{\epsilon_0^{2n}}{(1 - bs - bc)^{2n-1}} \]

\[ (1 - bs - bc)d(x_{2n}, x_{2n+1}) \leq (a + bs)d(x_{2n}, x_{2n-1}) + \frac{\epsilon_0^{2n}}{(1 - bs - bc)^{2n-1}} \]

\[ \Rightarrow d(x_{2n}, x_{2n+1}) \leq \frac{(a + bs)}{(1 - bs - bc)}d(x_{2n}, x_{2n-1}) + \frac{\epsilon_0^{2n}}{(1 - bs - bc)^{2n}} \]

Also,

\[ d(x_{2n+1}, x_{2n+2}) \leq H(Tx_{2n}, Sx_{2n}) + \frac{\epsilon_0^{2n+1}}{(1 - bs - bc)^{2n}} \]

\[ \leq ad(x_{2n},Tx_{2n}) + b[d(x_{2n},Sx_{2n}) + d(Tx_{2n},Tx_{2n}) + cd(Tx_{2n},Sx_{2n})] + \frac{\epsilon_0^{2n+1}}{(1 - bs - bc)^{2n}} \]

\[ \leq ad(x_{2n},x_{2n+1}) + bd(x_{2n+1},x_{2n+2}) + bcd(x_{2n+1},x_{2n+2}) + \frac{\epsilon_0^{2n+1}}{(1 - bs - bc)^{2n}} \]

\[ (1 - bs - bc)d(x_{2n+1}, x_{2n+2}) \leq (a + bs)d(x_{2n+1}, x_{2n+1}) + \frac{\epsilon_0^{2n+1}}{(1 - bs - bc)^{2n+1}} \]

\[ \Rightarrow d(x_{2n+1}, x_{2n+2}) \leq \frac{(a + bs)}{(1 - bs - bc)}d(x_{2n+1}, x_{2n+1}) + \frac{\epsilon_0^{2n+1}}{(1 - bs - bc)^{2n+1}} \]

Therefore,

\[ d(x_n, x_{n+1}) \leq \frac{(a + bs)}{(1 - bs - bc)}d(x_{n-1}, x_n) + e^n \quad \forall n \in \mathbb{N}, \text{ where } \epsilon = \frac{\epsilon_0}{(1 - bs - bc)}. \]

Let \( k = \frac{(a + bs)}{(1 - bs - bc)} \), then for each \( n \in \mathbb{N} \), we have

\[ d(x_n, x_{n+1}) \leq k d(x_{n-1}, x_n) + e^n \]

\[ \leq k^2 d(x_{n-2}, x_{n-1}) + e^{n-1} + e^n \]

\[ \vdots \]

\[ \leq k^n d(x_0, x_1) + \sum_{r=0}^{n-1} k^r e^{n-r} \]

which shows that,

\[ \sum_{n=1}^{N} d(x_n, x_{n+1}) \leq \sum_{n=1}^{N} k^n d(x_0, x_1) + \sum_{n=1}^{N} \left( \sum_{r=0}^{n-1} k^r e^{n-r} \right) \]

\[ \leq \sum_{n=1}^{N} k^n d(x_0, x_1) + e^n \sum_{r=0}^{N-1} k^r \]

\[ = d(x_0, x_1) \sum_{n=1}^{N} k^n + \sum_{n=1}^{N} e^n \cdot \frac{1 - k^n}{1 - k} \]
Now, let \( d \) thus \( x \) and \( F \).

Hence, we get \( \lim_{n \to \infty} d(x_n, x_{n+1}) = 0 \).

Now we show that \( \{x_n\}_{n \in \mathbb{N}} \) is a Cauchy sequence in \( X \). Let \( m, n > 0 \) with \( m > n \), and so taking \( m = n + p \), where \( p \in \mathbb{N} \), we get

\[
d(x_n, x_m) = d(x_n, x_{n+p}) \leq s d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \ldots + s^p d(x_{n+p-1}, x_{n+p})
\]

taking \( n \to \infty \), we get \( \lim_{n \to \infty} d(x_n, x_m) = 0 \). Hence the sequence \( \{x_n\} \) is a Cauchy sequence. As \( (X, d, s) \) is complete, then there exists \( z \in X \) such that \( x_n \to z \) and so, \( \lim_{n \to \infty} d(x_n, z) = 0 \).

Now we show that \( z \) is a fixed point of \( T \) and \( S \). To see this, we have

\[
d(z, Tz) \leq s[d(z, x_{2n+2}) + d(x_{2n+2}, Tz)] \\
\leq s[d(z, x_{2n+2}) + H(Sx_{2n}, Tz)] \\
\leq s[d(z, x_{2n+2}) + H(Tz, Sx_{2n})] \\
\leq s[d(z, x_{2n+2}) + ad(z, Tz) + cd(Tz, Sx_{2n}) + \ldots + s^p d(x_{2n+p-1}, x_{2n+p})] \\
\leq s[d(z, x_{2n+2}) + ad(z, x_{2n+1}) + cd(z, x_{2n+2})] \\
\leq \sum_{k=1}^{\infty} d(z, x_{2n+2}) + d(z, Tz) \leq bcs[d(Tz, z) + d(z, x_{2n+2})].
\]

Letting \( n \to \infty \) in above inequality, we obtain \( d(z, Tz) \leq s^2(b + bc)d(z, Tz) \), then we have \( d(z, Tz) = 0 \) (since \( b + bc < \frac{1}{s^2} \)), i.e., \( z \in Tz \). Hence \( F(T) \neq \phi \), here \( F(T) \) is the set of fixed points of \( T \). Also,

\[
H(Tz, Sz) \leq ad(z, Tz) + b[d(z, Sz) + d(Tz, Tz) + cd(Tz, Sz)] \\
\leq bd(z, Sz) + bcd(z, Sz) \\
\leq (b + bc)H(Tz, Sz),
\]

thus \( H(Tz, Sz) = 0 \), i.e., \( Tz = Sz \). Hence \( F(S) \neq \phi \), where \( F(S) \) denotes the collection of fixed points of \( S \).

Now, we arrive at our final step which requires the following steps:

1. \( F(T) = Tz \),
2. \( Sx = Tx \) for all \( x \in F(T) \),
3. \( F(T) = F(S) \).

Firstly, let \( x \in F(T) \), i.e. \( x \in Tx \),

\[
d(x, Tz) \leq H(Tx, Tz) \\
\leq H(Tx, Sx) \\
\leq ad(x, Tz) + b[d(x, Sx) + d(Tz, Tx) + cd(Tx, Sx)] \\
\leq ad(x, Tz) + bd(x, Tz) + bd(Tx, Sz) + bcd(x, Sx) \\
\leq ad(x, Tz) + bd(x, Tz) + bd(x, Tz) + bcd(x, Tz),
\]

thus \( d(x, Tz) = 0 \), i.e., \( x \in Tz \), and hence \( Tx \subset Tz \) and \( F(T) \subset Tz \).

Now, let \( x \in Tz \). Then,

\[
d(x, Tx) \leq H(Tz, Tx) \\
\leq H(Tz, Sz) \\
\leq ad(x, Tz) + b[d(x, Sz) + d(Tz, Tx) + cd(Tx, Sz)] \\
\leq ad(x, Tz) + bd(x, Sz) + bd(Tz, Tz) + bcd(Tx, Sz) \\
\leq ad(x, Tz) + bd(x, Tz) + bd(x, Tz) + bcd(Tx, Tz) \\
\leq ad(x, Tz) + bd(x, Tz) + bd(x, Tx) + bcd(Tx, x),
\]

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thus \( d(x, Tx) = 0 \), i.e., \( x \in Tx \). Hence, we get \( Tz \subset Tx, Tz \subset F(T) \), and so \( F(T) = Tz \).

Next, we show that \( Tx = Sx \). For all \( x \in F(T) \), we have
\[
H(Tx, Sx) \leq ad(x, Tx) + b[d(x, Sx) + d(Tx, Tx) + cd(Tx, Sx)] \\
\leq ad(x, Tx) + bd(x, Sx) + bcd(x, Sx) \\
\leq (b + bc)d(x, Sx) \\
\leq (b + c)H(Tx, Sx),
\]
thus \( H(Tx, Sx) = 0 \), i.e., \( Tx = Sx \) for all \( x \in F(T) \).

Now, we show that \( F(T) = F(S) \). Let \( x \in F(T) \), i.e., \( x \in Tx \). By previous result \( Tx = Sx \), we find \( x \in Sx \Rightarrow x \in F(S) \), so we automatically get \( F(T) \subset F(S) \).

Further, it remains to show that \( F(S) \subset F(T) \). Let \( x \in F(s) \), i.e., \( x \in Sx \)
\[
d(x, Tx) \leq H(Tx, Sx) \\
\leq ad(x, Tx) + b[d(x, Sx) + d(Tx, Tx) + cd(Tx, Sx)] \\
\leq ad(x, Tx) + bcd(Tx, Sx) \\
\leq ad(x, Tx) + bcd(Tx, x) \\
\leq (a + bc)d(Tx, x),
\]
thus \( d(x, Tx) = 0 \), i.e., \( x \in Tx \). Hence, \( F(T) = F(S) \neq \emptyset \) and \( Sx = Tx = F(T) \) for all \( x \in F(T) \).

At last, we have to show that the common fixed point is unique. Let \( z, v \) be two common fixed points of \( T \) and \( S \) such that \( z \neq v \). Then,
\[
d(z, v) \leq H(Tz, Tv) \\
\leq H(Tz, Sv) \\
\leq ad(z, Tv) + bd(z, S v) + cd(Tz, Sv) \\
\leq ad(z, Tv) + bd(z, Sv) + bd(v, z) + bcd(z, Sv) \\
\Rightarrow (1 - b - bc)d(z, v) \leq ad(z, Tv) + bd(z, Sv) \\
\leq ad(z, Tv) + bd(z, Tv) \\
\leq (a + b)d(z, v) \\
\Rightarrow (1 - a - 2b - bc)d(z, v) \leq 0 \\
\Rightarrow d(z, v) = 0, \text{ i.e., } z = v.
\]
which completes the proof.

Remark 2.1. Our result (Theorem 2.1) generalizes the result of [12].

As a consequence of Theorem 2.1, we have the following corollary.

**Corollary 2.1.** Let \((X, d, s)\) be a complete \( b \)-metric space and let \( S, T : X \rightarrow CB(X) \) be two multivalued mappings satisfying for all \( x, y \in X \),
\[
H(Tx, S y) \leq ad(x, Ty) + b[d(x, S y) + cd(Tx, S y)],
\]
where \( a, b, c \geq 0 \) with \( a + 2bs + bc < 1 \) and \( b + bc < 1/s^2 \). Then \( S \) and \( T \) have a unique common fixed point in \( X \).

Also, for different cases in Theorem 2.1 we obtain the following particular results as corollaries.

**Corollary 2.2** ([15]). Let \((X, d)\) be complete metric space and let \( S, T : X \rightarrow CB(X) \) be mappings satisfying \( H(Tx, Ty) \leq r d(x, Ty) \), for all \( x, y \in X \) with \( r \in (0, 1) \). Then \( F(T) = F(S) \neq \emptyset \) and \( Tx = Sx = F(T) \) for all \( x \in F(T) \).

**Corollary 2.3.** Let \((X, d)\) be complete metric space and let \( S, T \) be self mappings on \( X \) and if there exists \( r \in (0, 1) \) such that \( d(Tx, Sy) \leq r d(x, Ty) \) for all \( x, y \in X \). Then \( S \) and \( T \) have a unique common fixed point.

**Corollary 2.4.** Let \((X, d)\) be complete metric space and let \( T : X \rightarrow C(X) \) be a mapping satisfying \( H(Tx, T^2y) \leq r d(x, Ty) \), for all \( x, y \in X \) with \( r \in [0, 1) \). Then \( F(T) \neq \emptyset \) and \( Tx = F(T) \) for all \( x \in F(T) \).

Now, we present the following illustration in the support of Theorem 2.1.
Example 2.1. Let $X = [0, \infty)$ with b-metric defined by $d(x, y) = |x - y|^2$ with $s = 2$. Let $T, S : X \to CB(X)$ defined by

\[ T(x) = \begin{cases} \frac{1}{4}, & \text{if } x \in [2, \infty), \\ 0, & \text{if } x \not\in [2, \infty) \end{cases} \quad \text{and} \quad S(x) = \begin{cases} \frac{1}{3}, & \text{if } x \in [2, \infty), \\ 0, & \text{if } x \not\in [2, \infty). \end{cases} \]

Now, we consider the following case:

Case 1: If $x \not\in [2, \infty)$, $y \not\in [2, \infty)$, then $T(x) = 0, S(y) = 0$, and the condition (2.1) is obviously true because $H(T, S) = 0$.

Case 2: If $x \in [2, \infty)$, $y \in [2, \infty)$, we have $T(x) = \frac{1}{4}, S(y) = \frac{1}{3}$, and

\[
H(T, S) = \max \left\{ \sup_{a \in T} d(a, \left[ \frac{1}{16}, \frac{1}{25} \right]), \sup_{b \in S} d(b, \left[ \frac{1}{4}, \frac{1}{2} \right]) \right\} = \max \left\{ \frac{1}{16}, \frac{1}{30} \right\} = \frac{529}{625}.
\]

Also, $d(x, Ty) = \frac{9}{4}, d(x, Sx) = \frac{2049}{625}, d(Tx, Ty) = 0$ and $d(Tx, Sx) = \frac{441}{10000}$. Thus the condition (2.1) satisfies for $a = \frac{1}{4}, b = \frac{1}{30}$ and $c = 3$ with $a + 2bs + bc < 1$ and $b + bc < \frac{1}{s}$.

Case 3: If $x \in [2, \infty), y \not\in [2, \infty)$, we get $T(x) = \frac{1}{4}, S(y) = 0, H(T, S) = \frac{1}{4}, d(x, Ty) = 4, d(x, Sx) = 4, d(Tx, Ty) = \frac{1}{16}$ and $d(Tx, Sx) = \frac{1}{10}$. Thus, the condition (2.1) is true for $a = \frac{1}{4}, b = \frac{1}{30}$ and $c = 3$ with $a + 2bs + bc < 1$ and $b + bc < \frac{1}{s}$.

Case 4: If $x \not\in [2, \infty), y \in [2, \infty)$, we get $T(x) = 0, S(y) = \frac{1}{3}, H(T, S) = \frac{1}{4}, d(x, Ty) = 0, d(x, Sx) = 0$ and $d(Tx, Ty) = \frac{1}{16}$ and $d(Tx, Sx) = \frac{1}{10}$. Thus, the condition (2.1) holds for $a = \frac{1}{4}, b = \frac{1}{30}$ and $c = 3$ with $a + 2bs + bc < 1$ and $b + bc < \frac{1}{s}$.

Hence, in all cases we have the condition (2.1) of Theorem 2.1 satisfied for $a = \frac{1}{4}, b = \frac{1}{30}$ and $c = 3$ with $a + 2bs + bc < 1$ and $b + bc < \frac{1}{s}$, and $0 \in X$ is the only common fixed point of $S$ and $T$.

Theorem 2.2. Let $(X, d_b, s)$ be a complete dislocated b-metric space and let $S, T : X \to CB(X)$ be two multivalued mappings satisfying for all $x, y \in X$,

\[
H_b(Tx, Sx) \leq a d_b(x, Ty) + b [d_b(x, Sx) + c d_b(Tx, Sx)],
\]

where $a, b, c \geq 0$ with $2as + 3bs + bc < 1$ and $b + bc < 1/s^2$. Then $S$ and $T$ have a unique common fixed point in $X$.

Proof. Fix $x \in X$. Without loss of generality, we choose $\epsilon_0$ such that $0 < \epsilon_0 < 1 - 2bs - bc$. Then, for $x \in X$, define $x_0 = x$ and let $x_1 \in Tx_0$. By Lemma 2.1, we may choose $x_2 \in Sx_0$ such that

\[
d_b(x_1, x_2) \leq H_b(Tx_0, Sx_0) + \epsilon_0
\]

\[
\leq a d_b(x_0, x_1) + b [d_b(x_0, x_2) + c d_b(Tx_0, Sx_0)] + \epsilon_0
\]

\[
\leq a d_b(x_0, x_1) + bd_b(x_0, x_2) + bcd_b(x_1, x_2) + \epsilon_0
\]

\[
\Rightarrow d_b(x_1, x_2)(1 - 2bs - bc) \leq ad_b(x_0, x_1) + bsd_b(x_0, x_1) + \epsilon_0
\]

\[
\text{Thus,} \quad d_b(x_1, x_2) \leq \frac{(a + bs)}{(1 - 2bs - bc)} d_b(x_0, x_1) + \epsilon_1, \quad \text{where} \quad \epsilon_1 = \frac{\epsilon_0}{(1 - 2bs - bc)}.
\]

Similarly, there exists $x_3 \in Tx_2$ such that

\[
d_b(x_2, x_3) \leq H_b(Tx_2, Sx_0) + \frac{\epsilon_0^2}{(1 - 2bs - bc)}
\]

\[
\leq ad_b(x_2, Tx_0) + b [d_b(x_2, Sx_0) + c d_b(Tx_2, Sx_0)]
\]

\[
+ \frac{\epsilon_0^2}{(1 - 2bs - bc)}
\]

\[
\leq ad_b(x_2, x_1) + bd_b(x_2, x_2) + bcd_b(x_3, x_2) + \frac{\epsilon_0^2}{(1 - 2bs - bc)}
\]

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Let $N$ be a positive integer, we have
\[
\frac{1}{2} \leq ad_b(x_2, x_1) + bs[dl_b(x_2, x_3) + dl_b(x_3, x_2)] + bcd_b(x_3, x_2) + \frac{\epsilon_0^2}{(1 - 2bs - bc)}
\]
\[
\Rightarrow \quad d_b(x_2, x_3)(1 - 2bs - bc) \leq ad_b(x_2, x_1) + bsdl_b(x_2, x_1) + \frac{\epsilon_0^2}{(1 - 2bs - bc)}
\]
\[
\Rightarrow \quad d_b(x_2, x_3) \leq (a + bs)(1 - 2bs - bc) - d_b(x_1, x_2) + \frac{\epsilon_0^2}{(1 - 2bs - bc) - 2}. \]

Continuing in this way, we obtain a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that
\[
d_b(x_n, x_{n+1}) \leq \frac{(a + bs)}{(1 - 2bs - bc)} d_b(x_{n-1}, x_n) + \epsilon_1^n \quad \forall n \in \mathbb{N}.
\]

Let $k = \frac{(a + bs)}{(1 - 2bs - bc)}$, then for each $n \in \mathbb{N}$, we have
\[
d_b(x_n, x_{n+1}) \leq k d_b(x_{n-1}, x_n) + \epsilon_1^n
\]
\[
\leq k^2 d_b(x_{n-2}, x_{n-1}) + \epsilon_1^n + \ldots
\]
\[
\leq k^n d_b(x_0, x_1) + \sum_{r=0}^{n-1} k^r \epsilon_1^n
\]

which shows that,
\[
\sum_{n=1}^{N} d_b(x_n, x_{n+1}) \leq \sum_{n=1}^{N} k^n d_b(x_0, x_1) + \sum_{n=1}^{N} \sum_{r=0}^{n-1} k^r \epsilon_1^n
\]
\[
\leq \sum_{n=1}^{N} k^n d_b(x_0, x_1) + \sum_{n=1}^{N} \epsilon_1^n \cdot \sum_{r=0}^{n-1} k^r
\]
\[
\leq d_b(x_0, x_1) \sum_{n=1}^{N} k^n + \sum_{n=1}^{N} \epsilon_1^n \cdot \frac{1 - k^n}{1 - k}
\]
\[
< d_b(x_0, x_1) \sum_{n=1}^{N} k^n + \sum_{n=1}^{N} \epsilon_1^n \cdot \frac{1 - k}{1 - k}
\]
\[
\Rightarrow \quad \sum_{n=1}^{\infty} d_b(x_n, x_{n+1}) = \sum_{n=1}^{\infty} k^n \frac{d_b(x_0, x_1)}{1 - k} + \frac{\epsilon_1}{1 - k} < \infty.
\]

Hence, we get $\lim_{n \to \infty} d_b(x_n, x_{n+1}) = 0$.

Now we show that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $X$. Let $m, n > 0$ with $m > n$, and so taking $m = n + p$, where $p \in \mathbb{N}$, we have
\[
d_b(x_n, x_{n+p}) = d_b(x_n, x_{n+p-1}) + s^2 d_b(x_{n+p-1}, x_{n+p-2}) + \ldots + s^p d_b(x_{n+p-1}, x_{n+p}),
\]
making $n \to \infty$, we get $\lim_{n \to \infty} d(x_n, x_m) = 0$. Hence the sequence $\{x_n\}$ is a Cauchy sequence. As $(X, d_b, s)$ is complete, then there exists $z \in X$ such that $x_n \to z$ and so, $\lim_{n \to \infty} d_b(x_n, z) = 0$.

Now we show that $z$ is a fixed point of $T$ and $S$. To see this, we have
\[
d_b(z, Tz) \leq s[d_b(z, x_{2n+2}) + d_b(x_{2n+2}, Tz)]
\]
\[
\leq s[d_b(z, x_{2n+1}) + H_b(Sx_{2n}, Tz)]
\]
\[
\leq s[d_b(z, x_{2n+2}) + H_b(Tz, Sx_{2n})]
\]
\[
\leq s[d_b(z, x_{2n+2}) + ad_b(z, Tz) + b|d_b(z, Sx_{2n})| + c|d_b(Tz, Sx_{2n})|]
\]
\[
\leq s[d_b(z, x_{2n+2}) + ad_b(z, x_{2n+1}) + b|d_b(z, x_{2n+2})| + c|d_b(Tz, x_{2n+2})|]
\]
\[
\leq s[d_b(z, x_{2n+2}) + ad_b(z, x_{2n+1}) + bd_b(z, x_{2n+2})
\]
\[
+ bcsd_b(z, Tz) + d_b(z, x_{2n+2})).
\]

Letting $n \to \infty$ in the inequality above, we obtain $d_b(z, Tz) \leq s^2 bcd_b(z, Tz)$, then we have $d_b(z, Tz) = 0$ (since $b + bc < \frac{1}{p}$), i.e. $z \in Tz$. Hence $F(T) \neq \emptyset$, where $F(T)$ denotes the collection of fixed points of $T$. Also,
\[
H_b(Tz, Sz) \leq ad_b(z, Tz) + b|d_b(z, Sz)| + c|d_b(Tz, Sz)|
\]
Thus $H_b(T_z, S_z) = 0$, i.e., $T_z = S_z$. Hence $F(S) \neq \phi$, here $F(S)$ is the set of fixed points of $S$.

We arrive at the proof of our final result which require the following steps:

1. $F(T) = T_z$,
2. $S x = T x$ for all $x \in F(T)$,
3. $F(T) = F(S)$.

Firstly, let $x \in F(T)$, i.e. $x \in T x$,

$$
\begin{align*}
    d_b(x, T z) & \leq H_b(T x, T z) \\
    & \leq H_b(T x, S z) \\
    & \leq ad_b(x, T z) + b[d_b(x, S z) + cd_b(T x, S z)] \\
    & \leq ad_b(x, T z) + bd_b(x, T z) + bcd_b(x, S z) \\
    & \leq ad_b(x, T z) + bd_b(x, T z) + +bcd_b(T x, S z),
\end{align*}
$$

thus $d_b(x, T z) = 0$, i.e., $x \in T z$. Hence $T x \subset T z$ and $F(T) \subset T z$.

Now, let $x \in T z$. We show that $x \in T x$,

$$
\begin{align*}
    d_b(x, T x) & \leq H_b(T z, T x) \\
    & \leq H_b(S z, T x) \\
    & \leq H_b(T x, S z) \\
    & \leq ad_b(x, T z) + b[d_b(x, S z) + cd_b(T x, S z)] \\
    & \leq ad_b(x, T z) + bd_b(x, T z) + bcd_b(x, S z) \\
    & \leq ad_b(x, x) + bd_b(x, x) + bcd_b(T x, x),
\end{align*}
$$

Thus $d_b(x, T x) = 0$, i.e., $x \in T x$. Hence $T z \subset T x$, $T z \subset F(T)$, and so $F(T) = T z$.

Next, we show that $T x = S x$. For all $x \in F(T)$, we get

$$
\begin{align*}
    H_b(T x, S x) & \leq ad_b(x, T x) + b[d_b(x, S x) + cd_b(T x, S x)] \\
    & \leq ad_b(x, T x) + bd_b(x, S x) + bcd_b(x, S x) \\
    & \leq bd_b(x, S x) + bcd_b(x, S x) \\
    & \leq (b + bc)d_b(x, S x) \\
    & \leq (b + bc)H_b(T x, S x),
\end{align*}
$$

thus $H_b(T x, S x) = 0$, i.e., $T x = S x$ for all $x \in F(T)$.

Now, we show that $F(T) = F(S)$. Let $x \in F(T)$, i.e., $x \in T x$. By previous result $T x = S x$, we get $x \in S x \Rightarrow x \in F(S)$, so we automatically get $F(T) \subset F(S)$.

It remains to show that $F(S) \subset F(T)$. Let $x \in F(s)$, i.e. $x \in S x$

$$
\begin{align*}
    d_b(x, T x) & \leq H_b(S x, T x) \\
    & \leq ad_b(x, T x) + b[d_b(x, S x) + cd_b(T x, S x)] \\
    & \leq ad_b(x, T x) + bd_b(x, S x) + bcd_b(T x, S x) \\
    & \leq ad_b(x, T x) + bd_b(x, T x) + bcd_b(T x, x) \\
    & \leq (a + b + bc)d_b(T x, x),
\end{align*}
$$

thus $d_b(x, T x) = 0$, i.e., $x \in T x$, and we get $F(S) = F(T)$. Hence $F(T) = F(S) \neq \phi$ and $S x = T x = F(T)$ for all $x \in F(T)$.

At last, we have to show that the common fixed point is unique. Let $z, v$ be two common fixed points of $T$ and $S$ such that $u \neq v$. Then,

$$
\begin{align*}
    d_b(u, v) & \leq H_b(T z, T v) \\
    & \leq H_b(T z, S v) \\
    & \leq ad_b(z, T v) + b[d_b(z, S v) + cd_b(T z, S v)] \\
    & \leq ad_b(z, T v) + bd_b(z, S v) + bcd_b(z, v)
\end{align*}
$$
⇒ \( (1 - bc)d_b(z, v) \leq ad_b(z, Tv) + bd_b(z, S v) \)

\[ \leq ad_b(z, Tv) + bd_b(z, Tv) \]

⇒ \( d_b(z, v) = 0 \), i.e., \( z = v \).

which completes the proof.

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**References**


