ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 52(2) (2022), 38-47**
(Dedicated to Professor D. S. Hooda on His 80^{th} Birth Anniversary Celebrations)

A STUDY ON THE GROWTH OF GENERALISED ITERATED ENTIRE FUNCTIONS - I

Ratan Kumar Dutta
Department of Mathematics
Rishi Bankim Chandra College
Naihati - 743165, West Bengal, India
Email: ratan_3128@yahoo.com

(Received: May 14, 2022; Revised : August 8, 2022; Accepted : September 24, 2022)

DOI: https://doi.org/10.58250/jnanabha.2022.52204

Abstract

In this paper, we study the growth of generalised iterated entire functions which improve and generalise some earlier results.

2020 Mathematical Sciences Classification: 30D35.
Keywords and Phrases: Entire functions, Growth, Iteration, Order, Hyper order.

1. Introduction

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function defined in the open complex plane C. Then $M(r, f) = \max_{|z|=r} |f(z)|$ is called maximum modulus function and

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} log^+|f(re^{i\theta})|d\theta$$

is Nevanlinna characteristic function of $f(z)$.

For any two transcendental entire functions $f(z)$ and $g(z)$, J. Cluni [3] proved that $\lim_{r\to\infty} \frac{T(r,f\circ g)}{T(r,f)} = \infty$ and $\lim_{r\to\infty} \frac{T(r,f\circ g)}{T(r,g)} = \infty$. Later on Singh [14] investigated some comparative growth of $logT(r, f_o g)$ and $T(r, f)$. Further in [14] he raised the problem of investing the comparative growth of $logT(r, f_o g)$ and $T(r, g)$. However some results on the comparative growth of $logT(r, f_o g)$ and $T(r, g)$ are proved in [9].

Recently Banerjee and Dutta [1], and Dutta [4, 5, 6] made close investigation on comparative growth properties of iterated entire functions to generalise some earlier results. Also Dutta [7], investigated comparative growth property of generalise iterated entire functions.

In order to study the growth properties of generalised iterated entire functions, it is very much necessary to mention some relevant notations and definitions which refer to [8].

Definition 1.1. *The order ρ_f and lower order λ_f of a meromorphic function $f(z)$ is defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

If $f(z)$ is entire then

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log\log M(r, f)}{\log r}.$$

Definition 1.2. *The hyper order $\overline{\rho_f}$ and hyper lower order $\overline{\lambda}_f$ of a meromorphic function $f(z)$ is defined as*

$$\overline{\rho_f} = \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}$$

and

$$\overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

If $f(z)$ *is entire then*

$$\overline{\rho_f} = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}$$

and

$$\overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

Definition 1.3. *A function* $\lambda_f(r)$ *is called a lower proximate order of a meromorphic function* $f(z)$ *if*

(i) $\lambda_f(r)$ *is nonnegative and continuous for* $r \geq r_0$*, say;*

(ii) $\lambda_f(r)$ *is differentiable for* $r \geq r_0$ *except possibly at isolated points at which* $\lambda_f'(r-0)$ *and* $\lambda_f'(r+0)$ *exist;*

(iii) $\lim_{r\to\infty} \lambda_f(r) = \lambda_f < \infty$;

(iv) $\lim_{r\to\infty} r\lambda_f'(r)\log r = 0$*; and*

(v) $\liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f(r)}} = 1$.

In 2011, Banerjee and Dutta [1] proved the following results:

Theorem A ([1]). *Let* f *and* g *be two non-constant entire functions such that* $0 < \lambda_f \leq \rho_f < \infty$ *and* $0 < \lambda_g \leq \rho_g < \infty$. *Then for* $k = 0, 1, 2, 3, ...$

$$\frac{\overline{\lambda}_g}{\rho_g} \leq \liminf_{r\to\infty} \frac{\log^{[n+1]} T(r,f_n)}{\log\ T(r,g^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[n+1]} T(r,f_n)}{\log\ T(r,g^{(k)})} \leq \frac{\overline{\rho_g}}{\lambda_g},$$

when n *is even and*

$$\frac{\overline{\lambda}_f}{\rho_f} \leq \liminf_{r\to\infty} \frac{\log^{[n+1]} T(r,f_n)}{\log\ T(r,f^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[n+1]} T(r,f_n)}{\log\ T(r,f^{(k)})} \leq \frac{\overline{\rho_f}}{\lambda_f},$$

when n *is odd, where* $f^{(k)}$ *denote the* k*-th derivative of* f.

Theorem B ([1]). *Let* f *and* g *be two non-constant entire functions such that* $0 < \lambda_f \leq \rho_f < \infty$ *and* $0 < \lambda_g \leq \rho_g < \infty$. *Then*

$$(i)\ \ \frac{\lambda_g}{\rho_g} \leq \liminf_{r\to\infty} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,g)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,g)} \leq \frac{\rho_g}{\lambda_g},$$

when n *is even and*

$$(ii)\ \ \frac{\lambda_f}{\rho_f} \leq \liminf_{r\to\infty} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,f)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,f)} \leq \frac{\rho_f}{\lambda_f},$$

when n *is odd.*

Theorem C ([1]). *Let* f *and* g *be two entire functions such that* $0 < \lambda_f \leq \rho_f < \infty$ *and* $0 < \lambda_g \leq \rho_g < \infty$. *Then for* $k = 0, 1, 2, 3, ...$

$$(i)\ \ \frac{\lambda_g}{\rho_f} \leq \liminf_{r\to\infty} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,f^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,f^{(k)})} \leq \frac{\rho_g}{\lambda_f},$$

when n *is even and*

$$(ii)\ \ \frac{\lambda_f}{\rho_g} \leq \liminf_{r\to\infty} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,g^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,g^{(k)})} \leq \frac{\rho_f}{\lambda_g},$$

when n *is odd.*

In this paper we consider three entire functions $f(z)$, $g(z)$ and $h(z)$ and following Banerjee and Mandal [2] form the iterations of $f(z)$ with respect to $g(z)$ and $h(z)$ [define below] and generalise the Theorem A, Theorem B and Theorem C of Banerjee and Dutta [1] in this direction.

Definition 1.4 ([2]). *Let* $f(z)$, $g(z)$ *and* $h(z)$ *be three entire functions defined in the open complex plane. Then the generalise iterations of* $f(z)$ *with respect to* $g(z)$ *and* $h(z)$ *are defined as follows:*

$$\begin{aligned} f_1(z) &= f(z), \\ f_2(z) &= f(g(z)) = f(g_1(z)), \end{aligned}$$

$$
\begin{aligned}
f_3(z) &= f(g(h(z))) = f(g(h_1(z))) = f(g_2(z)), \\
f_4(z) &= f(g(h(f(z)))) = f(g(h_2(z))) = f(g_3(z)) \\
&\vdots \\
f_n(z) &= f(g(h(f..(f(z) \text{ or } g(z) \text{ or } h(z) \text{ according as } n = 3m-2 \text{ or } 3m-1 \\
&\quad \text{or } 3m)...))) \\
&= f(g_{n-1}(z)) = f(g(h_{n-2}(z))).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
g_1(z) &= g(z), \\
g_2(z) &= g(h(z)) = g(h_1(z)), \\
g_3(z) &= g(h(f(z))) = g(h(f_1(z))) = g(h_2(z)), \\
g_4(z) &= g(h(f(g(z)))) = g(h(f_2(z))) = g(h_3(z)), \\
&\vdots \\
g_n(z) &= g(h(f(g...(g(z) \text{ or } h(z) \text{ or } f(z) \text{ according as } n = 3m-2 \text{ or } 3m-1 \\
&\quad \text{or } 3m)...))) \\
&= g(h_{n-1}(z)) = g(h(f_{n-2}(z))),
\end{aligned}
$$

and

$$
\begin{aligned}
h_1(z) &= h(z), \\
h_2(z) &= h(f(z)) = h(f_1(z)), \\
h_3(z) &= h(f(g(z))) = h(f(g_1(z))) = h(f_2(z)), \\
h_4(z) &= h(f(g(h(z)))) = h(f(g_2(z))) = h(f_3(z)), \\
&\vdots \\
h_n(z) &= h(f(g(h...(h(z) \text{ or } f(z) \text{ or } g(z) \text{ according as } n = 3m-2 \text{ or } 3m-1 \\
&\quad \text{or } 3m)...))) \\
&= h(f_{n-1}(z)) = h(f(g_{n-2}(z))).
\end{aligned}
$$

Clearly all $f_n(z)$, $g_n(z)$ and $h_n(z)$ are entire functions.

Notation 1.1 ([13]). *Let* $log^{[0]}x = x$, $exp^{[0]}x = x$ *and for positive integer m,* $log^{[m]}x = log(log^{[m-1]}x)$, $exp^{[m]}x = exp(exp^{[m-1]}x)$.

Throughout we assume $f(z), g(z), h(z)$ etc. are non constant entire functions having respective orders ρ_f, ρ_g, ρ_h and respective lower orders $\lambda_f, \lambda_g, \lambda_h$. Also we do not explain the standard notations and definitions of the theory of entire and meromorphic functions because those are available in [8].

2. Main results

The following lemmas will be needed in the sequel.

Lemma 2.1 ([8]). *Let* $f(z)$ *be an entire function. For* $0 \leq r < R < \infty$, *we have*

$$T(r, f) \leq \log^+ M(r, f) \leq \frac{R + r}{R - r} T(R, f).$$

Lemma 2.2 ([12]). *Let* $f(z)$ *and* $g(z)$ *be two entire functions. Then we have*

$$T(r, f(g)) \geq \frac{1}{3} \log M\left(\frac{1}{8} M\left(\frac{r}{4}, g\right) + O(1), f\right).$$

Lemma 2.3 ([10]). *Let* f *be an entire function. Then for* $k > 2$,

$$\liminf_{r\to\infty} \frac{\log^{[k-1]} M(r, f)}{\log^{[k-2]} T(r, f)} = 1.$$

Lemma 2.4 ([11]). *Let $f(z)$ be a meromorphic function. Then for $\delta(>0)$ the function $r^{\lambda_f+\delta-\lambda_f(r)}$ is an increasing function of r.*

Lemma 2.5 ([7]). *Let $f(z)$, $g(z)$ and $h(z)$ be three non-constant entire functions of finite order and nonzero lower order. Then for any ε $(0<\varepsilon<min\{\lambda_f,\lambda_g,\lambda_h\})$,*

$$\log^{[n-1]} T(r,f_n) \le \begin{cases} (\rho_g+\varepsilon)\log M(r,h)+O(1) & \text{when } n=3m, \\ (\rho_h+\varepsilon)\log M(r,f)+O(1) & \text{when } n=3m+1, \\ (\rho_f+\varepsilon)\log M(r,g)+O(1) & \text{when } n=3m+2 \end{cases}$$

and

$$\log^{[n-1]} T(r,f_n) \ge \begin{cases} (\lambda_g-\varepsilon)\log M\left(\frac{r}{4^{n-1}},h\right)+O(1) & \text{when } n=3m, \\ (\lambda_h-\varepsilon)\log M\left(\frac{r}{4^{n-1}},f\right)+O(1) & \text{when } n=3m+1, \\ (\lambda_f-\varepsilon)\log M\left(\frac{r}{4^{n-1}},g\right)+O(1) & \text{when } n=3m+2. \end{cases}$$

Theorem 2.1. *Let $f(z)$, $g(z)$ and $h(z)$ be three non-constant entire functions of finite order and nonzero lower order. Then for $k=0,1,2,3,...$*

$$\frac{\overline{\lambda}_h}{\rho_h} \le \liminf_{r\to\infty} \frac{\log^{[n+1]} T(r,f_n)}{\log\ T(r,h^{(k)})} \le \limsup_{r\to\infty} \frac{\log^{[n+1]} T(r,f_n)}{\log\ T(r,h^{(k)})} \le \frac{\overline{\rho_h}}{\lambda_h},$$

when $n=3m$
and

$$\frac{\overline{\lambda}_f}{\rho_f} \le \liminf_{r\to\infty} \frac{\log^{[n+1]} T(r,f_n)}{\log\ T(r,f^{(k)})} \le \limsup_{r\to\infty} \frac{\log^{[n+1]} T(r,f_n)}{\log\ T(r,f^{(k)})} \le \frac{\overline{\rho_f}}{\lambda_f},$$

when $n=3m+1$. Also when $n=3m+2$,

$$\frac{\overline{\lambda}_g}{\rho_g} \le \liminf_{r\to\infty} \frac{\log^{[n+1]} T(r,f_n)}{\log\ T(r,g^{(k)})} \le \limsup_{r\to\infty} \frac{\log^{[n+1]} T(r,f_n)}{\log\ T(r,g^{(k)})} \le \frac{\overline{\rho_g}}{\lambda_g},$$

where $f^{(k)}, g^{(k)}, h^{(k)}$ denote the k-th derivative of f, g, h respectively and m is positive integer.

Proof. First suppose that $n=3m$, then for given $\varepsilon(0<\varepsilon<min\{\lambda_f,\lambda_g,\lambda_h\})$, we get from Lemma 2.5, for all large values of r,

$$\begin{aligned} \log^{[n-1]}\ T(r,f_n) &\ge (\lambda_g-\varepsilon)\log M\left(\frac{r}{4^{n-1}},h\right)+O(1) \\ &\ge (\lambda_g-\varepsilon)T\left(\frac{r}{4^{n-1}},h\right)+O(1), \\ \text{that is, } \log^{[n]}\ T(r,f_n) &\ge \log T\left(\frac{r}{4^{n-1}},h\right)+O(1). \\ \text{So, } \log^{[n+1]}\ T(r,f_n) &\ge \log^{[2]} T\left(\frac{r}{4^{n-1}},h\right)+O(1). \end{aligned}$$

Therefore for all large values of r,

$$\frac{\log^{[n+1]} T(r,f_n)}{\log\ T(r,h^{(k)})} \ge \frac{\log^{[2]} T\left(\frac{r}{4^{n-1}},h\right)}{\log\ \frac{r}{4^{n-1}}}.\frac{\log\ \frac{r}{4^{n-1}}}{\log\ T(r,h^{(k)})}+o(1). \tag{2.1}$$

Since

$$\limsup_{r\to\infty} \frac{\log T(r,h^{(k)})}{\log r} = \rho_h,$$

so for all large values of r and arbitrary $\varepsilon>0$, we have

$$\log T(r,h^{(k)}) < (\rho_h+\varepsilon)\log r. \tag{2.2}$$

Since $\varepsilon>0$ is arbitrary, so from (2.1) and (2.2) we have

$$\begin{aligned} \liminf_{r\to\infty} \frac{\log^{[n+1]} T(r,f_n)}{\log\ T(r,h^{(k)})} &\ge \liminf_{r\to\infty} \frac{\log^{[2]} T\left(\frac{r}{4^{n-1}},h\right)}{\log\ \frac{r}{4^{n-1}}}.\left(\frac{\log r-\log 4^{n-1}}{\rho_h \log\ r}\right) \\ &\ge \frac{\overline{\lambda}_h}{\rho_h}. \end{aligned} \tag{2.3}$$

Again from Lemma 2.5, for all large values of r,

$$\log^{[n-1]} T(r, f_n) \leq (\rho_f + \varepsilon) \log M(r, h) + O(1)$$

$$\text{i.e. } \frac{\log^{[n+1]} T(r, f_n)}{\log T(r, h^{(k)})} \leq \frac{\log^{[3]} M(r, h)}{\log T(r, h^{(k)})} + o(1). \tag{2.4}$$

Therefore

$$\liminf_{r\to\infty} \frac{\log T(r, h^{(k)})}{\log r} = \lambda_h,$$

so for all large values of r and arbitrary $\varepsilon(0 < \varepsilon < \lambda_h)$, we have

$$\log T(r, h^{(k)}) > (\lambda_h - \varepsilon) \log r. \tag{2.5}$$

Since $\varepsilon > 0$ is arbitrary, so from (2.4) and (2.5) we have

$$\limsup_{r\to\infty} \frac{\log^{[n+1]} T(r, f_n)}{\log T(r, h^{(k)})} \leq \frac{\overline{\rho_h}}{\lambda_h}. \tag{2.6}$$

Combining (2.3) and (2.6) we obtain the first part of the theorem.

Similarly, for $n = 3m + 1$ and $3m + 2$ we get the other results.

This proves the theorem.

Example 2.1. Let $f(z) = g(z) = h(z) = \exp z$. Then $\lambda_f = \lambda_g = \lambda_h = \rho_f = \rho_g = \rho_h = 1$ and $\overline{\lambda_f} = \overline{\lambda_g} = \overline{\lambda_h} = \overline{\rho_f} = \overline{\rho_g} = \overline{\rho_h} = 0$.

Now

$$f_n(z) = \exp^n z.$$

Therefore,

$$T(r, f_n) \leq \log M(r, f_n) = \exp^{[n-1]} r.$$

So,

$$\log^{[n+1]} T(r, f_n) \leq \log^{[2]} r.$$

Now

$$\log T(r, f) = logT(r, g) = logT(r, h) = \log r - \log \pi.$$

Therefore

$$\frac{\log^{[n+1]} T(r, f_n)}{\log T(r, h)} \leq \frac{\log^{[2]} r}{\log r - \log \pi} \to 0 \text{ as } r \to \infty \text{ and } n = 3m,$$

$$\frac{\log^{[n]} T(r, f_n)}{\log T(r, f)} \leq \frac{\log^{[2]} r}{\log \log r} \to 0 \text{ as } r \to \infty \text{ and } n = 3m + 1,$$

$$\frac{\log^{[n]} T(r, f_n)}{\log T(r, g)} \leq \frac{\log^{[2]} r}{\log r - \log \pi} \to 0 \text{ as } r \to \infty \text{ and } n = 3m + 2.$$

Theorem 2.2. *Let $f(z)$, $g(z)$ and $h(z)$ be three non-constant entire functions of finite order and nonzero lower order. Then*

$$(i)\ \frac{\lambda_h}{\rho_h} \leq \liminf_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log T(r, h)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log T(r, h)} \leq \frac{\rho_h}{\lambda_h},$$

when $n = 3m$
and

$$(ii)\ \frac{\lambda_f}{\rho_f} \leq \liminf_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log T(r, f)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log T(r, f)} \leq \frac{\rho_f}{\lambda_f},$$

when $n = 3m + 1$.

Also when $n = 3m + 2$,

$$(iii)\ \frac{\lambda_g}{\rho_g} \leq \liminf_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log T(r, g)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log T(r, g)} \leq \frac{\rho_g}{\lambda_g}.$$

Proof. First suppose that $n = 3m$, then for given $\varepsilon(0 < \varepsilon < min\{\lambda_f, \lambda_g, \lambda_h\})$, we get from Lemma 2.5, for all large values of r

$$\begin{aligned} \log^{[n-1]} T(r, f_n) &\leq (\rho_g + \varepsilon) \log M(r, h) + O(1) \\ \text{i.e.} \quad \log^{[n]} T(r, f_n) &\leq \log^{[2]} M(r, h) + O(1) \\ \text{i.e.} \quad \frac{\log^{[n]} T(r, f_n)}{\log T(r, h)} &\leq \frac{\log^{[2]} M(r, h)}{\log T(r, h)} + o(1), \end{aligned} \tag{2.7}$$

$$\text{i.e.} \liminf_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log T(r, h)} \leq 1 \quad \text{[by Lemma 2.3].} \tag{2.8}$$

Also,

$$\begin{aligned} \log^{[n-1]} T(r, f_n) &\geq (\lambda_f - \varepsilon) \log M\left(\frac{r}{4^{n-1}}, h\right) + O(1), \\ \text{i.e.} \quad \log^{[n]} T(r, f_n) &\geq \log^{[2]} M\left(\frac{r}{4^{n-1}}, h\right) + O(1). \end{aligned}$$

Therefore,

$$\frac{\log^{[n]} T(r, f_n)}{\log T(r, h)} \geq \frac{\log T\left(\frac{r}{4^{n-1}}, h\right)}{\log \frac{r}{4^{n-1}}} \cdot \left(\frac{\log r - \log 4^{n-1}}{\rho_h \log r}\right) + o(1)$$

$$\text{i.e.} \quad \liminf_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log T(r, h)} \geq \frac{\lambda_h}{\rho_h}. \tag{2.9}$$

Also from (2.7), we get for all large values of r,

$$\frac{\log^{[n]} T(r, f_n)}{\log T(r, h)} \leq \frac{\log^{[2]} M(r, h)}{\log r} \frac{\log r}{\log T(r, h)} + o(1).$$

Therefore

$$\limsup_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log T(r, h)} \leq \frac{\rho_h}{\lambda_h}. \tag{2.10}$$

Again from Lemma 2.5,

$$\begin{aligned} \log^{[n-1]} T(r, f_n) &\geq (\lambda_f - \varepsilon) \log M\left(\frac{r}{4^{n-1}}, h\right) + O(1) \\ \text{i.e.} \quad \log^{[n]} T(r, f_n) &\geq \log^{[2]} M\left(\frac{r}{4^{n-1}}, h\right) + O(1). \end{aligned} \tag{2.11}$$

Let $0 < \varepsilon < \min\{1, \lambda_f, \lambda_g, \lambda_h\}$. Since

$$\liminf_{r\to\infty} \frac{T(r, h)}{r^{\lambda_h(r)}} = 1,$$

there is a sequence of values of r tending to infinity, for which

$$T(r, h) < (1 + \varepsilon) r^{\lambda_h(r)} \tag{2.12}$$

and for all large value of r,

$$T(r, h) > (1 - \varepsilon) r^{\lambda_h(r)}. \tag{2.13}$$

From (2.13), we obtain for all large values of r and $\delta > 0$, also for $\varepsilon(0 < \varepsilon < 1)$

$$\begin{aligned} \log M\left(\frac{r}{4^{n-1}}, h\right) &> (1 - \varepsilon) \frac{\left(\frac{r}{4^{n-1}}\right)^{\lambda_h + \delta}}{\left(\frac{r}{4^{n-1}}\right)^{\lambda_h + \delta - \lambda_h\left(\frac{r}{4^{n-1}}\right)}} \\ &\geq \frac{1 - \varepsilon}{(4^{n-1})^{\lambda_h + \delta}} r^{\lambda_h(r)}, \end{aligned}$$

because $r^{\lambda_h + \delta - \lambda_h(r)}$ is an increasing function of r.

So by (2.12), we get for a sequence of value of r tending to infinity,

$$\begin{aligned} \log M\left(\frac{r}{4^{n-1}}, h\right) &\geq \frac{1 - \varepsilon}{1 + \varepsilon} \frac{1}{(4^{n-1})^{\lambda_h + \delta}} T(r, h) \\ \text{i.e.} \quad \log^{[2]} M\left(\frac{r}{4^{n-1}}, h\right) &\geq \log T(r, h) + O(1). \end{aligned} \tag{2.14}$$

Now from (2.11) and (2.14),

$$\limsup_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log T(r, h)} \geq 1. \tag{2.15}$$

So the theorem follows from (2.8), (2.9), (2.10) and (2.15), when n is even.

Similarly, when for $n = 3m + 1$ and $3m + 2$ we get the results (ii) and (iii) respectively.

Corollary 2.1. *Using the hypothesis of Theorem* 2.3 *if* f, g *and* h *are of regular growth then*

$$\lim_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log\ T(r, f)} = \lim_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log\ T(r, g)} = \lim_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log\ T(r, h)} = 1.$$

Remark 2.1. The conditions $\lambda_f, \lambda_g, \lambda_h > 0$ and $\rho_f, \rho_g, \rho_h < \infty$ are necessary for Theorem 2.3 and Corollary 2.1, which are shown by the following examples.

Example 2.2. Let $f(z) = z, g(z) = h(z) = \exp z$. Then $\lambda_f = \rho_f = 0$ and $0 < \lambda_g = \lambda_h = \rho_g = \rho_h < \infty$.
Now when $n = 3m$,

$$f_n(z) = \exp^{[\frac{2n}{3}]} z.$$

Therefore,

$$T(r, f_n) \leq \log M(r, f_n) = \exp^{[\frac{2n}{3}-1]} r.$$

So,

$$\begin{aligned} \log^{[n]} T(r, f_n) &\leq \log^{[n]}(\exp^{[\frac{2n}{3}-1]} r) \\ &= \log^{[n-\frac{2n}{3}+1]} r \\ &= \log^{[\frac{n}{3}+1]} r. \end{aligned}$$

Also when $n = 3m + 1$,

$$f_n(z) = \exp^{[\frac{2(n-1)}{3}]} z.$$

Therefore,

$$T(r, f_n) \leq \log M(r, f_n) = \exp^{[\frac{2(n-1)}{3}-1]} r.$$

So,

$$\begin{aligned} \log^{[n]} T(r, f_n) &\leq \log^{[n]}(\exp^{[\frac{2(n-1)}{3}-1]} r) \\ &= \log^{[n-\frac{2(n-1)}{3}+1]} r \\ &= \log^{[\frac{n+2}{3}+1]} r. \end{aligned}$$

If $n = 3m + 2$,

$$f_n(z) = \exp^{[\frac{2n-1}{3}]} z.$$

Therefore,

$$T(r, f_n) \leq \log M(r, f_n) = \exp^{[\frac{2n-1}{3}-1]} r.$$

So,

$$\begin{aligned} \log^{[n]} T(r, f_n) &\leq \log^{[n]}(\exp^{[\frac{2n-1}{3}-1]} r) \\ &= \log^{[n-\frac{2n-1}{3}+1]} r \\ &= \log^{[\frac{n+1}{3}+1]} r. \end{aligned}$$

Now

$$\log T(r, f) = \log\log r, \quad \log T(r, g) = \log r - \log\pi \ \text{ and } \ \log T(r, h) = \log r - \log\pi.$$

Therefore

$$\begin{aligned} \frac{\log^{[n]} T(r, f_n)}{\log T(r, h)} &\leq \frac{\log^{[\frac{n}{3}+1]} r}{\log r - \log\pi} \to 0 \ \text{ as } \ r \to \infty \text{ and } n = 3m, \\ \frac{\log^{[n]} T(r, f_n)}{\log T(r, f)} &\leq \frac{\log^{[\frac{n+2}{3}+1]} r}{\log\log r} \to 0 \ \text{ as } \ r \to \infty \text{ and } n = 3m + 1, \\ \frac{\log^{[n]} T(r, f_n)}{\log T(r, g)} &\leq \frac{\log^{[\frac{n+1}{3}+1]} r}{\log r - \log\pi} \to 0 \ \text{ as } \ r \to \infty \text{ and } n = 3m + 2. \end{aligned}$$

Example 2.3. Let $f(z) = \exp^{[2]} z, g(z) = h(z) = \exp z$. Then $\lambda_f = \rho_f = \infty, \ \ \lambda_g = \ \lambda_h = \rho_g = \rho_h = 1$.

Now when $n = 3m$,

$$f_n(z) = \exp^{[\frac{4n}{3}]} z.$$

Therefore

$$\begin{aligned} 3T(2r, f_n) &\geq \log M(r, f_n) = \exp^{[\frac{4n}{3}-1]} r \\ \text{i.e.} \quad T(r, f_n) &\geq \frac{1}{3}\exp^{[\frac{4n}{3}-1]} \frac{r}{2} \\ \therefore \quad \log^{[n]} T(r, f_n) &\geq \log^{[n]}(\exp^{[\frac{4n}{3}-1]} \frac{r}{2}) + o(1) \\ &= \exp^{[\frac{n}{3}-1]} \frac{r}{2} + o(1). \end{aligned}$$

Also when $n = 3m + 1$,

$$f_n(z) = \exp^{[\frac{4n+2}{3}]} z.$$

Therefore

$$\begin{aligned} 3T(2r, f_n) &\geq \log M(r, f_n) = \exp^{[\frac{4n+2}{3}-1]} r \\ \text{i.e.} \quad T(r, f_n) &\geq \frac{1}{3}\exp^{[\frac{4n-1}{3}]} \frac{r}{2} \\ \therefore \quad \log^{[n]} T(r, f_n) &\geq \log^{[n]}(\exp^{[\frac{4n-1}{3}]} \frac{r}{2}) + o(1) \\ &= \exp^{[\frac{n-1}{3}]} \frac{r}{2} + o(1). \end{aligned}$$

If $n = 3m + 2$,

$$f_n(z) = \exp^{[\frac{4n+1}{3}]} z.$$

Therefore

$$\begin{aligned} 3T(2r, f_n) &\geq \log M(r, f_n) = \exp^{[\frac{4n+1}{3}-1]} r \\ \text{i.e.} \quad T(r, f_n) &\geq \frac{1}{3}\exp^{[\frac{4n-2}{3}]} \frac{r}{2} \\ \therefore \quad \log^{[n]} T(r, f_n) &\geq \log^{[n]}(\exp^{[\frac{4n-2}{3}]} \frac{r}{2}) + o(1) \\ &= \exp^{[\frac{n-2}{3}]} \frac{r}{2} + o(1). \end{aligned}$$

If

$$T(r, f) \leq e^r \text{ and } T(r, g) = \ T(r, h) = \frac{r}{\pi}.$$

Therefore

$$\begin{aligned} \frac{\log^{[n]} T(r, f_n)}{\log T(r, h)} &\geq \frac{\exp^{[\frac{n}{3}-1]} \frac{r}{2} + o(1)}{\log r - \log \pi} \to \infty \ \text{ as } \ r \to \infty \text{ and } n = 3m, \\ \frac{\log^{[n]} T(r, f_n)}{\log T(r, f)} &\geq \frac{\exp^{[\frac{n-1}{3}]} \frac{r}{2} + o(1)}{r} \to \infty \ \text{ as } \ r \to \infty \text{ and } n = 3m + 1, \\ \frac{\log^{[n]} T(r, f_n)}{\log T(r, g)} &\geq \frac{\exp^{[\frac{n-2}{3}]} \frac{r}{2} + o(1)}{\log r - \log \pi} \to \infty \ \text{ as } \ r \to \infty \text{ and } n = 3m + 2. \end{aligned}$$

Theorem 2.3. *Let f, g and h be three entire functions such that $0 < \lambda_f \leq \rho_f < \infty, 0 < \lambda_g \leq \rho_g < \infty$ and $0 < \lambda_h \leq \rho_h < \infty$. Then for $k = 0, 1, 2, 3, ...$*

$$(i) \ \ \frac{\lambda_h}{\rho_f} \leq \liminf_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log \ T(r, f^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log \ T(r, f^{(k)})} \leq \frac{\rho_h}{\lambda_f},$$

$$(ii) \ \ \frac{\lambda_h}{\rho_g} \leq \liminf_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log \ T(r, g^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log \ T(r, g^{(k)})} \leq \frac{\rho_h}{\lambda_g},$$

when $n = 3m$ and

$$(iii) \ \ \frac{\lambda_f}{\rho_g} \leq \liminf_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log \ T(r, g^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[n]} T(r, f_n)}{\log \ T(r, g^{(k)})} \leq \frac{\rho_f}{\lambda_g},$$

$$(iv)\ \frac{\lambda_f}{\rho_h} \leq \liminf_{r\to\infty} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,h^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,h^{(k)})} \leq \frac{\rho_f}{\lambda_h},$$

when $n = 3m + 1$.

Also when $n = 3m + 2$,

$$(v)\ \frac{\lambda_g}{\rho_f} \leq \liminf_{r\to\infty} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,f^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,f^{(k)})} \leq \frac{\rho_g}{\lambda_f},$$

$$(vi)\ \frac{\lambda_g}{\rho_h} \leq \liminf_{r\to\infty} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,h^{(k)})} \leq \limsup_{r\to\infty} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,h^{(k)})} \leq \frac{\rho_g}{\lambda_h}.$$

Proof. First suppose that $n = 3m$, then for given ε $(0 < \varepsilon < min\{\lambda_f, \lambda_g, \lambda_h\})$, we have from Lemma 2.5, for all large values of r,

$$\begin{aligned} \log^{[n-1]}\ T(r,f_n) &\leq (\rho_g + \varepsilon)\log M(r,h) + O(1) \\ \text{i.e.}\quad \log^{[n]}\ T(r,f_n) &\leq \log^{[2]} M(r,h) + O(1). \end{aligned}$$

Also we know that

$$\liminf_{r\to\infty} \frac{\log T(r,f^{(k)})}{\log r} = \lambda_f.$$

Now

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,f^{(k)})} &\leq \limsup_{r\to\infty} \frac{\log^{[2]} M(r,h)}{\log\ T(r,f^{(k)})} \\ &\leq \limsup_{r\to\infty} \left[\frac{\log^{[2]} M(r,h)}{\log r}.\frac{\log r}{\log T(r,f^{(k)})}\right] \\ &= \frac{\rho_h}{\lambda_f}. \end{aligned} \tag{2.16}$$

Again from Lemma 2.5, we have for all large values of r,

$$\begin{aligned} \log^{[n-1]}\ T(r,f_n) &\geq (\lambda_g - \varepsilon)\log M\left(\frac{r}{4^{n-1}},h\right) + O(1) \\ &\geq (\lambda_g - \varepsilon)\left(\frac{r}{4^{n-1}}\right)^{\lambda_h-\varepsilon} + O(1) \\ \text{i.e.,}\quad \log^{[n]}\ T(r,f_n) &\geq (\lambda_h - \varepsilon)\log\ r + O(1). \end{aligned}$$

Also

$$\log\ T(r,f^{(k)}) < (\rho_f + \varepsilon)\log\ r.$$

Therefore,

$$\frac{\log^{[n]} T(r,f_n)}{\log\ T(r,f^{(k)})} \geq \frac{(\lambda_h - \varepsilon)\log\ r + O(1)}{(\rho_f + \varepsilon)\log\ r}.$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\liminf_{r\to\infty} \frac{\log^{[n]} T(r,f_n)}{\log\ T(r,f^{(k)})} \geq \frac{\lambda_h}{\rho_f}. \tag{2.17}$$

Therefore from (2.16) and (2.17), we have the result (i) for $n = 3m$.

Similarly we get other results.

This proves the Theorem.

3. Conclusion

Some growth properties of iterated entire functions with non zero finite iterated order with respect to their Nevanlinna characteristic function have been discussed in this article. Similar study may be done with respect to maximum terms. These results may be applied to the growth of entire solution of complex differential equations. These applications are open for further research works.

Acknowledgement. The author is grateful to the referees for their valuable suggestions to improve the paper in present form. The author is also very much thankful to the Editor for his valuable comments to standardize it.

References

[1] D. Banerjee and R. K. Dutta, The growth of iterated entire functions, *Bull. Math. Anal. and Appl.*, **3**(3) (2011), 35-49.

[2] D. Banerjee and B. Mandal, Relative fixed points of a certain class of complex functions, *Istanbul Univ. Sci. Fac. J. Math. Phys. Astr.*, **6** (2015), 15-25.

[3] J. Clunie, The composition of entire and meromorphic functions, *Mathematical Essays* (Dedicated to A. J. Macintyre), Ohio Univ. Press, 1970, 75-92.

[4] R. K. Dutta, Further growth of iterated entire functions-I, *J. Math. Ineq.*, **5**(4) (2011), 533-550.

[5] R. K. Dutta, Growth of iterated entire functions in terms of its maximum term, *Acta Univ. Apu.*, **30** (2012), 209-219.

[6] R. K. Dutta, The growth estimate of iterated entire functions, *Acta Univ. Apu.*, **34** (2013), 81-87.

[7] R. K. Dutta, A study on the growth of generalist iterated entire functions, *J. Hyper.*, **9**(2) (2020), 115-125.

[8] W. K. Hayman, *Meromorphic Functions*, Oxford University Press, 1964.

[9] I. Lahiri, Growth of composite integral functions, *Indian J. Pure and Appl. Math.*, **20**(9) (1989), 899-907.

[10] I. LAHIRI, Generalised proximate order of meromorphic functions, *Matematnykn Bechnk*, **41** (1989), 9-16.

[11] I. Lahiri and S. K. Datta, On the growth of composite entire and meromorphic functions, *Indian J. Pure and Appl. Math.*, **35**(4) (2004), 525-543.

[12] K. Niino and C. C. Yang, *Some growth relationships on factors of two composite entire functions, Factorization Theory of Meromorphic Functions and Related Topics*, Marcel Dekker Inc. (New York and Basel), (1982), 95-99.

[13] D. Sato, On the rate of growth of entire functions of fast growth, *Bull. Amer. Math. Soc.*, **69** (1963), 411-414.

[14] A. P. Singh, Growth of composite entire functions, *Kodai Math. J.*, **8** (1985), 99-102.