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(Dedicated to Professor D. S. Hooda on His 80^{th} Birth Anniversary Celebrations)

COMMON FIXED POINT THEOREMS FOR GENERALIZED MULTI-VALUED CONTRACTIONS IN *b*-METRIC AND DISLOCATED *b*-METRIC SPACES

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Abstract

In this paper, we extend the contraction to find some common fixed point theorems of multivalued contraction in b-metric and dislocated b-metric spaces. Also, we give some examples to vindicate our results. Moreover, the obtained results extend and improve some well known results of the literature.

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1. Introduction and Preliminaries

Let (X, d) be a metric space, then a mapping $T : X \to X$ is said to be contraction if there exists a positive real number $r < 1$ such that $d(Tx, Ty) \leq r\,d(x, y)$ for all $x, y \in X$. In the literature of fixed point theory, Banach contraction principle (*BCP*) plays an improtant role which states that every contraction on a complete metric space (X, d) has a unique fixed point, i.e., there is a point $z \in X$ such that $Tz = z$. It has a wide range of applications in physical sciences, computer sciences and Engineering Sciences. *BCP* is further improved, extended and generalized by many researchers in the fixed point theory by weakening the contractive condition and structure of the metric space [1, 18]. In 1969, Nadler[17] introduced the notion of multi-valued contractive mapping in a complete metric space, and established a fixed point theorem for multivalued mapping which is a generalization of *BCP*. In fact, Nadler proved the following result:

Theorem 1.1 ([17]). *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a mapping. Assume there exists $r \in [0, 1)$ such that $H(Tx, Ty) \leq r\,d(x, y)$ for all $x, y \in X$. Then, there exists $z \in X$ such that $z \in Tz$.*

Thereafter, Nadler's fixed point theorem, many authors have given results for muti-valued and hybrid contractive mappings [8, 9, 10, 11, 19, 4] and references therein.

In generalization process of metric structure, the concept of b-metric space was introduced by Czerwik[7]. Since then many authors used notion of b-metric to obtain various fixed point theorems. Moreover, several results of metric spaces are presented for b-metric spaces, see [3, 13, 12, 16] and references therein. Similarly, some interesting results in dislocated b-metric space are also proved by researchers, see [14, 20].

Now we give some definitions and important theorems of literature which will be useful for our main results.

Definition 1.1 ([5]). *Let X be a non-empty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is called a b-metric iff for all $x, y, z \in X$, the following conditions are satisfied:*

(i) $d(x, y) = 0$ *iff* $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$,

the triplet (X, d, s) is known as b-metric space.

Definition 1.2 ([5]). *Let (X, d, s) be a b-metric space. A sequence x_n in X is said to be:*

1. *Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$;*
2. *convergent iff there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$ (or if for all $\epsilon > 0$, there exists $k \in \mathbb{N}$ we have such that $d(x_n, x) < \epsilon$ for all $n \geq k$), and we write $\lim_{n\to\infty} x_n = x$;*
3. *complete if every Cauchy sequence in X is convergent.*

Definition 1.3 ([3]). *Let (X, d, s) be a b-metric space. A subset Y of X is said to be:*

1. *closed iff for each sequence x_n in Y, which converges to an element x, we have $x \in Y$;*
2. *compact iff for every sequence of element in Y there exists a subsequence that converges to an element in Y;*
3. *bounded iff* $\sup\{d(x, y) : x, y \in Y\} < \infty$.

The extension of Banach contraction principle in b-metric spaces as follows:

Theorem 1.2 ([5]). *Let (X, d) be a complete b-metric space with constant $s \geq 1$ such that b-metric is a continuous functional. Let $T : X \to X$ be a contraction having contraction constant $k \in [0, 1)$ such that $ks < 1$. Then T has a unique fixed point.*

Definition 1.4 ([2]). *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d_b : X \times X \to [0, \infty)$ is called a dislocated b-metric iff for all $x, y, z \in X$, the following conditions are satisfied:*

(i) $d_b(x, y) = 0 \Rightarrow x = y$;
(ii) $d_b(x, y) = d(y, x)$;
(iii) $d_b(x, z) \leq s[d_b(x, y) + d_b(y, z)]$,

the triplet (X, d_b, s) is known as dislocated b-metric space. If $s = 1$, the disloctated b-metric space (X, d_b, s) is called a dislocated metric space.

Definition 1.5 ([2]). *Let (X, d_b) be a dislocated b-metric space. A sequence $\{x_n\}$ in X is said to be:*

1. *Cauchy if and only if $d_b(x_n, x_m) \to 0$ as $n, m \to \infty$;*
2. *convergent iff there exist $x \in X$ such that $d_b(x_n, x) \to 0$ as $n \to \infty$ and we write* $\lim_{n\to\infty} x_n = x$;
3. *complete if every Cauchy sequence in X is convergent.*

Example 1.1. Let $X = \mathbb{Q}^+ \cup \{0\}$ and let $d_b : X \times X \to X$ defined by $d_b(x, y) = (x + y)^2$ for all $x, y \in X$. Then d_b is dislocated b-metric on X with $s = 2$.

Let (X, d) be a metric (resp. b-metric, dislocated b-metric) space and $CB(X)$ the collection of all non-empty closed and bounded subsets of X. The Hausdorff metric $H(resp. H, H_b)$ on $CB(X)$ induced by the metric d is given by

$$H(A, B) = \max\left\{\sup_{p\in A} d(p, B), \sup_{q\in B} d(q, A)\right\}$$

for $A, B \in CB(X)$, where $d(x, A) = \inf_{y\in A} d(x, y)$. Moreover, the distance between sets $A, B \in CB(X)$ is defined as $d(A, B) = \inf\{d(p, q) : p \in A, q \in B\}$.

We recall the following properties from [3, 6] and the references therein.

Lemma 1.1. *Let (X, d, s) be a b-metric space. For any $A, B, C \in CB(X)$ and any $x, y \in X$, the following statements are true:*

1. *$d(x, B) \leq d(x, q)$, for any $q \in B$;*
2. $\sup_{p\in A} d(p, B) \leq H(A, B)$;
3. *$H(A, B) = 0$ iff $A = B$;*
4. *$d(x, B) \leq H(A, B)$, for any $x \in A$;*
5. *$H(A, B) = H(B, A)$;*
6. *$H(A, C) \leq s[H(A, B) + H(B, C)]$*
7. *$d(x, A) \leq s[d(x, y) + d(y, A)]$.*

Following the Nadler's fixed point theorem, Czerwik [6] established the following theorem for mulitvalued mapping.

Theorem 1.3 ([6]). *Let (X, d, s) be a complete b-metric space and let $T : X \to CB(X)$ be a multi-valued mapping such that T satisfies the inequality $H(Tx, Ty) \leq ad(x, y)$ for all $x, y \in X$, where $0 < a < \frac{1}{s^2}$.Then T has a fixed point.*

Subsequently a number of fixed point theorems have been obtained by the researchers for multivalued mappings in different settings of spaces see [12, 19, 18, 4] and references therein. In this paper, we extend and generalize the result of [12] for multivalued mappings satisfying generalized contractive type condition in complete b-metric and dislocated b-metric spaces.

2. Main Results

Lemma 2.1 ([7]). *Let (X, d, s) be a b-metric (or dislocated b-metric) space and $A, B \in CB(X)$. Then, for each $\epsilon > 0$ and for all $p \in A$, there exists $q \in B$ such that $d(p, q) \le H(A, B) + \epsilon$.*

Theorem 2.1. *Let (X, d, s) be a complete b-metric space and let $S, T : X \to CB(X)$ be two multivalued mappings satisfying for all $x, y \in X$,*

$$H(Tx, Sy) \le a\,d(x, Ty) + b\,[d(x, Sy) + d(Tx, Ty) + c\,d(Tx, Sy)], \tag{2.1}$$

where $a, b, c \ge 0$ with $a + 2bs + bc < 1$ and $b + bc < 1/s^2$. Then S and T have a unique common fixed point in X.

Proof. Fix $x \in X$. Without loss of generality, we choose ϵ_0 such that $0 < \epsilon_0 < 1 - bs - bc$. Then, for $x \in X$, define $x_0 = x$ and let $x_1 \in Tx_0$. By Lemma 2.1, we may choose $x_2 \in Sx_0$ such that

$$\begin{aligned}
d(x_1, x_2) &\le H(Tx_0, Sx_0) + \epsilon_0 \\
&\le a\,d(x_0, Tx_0) + b\,[d(x_0, Sx_0) + d(Tx_0, Tx_0) + c\,d(Tx_0, Sx_0] + \epsilon_0 \\
&\le ad(x_0, x_1) + bd(x_0, x_2) + bcd(x_1, x_2) + \epsilon_0 \\
&\le a\,d(x_0, x_1) + bs\,[d(x_0, x_1) + d(x_1, x_2)] + bc\,d(x_1, x_2) + \epsilon_0 \\
\Rightarrow d(x_1, x_2)(1 - bs - bc) &\le ad(x_0, x_1) + bsd(x_0, x_1) + \epsilon_0 \\
&\le (a + bs)d(x_0, x_1) + \epsilon_0 \\
\Rightarrow d(x_1, x_2) &\le \frac{(a + bs)}{(1 - bs - bc)}\,d(x_0, x_1) + \frac{\epsilon_0}{(1 - bs - bc)}. \\
\text{Thus,}\quad d(x_1, x_2) &\le \frac{(a + bs)}{(1 - bs - bc)}d(x_0, x_1) + \epsilon, \quad \text{where } \epsilon = \frac{\epsilon_0}{(1 - bs - bc)}.
\end{aligned}$$

Similarly, there exists $x_3 \in Tx_2$ such that

$$\begin{aligned}
d(x_2, x_3) &\le H(Tx_2, Sx_0) + \frac{{\epsilon_0}^2}{(1 - bs - bc)} \\
&\le ad(x_2, Tx_0) + b[d(x_2, Sx_0) + d(Tx_0, Tx_2) + cd(Tx_2, Sx_0)] \\
&\quad + \frac{{\epsilon_0}^2}{(1 - bs - bc)} \\
\Rightarrow d(x_2, x_3) &\le ad(x_2, x_1) + bd(x_1, x_3) + bcd(x_3, x_2) + \frac{{\epsilon_0}^2}{(1 - bs - bc)} \\
&\le ad(x_2, x_1) + bs[d(x_1, x_2) + d(x_2, x_3)] + bcd(x_3, x_2) + \frac{{\epsilon_0}^2}{(1 - bs - bc)} \\
\Rightarrow d(x_2, x_3)(1 - bs - bc) &\le ad(x_2, x_1) + bsd(x_1, x_2) + \frac{{\epsilon_0}^2}{(1 - bs - bc)} \\
\Rightarrow d(x_2, x_3) &\le \frac{(a + bs)}{(1 - bs - bc)}d(x_1, x_2) + \frac{{\epsilon_0}^2}{(1 - bs - bc)^2}
\end{aligned}$$

Continuing in this way, we obtain by induction a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_{2n} \in Sx_{2n-2}, x_{2n+1} \in Tx_{2n}$, such that

$$\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &\le H(Tx_{2n}, Sx_{2n}) + \frac{\epsilon_0^{2n+1}}{(1 - bs - bc)^{2n}} \\
d(x_{2n}, x_{2n+1}) &\le H(Sx_{2n-2}, Tx_{2n}) + \frac{\epsilon_0^{2n}}{(1 - bs - bc)^{2n-1}}
\end{aligned}$$

Now,

$$\begin{aligned}
d(x_{2n}, x_{2n+1}) &\le H(Sx_{2n-2}, Tx_{2n}) + \frac{\epsilon_0^{2n}}{(1 - bs - bc)^{2n-1}} \\
&\le ad(x_{2n}, Tx_{2n-2}) + b[d(x_{2n}, Sx_{2n-2}) + d(Tx_{2n}, Tx_{2n-2}) + cd(Tx_{2n}, Sx_{2n-2})] \\
&\quad + \frac{\epsilon_0^{2n}}{(1 - bs - bc)^{2n-1}}
\end{aligned}$$

$$
\begin{aligned}
d(x_{2n}, x_{2n+1}) &\leq ad(x_{2n}, x_{2n-1}) + b[d(x_{2n}, x_{2n}) + d(x_{2n+1}, x_{2n-1}) + cd(x_{2n+1}, x_{2n})] \\
&\quad + \frac{\epsilon_0^{2n}}{(1-bs-bc)^{2n-1}} \\
&\leq ad(x_{2n}, x_{2n-1}) + bd(x_{2n+1}, x_{2n-1}) + bcd(x_{2n+1}, x_{2n}) + \frac{\epsilon_0^{2n}}{(1-bs-bc)^{2n-1}} \\
d(x_{2n}, x_{2n+1}) &\leq ad(x_{2n}, x_{2n-1}) + bs[d(x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n-1})] + bcd(x_{2n+1}, x_{2n}) \\
&\quad + \frac{\epsilon_0^{2n}}{(1-bs-bc)^{2n-1}} \\
(1-bs-bc)d(x_{2n}, x_{2n+1}) &\leq (a+bs)d(x_{2n}, x_{2n-1}) + \frac{\epsilon_0^{2n}}{(1-bs-bc)^{2n-1}} \\
\Rightarrow \qquad d(x_{2n}, x_{2n+1}) &\leq \frac{(a+bs)}{(1-bs-bc)} d(x_{2n}, x_{2n-1}) + \frac{\epsilon_0^{2n}}{(1-bs-bc)^{2n}}
\end{aligned}
$$

Also,

$$
\begin{aligned}
d(x_{2n+1}, x_{2n+2}) &\leq H(Tx_{2n}, Sx_{2n}) + \frac{\epsilon_0^{2n+1}}{(1-bs-bc)^{2n}} \\
&\leq ad(x_{2n}, Tx_{2n}) + b[d(x_{2n}, Sx_{2n}) + d(Tx_{2n}, Tx_{2n}) + cd(Tx_{2n}, Sx_{2n})] \\
&\quad + \frac{\epsilon_0^{2n+1}}{(1-bs-bc)^{2n}} \\
&\leq ad(x_{2n}, x_{2n+1}) + bd(x_{2n}, x_{2n+2}) + bcd(x_{2n+1}, x_{2n+2}) + \frac{\epsilon_0^{2n+1}}{(1-bs-bc)^{2n}} \\
&\leq ad(x_{2n}, x_{2n+1}) + bs[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})] + bcd(x_{2n+1}, x_{2n+2}) \\
&\quad + \frac{\epsilon_0^{2n+1}}{(1-bs-bc)^{2n}} \\
(1-bs-bc)d(x_{2n+1}, x_{2n+2}) &\leq (a+bs)d(x_{2n}, x_{2n+1}) + \frac{\epsilon_0^{2n+1}}{(1-bs-bc)^{2n}} \\
\Rightarrow d(x_{2n+1}, x_{2n+2}) &\leq \frac{(a+bs)}{(1-bs-bc)} d(x_{2n}, x_{2n+1}) + \frac{\epsilon_0^{2n+1}}{(1-bs-bc)^{2n+1}}
\end{aligned}
$$

Therefore,

$$
d(x_n, x_{n+1}) \leq \frac{(a+bs)}{(1-bs-bc)} d(x_{n-1}, x_n) + \epsilon^n \quad \forall n \in \mathbb{N}, \quad \text{where} \quad \epsilon = \frac{\epsilon_0}{(1-bs-bc)}.
$$

Let $k = \frac{(a+bs)}{(1-bs-bc)}$, then for each $n \in \mathbb{N}$, we have

$$
\begin{aligned}
d(x_n, x_{n+1}) &\leq k\, d(x_{n-1}, x_n) + \epsilon^n \\
&\leq k[kd(x_{n-2}, x_{n-1}) + \epsilon^{n-1}] + \epsilon^n \\
&\vdots \\
&\leq k^n d(x_0, x_1) + \sum_{r=0}^{n-1} k^n \epsilon^{n-r} \\
\text{which shows that, } \sum_{n=1}^{N} d(x_n, x_{n+1}) &\leq \sum_{n=1}^{N} k^n d(x_0, x_1) + \sum_{n=1}^{N} \left(\sum_{r=0}^{n-1} k^r \epsilon^{n-r}\right) \\
&\leq \sum_{n=1}^{N} k^n d(x_0, x_1) + \sum_{n=1}^{N} \epsilon^n \left(\sum_{r=0}^{n-1} k^r\right) \\
&= d(x_0, x_1) \sum_{n=1}^{N} k^n + \sum_{n=1}^{N} \epsilon^n \cdot \frac{1-k^n}{1-k}
\end{aligned}
$$

$$
\begin{aligned}
& & < & \; d(x_0, x_1)\sum_{n=1}^{N} k^n + \sum_{n=1}^{N} \epsilon^n \cdot \frac{1}{1-k} \\
\Rightarrow & \sum_{n=1}^{\infty} d(x_n, x_{n+1}) & = & \; d(x_0, x_1)\sum_{n=1}^{\infty} k^n + \frac{1}{1-k}\sum_{n=1}^{\infty} \epsilon^n \\
& & \leq & \; d(x_0, x_1)\frac{k}{1-k} + \frac{\epsilon}{(1-k)(1-\epsilon)} < \infty.
\end{aligned}
$$

Hence, we get $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Now we show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Let $m, n > 0$ with $m > n$, and so taking $m = n + p$, where $p \in \mathbb{N}$, we get

$$d(x_n, x_m) = d(x_n, x_{n+p}) \leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \ldots + s^p d(x_{n+p-1}, x_{n+p})$$

taking $n \to \infty$, we get $\lim_{n\to\infty} d(x_n, x_m) = 0$. Hence the sequence $\{x_n\}$ is a Cauchy sequence. As (X, d, s) is complete, then there exists $z \in X$ such that $x_n \to z$ and so, $\lim_{n\to\infty} d(x_n, z) = 0$.

Now we show that z is a fixed point of T and S. To see this, we have

$$
\begin{aligned}
d(z, Tz) & \leq s[d(z, x_{2n+2}) + d(x_{2n+2}, Tz)] \\
& \leq s[d(z, x_{2n+2}) + H(Sx_{2n}, Tz)] \\
& \leq s[d(z, x_{2n+2}) + H(Tz, Sx_{2n}] \\
& \leq s[d(z, x_{2n+2}) + ad(z, Tx_{2n}) + b\{d(z, Sx_{2n}) + d(Tx_{2n}, Tz) + cd(Tz, Sx_{2n}\}] \\
& \leq s[d(z, x_{2n+2}) + ad(z, x_{2n+1}) + b\{d(z, x_{2n+2}) + d(x_{2n+1}, Tz) + cd(Tz, x_{2n+2}\}] \\
& \leq s[d(z, x_{2n+2}) + ad(z, x_{2n+1}) + bd(z, x_{2n+2}) \\
& \quad + bs\{d(x_{2n+1}, z) + d(z, Tz)\} + bcs\{d(Tz, z) + d(z, x_{2n+2})\}].
\end{aligned}
$$

Letting $n \to \infty$ in above inequality, we obtain $d(z, Tz) \leq s^2(b + bc)d(z, Tz)$, then we have $d(z, Tz) = 0$ (since $b + bc < \frac{1}{s^2}$), i.e., $z \in Tz$. Hence $F(T) \neq \phi$, here $F(T)$ is the set of fixed points of T. Also,

$$
\begin{aligned}
H(Tz, Sz) & \leq ad(z, Tz) + b[d(z, Sz) + d(Tz, Tz) + cd(Tz, Sz)] \\
& \leq bd(z, Sz) + bcd(z, Sz) \\
& \leq (b + bc)H(Tz, Sz),
\end{aligned}
$$

thus $H(Tz, Sz) = 0$, i.e., $Tz = Sz$. Hence $F(S) \neq \phi$, where $F(S)$ denotes the collection of fixed points of S.
Now, we arrive at our final step which requires the following steps:

1. $F(T) = Tz$,
2. $Sx = Tx$ for all $x \in F(T)$,
3. $F(T) = F(S)$.

Firstly, let $x \in F(T)$, i.e. $x \in Tx$,

$$
\begin{aligned}
d(x, Tz) & \leq H(Tx, Tz) \\
& \leq H(Tx, Sz) \\
& \leq ad(x, Tz) + b[d(x, Sz) + d(Tz, Tx) + cd(Tx, Sz)] \\
& \leq ad(x, Tz) + bd(x, Tz) + bd(Tz, x) + bcd(x, Sz) \\
& \leq ad(x, Tz) + bd(x, Tz) + bd(x, Tz) + bcd(x, Tz),
\end{aligned}
$$

thus $d(x, Tz) = 0$, i.e., $x \in Tz$, and hence $Tx \subset Tz$ and $F(T) \subset Tz$.
Now, let $x \in Tz$. Then,

$$
\begin{aligned}
d(x, Tx) & \leq H(Tz, Tx) \\
& \leq H(Tx, Sz) \\
& \leq ad(x, Tz) + b[d(x, Sz) + d(Tz, Tx) + cd(Tx, Sz)] \\
& \leq ad(x, Tz) + bd(x, Sz) + bd(Tz, Tx) + bcd(Tx, Sz) \\
& \leq ad(x, Tz) + bd(x, Tz) + bd(x, Tx) + bcd(Tx, Tz) \\
& \leq ad(x, Tz) + bd(x, Tz) + bd(x, Tx) + bcd(Tx, x),
\end{aligned}
$$

thus $d(x, Tx) = 0$, i.e., $x \in Tx$. Hence, we get $Tz \subset Tx$, $Tz \subset F(T)$, and so $F(T) = Tz$.
Next, we show that $Tx = Sx$. For all $x \in F(T)$, we have

$$\begin{aligned} H(Tx, Sx) &\leq ad(x, Tx) + b[d(x, Sx) + d(Tx, Tx) + cd(Tx, Sx)] \\ &\leq ad(x, Tx) + bd(x, Sx) + bcd(x, Sx) \\ &\leq bd(x, Sx) + bcd(x, Sx) \\ &\leq (b + bc)d(x, Sx) \\ &\leq (b + c)H(Tx, Sx), \end{aligned}$$

thus $H(Tx, Sx) = 0$, i.e., $Tx = Sx$ for all $x \in F(T)$.

Now, we show that $F(T) = F(S)$. Let $x \in F(T)$, i.e. $x \in Tx$. By previous result $Tx = Sx$, we find $x \in Sx \Rightarrow x \in F(S)$, so we automatically get $F(T) \subset F(S)$.
Further, it remains to show that $F(S) \subset F(T)$. Let $x \in F(s)$, i.e., $x \in Sx$

$$\begin{aligned} d(x, Tx) &\leq H(Tx, Sx) \\ &\leq ad(x, Tx) + b[d(x, Sx) + d(Tx, Tx) + cd(Tx, Sx)] \\ &\leq ad(x, Tx) + bcd(Tx, Sx) \\ &\leq ad(x, Tx) + bcd(Tx, x) \\ &\leq (a + bc)d(Tx, x), \end{aligned}$$

thus $d(x, Tx) = 0$, i.e., $x \in Tx$. Hence, $F(T) = F(S) \neq \phi$ and $Sx = Tx = F(T)$ for all $x \in F(T)$.

At last, we have to show that the common fixed point is unique. Let z, v be two common fixed points of T and S such that $z \neq v$. Then,

$$\begin{aligned} d(z, v) &\leq H(Tz, Tv) \\ &\leq H(Tz, Sv) \\ &\leq ad(z, Tv) + b[d(z, Sv) + d(Tv, Tz) + cd(Tz, Sv)] \\ &\leq ad(z, Tv) + bd(z, Sv) + bd(v, z) + bcd(z, Sv) \\ \Rightarrow \quad (1 - b - bc)d(z, v) &\leq ad(z, Tv) + bd(z, Sv) \\ &\leq ad(z, Tv) + bd(z, Tv) \\ &\leq (a + b)d(z, v) \\ \Rightarrow \quad (1 - a - 2b - bc)d(z, v) &\leq 0 \\ \Rightarrow \quad d(z, v) = 0, \text{i.e., } z = v. & \end{aligned}$$

which completes the proof.

Remark 2.1. Our result (Theorem 2.1)*generalizes the result of* [12].

As a consequence of Theorem 2.1, we have the following corollary.

Corollary 2.1. *Let (X, d, s) be a complete b-metric space and let $S, T : X \to CB(X)$ be two multivalued mappings satisfying for all $x, y \in X$,*

$$H(Tx, Sy) \leq a\, d(x, Ty) + b\, [d(x, Sy) + c\, d(Tx, Sy)], \tag{2.2}$$

where $a, b, c \geq 0$ with $a + 2bs + bc < 1$ and $b + bc < 1/s^2$. Then S and T have a unique common fixed point in X.

Also, for different cases in the Theorem 2.1 we obtain the following particular results as corollaries.

Corollary 2.2 ([15])**.** *Let (X, d) be complete metric space and let $S, T : X \to CB(X)$ be mappings satisfying $H(Tx, Ty) \leq r\, d(x, Ty)$, for all $x, y \in X$ with $r \in [0, 1)$. Then $F(T) = F(S) \neq \phi$ and $Tx = Sx = F(T)$ for all $x \in F(T)$.*

Corollary 2.3. *Let (X, d) be complete metric space and let S, T be self mappings on X and if there exists $r \in [0, 1)$ such that $d(Tx, Sy) \leq r\, d(x, Ty)$ for all $x, y \in X$. Then S and T have a unique common fixed point.*

Corollary 2.4. *Let (X, d) be complete metric space and let $T : X \to C(X)$ be a mapping satisfying $H(Tx, T^2y) \leq r\, d(x, Ty)$, for all $x, y \in X$ with $r \in [0, 1)$.Then $F(T) \neq \phi$ and $Tx = F(T)$ for all $x \in F(T)$.*

Now, we present the following illustration in the support of Theorem 2.1.

Example 2.1. Let $X = [0, \infty)$ with b-metric defined by $d(x, y) = |x - y|^2$ with $s = 2$. Let $T, S : X \to CB(X)$ defined by

$$T(x) = \begin{cases} [\frac{1}{4}, \frac{1}{2}], \text{ if } x \in [2, \infty), \\ \{0\}, \text{ if } x \notin [2, \infty) \end{cases} \quad \text{and} \quad S(x) = \begin{cases} [\frac{1}{30}, \frac{1}{25}], \text{ if } x \in [2, \infty)], \\ \{0\}, \text{ if } x \notin [2, \infty). \end{cases}$$

Now, we consider the following case:

Case 1: If $x \notin [2, \infty), y \notin [2, \infty)$, then $T(x) = \{0\}, S(y) = \{0\}$, and the condition (2.1) is obviously true because $H(Tx, Sy) = 0$.

Case 2: If $x \in [2, \infty), y \in [2, \infty)$, we have $T(x) = [\frac{1}{4}, \frac{1}{2}], S(y) = [\frac{1}{30}, \frac{1}{25}]$, and

$$\begin{aligned} H(Tx, Sy) &= \max\left\{\sup_{a \in Tx} d(a, [\frac{1}{30}, \frac{1}{25}]), \sup_{b \in Sy} d(b, [\frac{1}{4}, \frac{1}{2}])\right\} \\ &= \max\left\{d(\frac{1}{2}, [\frac{1}{30}, \frac{1}{25}]), d(\frac{1}{30}, [\frac{1}{4}, \frac{1}{2}])\right\} \\ &= \max\left\{(\frac{1}{2} - \frac{1}{25})^2, (\frac{1}{30} - \frac{1}{4})^2\right\} = \frac{529}{625}. \end{aligned}$$

Also, $d(x, Ty) = \frac{9}{4}$, $d(x, Sy) = \frac{2401}{625}$, $d(Tx, Ty) = 0$ and $d(Tx, Sy) = \frac{441}{10000}$. Thus the condition (2.1) satisfies for $a = \frac{3}{5}, b = \frac{1}{20}$ and $c = 3$ with $a + 2bs + bc < 1$ and $b + bc < \frac{1}{s^2}$.

Case 3: If $x \in [2, \infty), y \notin [2, \infty)$, we get $T(x) = [\frac{1}{4}, \frac{1}{2}], S(y) = \{0\}$, $H(Tx, Sy) = \frac{1}{4}$, $d(x, Ty) = 4$, $d(x, Sy) = 4$, $d(Tx, Ty) = \frac{1}{16}$ and $d(Tx, Sy) = \frac{1}{16}$. Thus, the condition (2.1) is true for $a = \frac{3}{5}, b = \frac{1}{20}$ and $c = 3$ with $a + 2bs + bc < 1$ and $b + bc < \frac{1}{s^2}$.

Case 4: If $x \notin [2, \infty), y \in [2, \infty)$, we get $T(x) = \{0\}$, $S(y) = [\frac{1}{25}, \frac{1}{30}]$, $H(Tx, Sy) = \frac{1}{625}$, $d(x, Ty) = 0$, $d(x, Sy) = 0$ and $d(Tx, Ty) = \frac{1}{16}, d(Tx, Sy) = \frac{1}{900}$. Thus, the condition (2.1) holds for $a = \frac{3}{5}, b = \frac{1}{20}$ and $c = 3$ with $a + 2bs + bc < 1$ and $b + bc < \frac{1}{s^2}$.

Hence, in all cases we have the condition (2.1 of Theorem 2.1 is satisfied for $a = \frac{3}{5}, b = \frac{1}{20}$ and $c = 3$ with $a+2bs+bc < 1$ and $b + bc < \frac{1}{s^2}$, and $0 \in X$ is the only common fixed point of S and T.

Theorem 2.2. *Let (X, d_b, s) be a complete dislocated b-metric space and let $S, T : X \to CB(X)$ be two multivalued mappings satisfying for all $x, y \in X$,*

$$H_b(Tx, Sy) \leq a\, d_b(x, Ty) + b\, [d_b(x, Sy) + c\, d_b(Tx, Sy)], \tag{2.3}$$

where $a, b, c \geq 0$ with $2as + 3bs + bc < 1$ and $b + bc < 1/s^2$. Then S and T have a unique common fixed point in X.

Proof. Fix $x \in X$. Without loss of generality, we choose ϵ_0 such that $0 < \epsilon_0 < 1 - 2bs - bc$. Then, for $x \in X$, define $x_0 = x$ and let $x_1 \in Tx_0$. By Lemma 2.1, we may choose $x_2 \in Sx_0$ such that

$$\begin{aligned} d_b(x_1, x_2) &\leq H_b(Tx_0, Sx_0) + \epsilon_0 \\ &\leq a\, d_b(x_0, Tx_0) + b\, [d_b(x_0, Sx_0) + c\, d_b(Tx_0, Sx_0] + \epsilon_0 \\ &\leq ad_b(x_0, x_1) + bd_b(x_0, x_2) + bcd_b(x_1, x_2) + \epsilon_0 \\ &\leq a\, d_b(x_0, x_1) + bs\, [d_b(x_0, x_1) + d_b(x_1, x_2)] + bc\, d_b(x_1, x_2) + \epsilon_0 \\ \Rightarrow \quad d_b(x_1, x_2)(1 - 2bs - bc) &\leq ad_b(x_0, x_1) + bsd_b(x_0, x_1) + \epsilon_0 \\ \text{Thus,} \quad d_b(x_1, x_2) &\leq \frac{(a + bs)}{(1 - 2bs - bc)} d_b(x_0, x_1) + \epsilon_1, \quad \text{where } \epsilon_1 = \frac{\epsilon_0}{(1 - 2bs - bc)}. \end{aligned}$$

Similarly, there exists $x_3 \in Tx_2$ such that

$$\begin{aligned} d_b(x_2, x_3) &\leq H_b(Tx_2, Sx_0) + \frac{{\epsilon_0}^2}{(1 - 2bs - bc)} \\ &\leq ad_b(x_2, Tx_0) + b[d_b(x_2, Sx_0) + cd_b(Tx_2, Sx_0)] \\ &\quad + \frac{{\epsilon_0}^2}{(1 - 2bs - bc)} \\ &\leq ad_b(x_2, x_1) + bd_b(x_2, x_2) + bcd_b(x_3, x_2) + \frac{{\epsilon_0}^2}{(1 - 2bs - bc)} \end{aligned}$$

$$\leq \quad ad_b(x_2,x_1)+bs[d_b(x_2,x_3)+d_b(x_3,x_2)]+bcd_b(x_3,x_2)+\frac{{\epsilon_0}^2}{(1-2bs-bc)}$$

$$\Rightarrow \quad d_b(x_2,x_3)(1-2bs-bc) \leq ad_b(x_2,x_1)+bsd_b(x_2,x_1)+\frac{{\epsilon_0}^2}{(1-2bs-bc)}$$

$$\Rightarrow \quad d_b(x_2,x_3) \leq \frac{(a+bs)}{(1-2bs-bc)}d_b(x_1,x_2)+\frac{{\epsilon_0}^2}{(1-2bs-bc)^2}.$$

Continuing in this way, we obtain a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that

$$d_b(x_n,x_{n+1}) \leq \frac{(a+bs)}{(1-2bs-bc)}d_b(x_{n-1},x_n)+\epsilon_1^n \quad \forall n\in\mathbb{N}.$$

Let $k=\frac{(a+bs)}{(1-2bs-bc)}$, then for each $n\in\mathbb{N}$, we have

$$\begin{aligned}
d_b(x_n,x_{n+1}) &\leq k\,d_b(x_{n-1},x_n)+\epsilon_1^n\\
&\leq k[kd_b(x_{n-2},x_{n-1})+\epsilon_1^{n-1}]+\epsilon_1^n\\
&\vdots\\
&\leq k^n d_b(x_0,x_1)+\sum_{r=0}^{n-1}k^r\epsilon_1 n-r
\end{aligned}$$

$$\begin{aligned}
\text{which shows that, } \sum_{n=1}^{N}d_b(x_n,x_{n+1}) &\leq \sum_{n=1}^{N}k^n d_b(x_0,x_1)+\sum_{n=1}^{N}(\sum_{r=0}^{n-1}k^r\epsilon_1^{n-r})\\
&\leq \sum_{n=1}^{N}k^n d_b(x_0,x_1)+\sum_{n=1}^{N}\epsilon_1^n(\sum_{r=0}^{n-1}k^r)\\
&\leq d_b(x_0,x_1)\sum_{n=1}^{N}k^n+\sum_{n=1}^{N}\epsilon_1^n\cdot\frac{1-k^n}{1-k}\\
&< d_b(x_0,x_1)\sum_{n=1}^{N}k^n+\sum_{n=1}^{N}\epsilon_1^n\cdot\frac{1}{1-k}\\
\Rightarrow \quad \sum_{n=1}^{\infty}d_b(x_n,x_{n+1}) &= d_b(x_0,x_1)\sum_{n=1}^{\infty}k^n+\frac{1}{1-k}\sum_{n=1}^{\infty}\epsilon_1^n\\
&\leq d_b(x_0,x_1)\frac{k}{1-k}+\frac{\epsilon_1}{(1-k)(1-\epsilon_1)}<\infty.
\end{aligned}$$

Hence, we get $\lim_{n\to\infty}d_b(x_n,x_{n+1})=0$.

Now we show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Let $m,n>0$ with $m>n$, and so taking $m=n+p$, where $p\in\mathbb{N}$, we have

$$d_b(x_n,x_m)=d_b(x_n,x_{n+p})\leq sd_b(x_n,x_{n+1})+s^2d_b(x_{n+1},x_{n+2})+\ldots+s^pd_b(x_{n+p-1},x_{n+p}),$$

making $n\to\infty$, we get $\lim_{n\to\infty}d(x_n,x_m)=0$. Hence the sequence $\{x_n\}$ is a Cauchy sequence. As (X,d_b,s) is complete, then there exists $z\in X$ such that $x_n\to z$ and so, $\lim_{n\to\infty}d_b(x_n,z)=0$.

Now we show that z is a fixed point of T and S. To see this, we have

$$\begin{aligned}
d_b(z,Tz) &\leq s[d_b(z,x_{2n+2})+d_b(x_{2n+2},Tz)]\\
&\leq s[d_b(z,x_{2n+2})+H_b(Sx_{2n},Tz)]\\
&\leq s[d_b(z,x_{2n+2})+H_b(Tz,Sx_{2n}]\\
&\leq s[d_b(z,x_{2n+2})+ad_b(z,Tx_{2n})+b\{d_b(z,Sx_{2n})+cd_b(Tz,Sx_{2n}\}]\\
&\leq s[d_b(z,x_{2n+2})+ad_b(z,x_{2n+1})+b\{d_b(z,x_{2n+2})+cd_b(Tz,x_{2n+2}\}]\\
&\leq s[d_b(z,x_{2n+2})+ad_b(z,x_{2n+1})+bd_b(z,x_{2n+2})\\
&\quad +bcs\{d_b(Tz,z)+d_b(z,x_{2n+2})\}].
\end{aligned}$$

Letting $n\to\infty$ in the inequality above, we obtain $d_b(z,Tz)\leq s^2bcd_b(z,Tz)$, then we have $d_b(z,Tz)=0$ (since $b+bc<\frac{1}{s^2}$), i.e. $z\in Tz$. Hence $F(T)\neq\phi$, where $F(T)$ denotes the collection of fixed points of T . Also,

$$H_b(Tz,Sz) \leq ad_b(z,Tz)+b[d_b(z,Sz)+cd_b(Tz,Sz)]$$

$$
\begin{aligned}
&\leq bd_b(z,Sz)+bcd_b(z,Sz)\\
&\leq (b+bc)H_b(Tz,Sz),
\end{aligned}
$$

thus $H_b(Tz,Sz)=0$, i.e., $Tz=Sz$. Hence $F(S)\neq\phi$, here $F(S)$ is the set of fixed points of S .

We arrive at the proof of our final result which require the following steps:

1. $F(T)=Tz$,
2. $Sx=Tx$ for all $x\in F(T)$,
3. $F(T)=F(S)$.

Firstly, let $x\in F(T)$, i.e. $x\in Tx$,

$$
\begin{aligned}
d_b(x,Tz) &\leq H_b(Tx,Tz)\\
&\leq H_b(Tx,Sz)\\
&\leq ad_b(x,Tz)+b[d_b(x,Sz)+cd_b(Tx,Sz)]\\
&\leq ad_b(x,Tz)+bd_b(x,Tz)+bcd_b(x,Sz)\\
&\leq ad_b(x,Tz)+bd_b(x,Tz)++bcd_b(x,Tz),
\end{aligned}
$$

thus $d_b(x,Tz)=0$, i.e., $x\in Tz$. Hence $Tx\subset Tz$ and $F(T)\subset Tz$.

Now, let $x\in Tz$. We show that $x\in Tx$

$$
\begin{aligned}
d_b(x,Tx) &\leq H_b(Tz,Tx)\\
&\leq H_b(Sz,Tx)\\
&\leq H_b(Tx,Sz)\\
&\leq ad_b(x,Tz)+b[d_b(x,Sz)+cd_b(Tx,Sz)]\\
&\leq ad_b(x,Tz)+bd_b(x,Sz)++bcd_b(Tx,Sz)\\
&\leq ad_b(x,x)+bd_b(x,x)+bcd_b(Tx,x).
\end{aligned}
$$

Thus $d_b(x,Tx)=0$, i.e., $x\in Tx$. Hence $Tz\subset Tx$, $Tz\subset F(T)$, and so $F(T)=Tz$.

Next, we show that $Tx=Sx$. For all $x\in F(T)$, we get

$$
\begin{aligned}
H_b(Tx,Sx) &\leq ad_b(x,Tx)+b[d_b(x,Sx)+cd_b(Tx,Sx)]\\
&\leq ad_b(x,Tx)+bd_b(x,Sx)+bcd_b(x,Sx)\\
&\leq bd_b(x,Sx)+bcd_b(x,Sx)\\
&\leq (b+bc)d_b(x,Sx)\\
&\leq (b+bc)H_b(Tx,Sx),
\end{aligned}
$$

thus $H_b(Tx,Sx)]=0$, i.e., $Tx=Sx$ for all $x\in F(T)$.

Now, we show that $F(T)=F(S)$. Let $x\in F(T)$, i.e. $x\in Tx$. By previous result $Tx=Sx$, we get $x\in Sx\Rightarrow x\in F(S)$, so we automatically get $F(T)\subset F(S)$.

It remains to show that $F(S)\subset F(T)$. Let $x\in F(s)$, i.e. $x\in Sx$

$$
\begin{aligned}
d_b(x,Tx) &\leq H_b(Sx,Tx)\\
&\leq ad_b(x,Tx)+b[d_b(x,Sx)+cd_b(Tx,Sx)]\\
&\leq ad_b(x,Tx)+bd_b(x,Sx)+bcd_b(Tx,Sx)\\
&\leq ad_b(x,Tx)+bd_b(x,Tx)+bcd_b(Tx,x)\\
&\leq (a+b+bc)d_b(Tx,x),
\end{aligned}
$$

thus $d_b(x,Tx)=0$, i.e., $x\in Tx$, and we get $F(S)=F(T)$. Hence $F(T)=F(S)\neq\phi$ and $Sx=Tx=F(T)$ for all $x\in F(T)$.

At last, we have to show that the common fixed point is unique. Let z,v be two common fixed points of T and S such that $u\neq v$. Then,

$$
\begin{aligned}
d_b(u,v) &\leq H_b(Tz,Tv)\\
&\leq H_b(Tz,Sv)\\
&\leq ad_b(z,Tv)+b[d_b(z,Sv)+cd_b(Tz,Sv)]\\
&\leq ad_b(z,Tv)+bd_b(z,Sv)+bcd_b(z,v)
\end{aligned}
$$

$$\begin{aligned}\Rightarrow\qquad & (1-bc)d_b(z,v) \le ad_b(z,Tv) + bd_b(z,Sv)\\ & \le ad_b(z,Tv) + bd_b(z,Tv)\\ \Rightarrow\qquad & d_b(z,v) = 0,\ \text{i.e., } z = v.\end{aligned}$$

which completes the proof.

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