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(Dedicated to Professor D. S. Hooda on His 80^{th} Birth Anniversary Celebrations)

LP-KENMOTSU MANIFOLD ADMITTING SCHOUTEN-VAN KAMPEN CONNECTION

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Abstract

In this paper we study Schouten-van Kampen connection on a Lorentzian para-Kenmotsu manifolds M. We obtain curvature tensor $\bar{R}$, Ricci tensor $\bar{S}$ and scalar curvature $\bar{r}$, with respect to Schouten-van Kampen connection and study their properties. Further, we take some curvature conditions like $\bar{R}\cdot\bar{S} = 0$, $\bar{S}\cdot\bar{R} = 0$ etc., on M and prove $\bar{R}\cdot\bar{C} = \bar{R}\cdot\bar{R}$. We also consider the cases when M is ξ-concircularly flat, pseudo-concircularly flat, ϕ-concircularly semisymmetric and obtain some interesting results.
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1. Introduction

The Schouten-van Kampen connection has been introduced for studying non holomorphic manifolds. It is one of the most natural connections adapted to a pair of complementary distributions on a differentiable manifold endowed with an affine connection [3, 7, 18]. Olszak [15] has studied the Schouten-van Kampen connection to adapt an almost contact metric structure in 2014. He characterized some classes of almost contact metric manifolds with the Schouten-van Kampen connection and established certain curvature properties with respect to this connection. In 2018, Ghosh [4] studied the Schouten-van Kampen connection in Sasakian manifolds. Further, Nagaraja [14] in 2019 and Yildiz [21] in 2017 studied this connection in Kenmotsu manifold and f-Kenmotsu manifold respectively. Recently, the Schouten-van Kampen connection has been studied by Mondal [13] in f-kenmotsu manifold and by Zeren - Yildiz [22] in Trans-Sasakian 3-manifolds.

The notion of an almost para contact Riemannain manifold have been defined by Sato [17] in 1976. Adati and Matsumoto [1] defined and studied para-Sasakian and *SP* para-Sasakian manifolds which are regarded as a special kind of an almost contact Riemannian manifolds. Before Sato, Kenmotsu [8] defined a class of almost contact Riemannian manifolds which satisfies the two conditions viz.

$$\nabla_X\phi Y = -\eta(X)\phi X - g(X, \phi Y)\xi \quad \text{and} \quad \nabla_X\xi = X - \eta(X)\xi.$$

Sinha and Prasad [19] defined a class of almost para contact metric manifolds namely para Kenmotsu and special para Kenmotsu manifolds. In 1989, Matsumoto [9] introduced the notion of Lorentzian para-Sasakian manifold. Mihai and Rosca [11] introduced the same notion independently and obtained several results on this manifold. *LP*-Sasakian manifolds have also been studied by Matsumoto and Mihai [10], Mihai et al.[12], and Venkatesha and Bagewadi [20]. Recently, Haseeb and Prasad, [5, 6] studied Ricci-pseudosymmetricity, Ricci-generalized pseudosymmetricity etc., conditions to characterize *LP*-Kenmotsu manifolds. Moreover, they also explored Ricci solitons on *LP*-Kenmotsu manifolds. Pandey et al. [16] investigated the geometric properties of η-Ricci solitons on this manifolds.

In the present paper, we study Lorentzian para-Kenmotsu manifold (*LP*-Kenmotsu manifold, in short) admitting Schouten-van Kampen connection. After introduction in first section, the second section contains some basic results of *LP*-Kenmotsu manifold. Further, the Schouten-van Kampen connection is defined in third section. We study curvature properties of *LP*-Kenmotsu manifold with respect to Schouten-van Kampen connection and obtain some results in this section. We conclude this paper by giving an example in last section.

2. Preliminaries

Let M be an n-dimensional Lorentzian metric manifold. If it is endowed with a structure (ϕ, ξ, η, g), where ϕ is a (1, 1)-tensor field, ξ is a vector field, η is a 1-form on M and g is a Lorentzian metric, satisfying the following [2]

$$\phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$
$$\eta(\xi) = -1, \quad g(X, \xi) = \eta(X), \tag{2.3}$$
for any vector fields X, Y on M, then it is called Lorentzian almost paracontact manifold. In a Lorentzian almost paracontact manifold, the following relations hold:
$$\phi(\xi) = 0, \quad \eta(\phi X) = 0, \quad \Phi(X, Y) = \Phi(Y, X),$$
where the fundamental 2-form Φ is defined by $\Phi(X, Y) = g(X, \phi Y)$. If ξ is killing vector field, the para contact structure is called K-para contact structure. In such case we have
$$\nabla_X \xi = \phi X.$$
A Lorentzian almost paracontact manifold M is called a Lorentzian para-Sasakian manifold if, for any vector fields X and Y on M, we have:
$$(\nabla_X \phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi.$$
Now we give definition of Lorentzian para-Kenmotsu manifold.

Definition 2.1 ([5])**.** *A Lorentzian almost paracontact manifold M is called Lorentzian para-Kenmotsu Manifold if*
$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X,$$
for any vector field X and Y on M.

In a Lorentzian para-Kenmotsu manifold, we have
$$\nabla_X \xi = -X - \eta(X)\xi, \tag{2.4}$$
$$(\nabla_X \eta)Y = -g(X, Y) - \eta(X)\eta(Y), \tag{2.5}$$
where ∇ is the Levi-Civita connection with respect to the Lorentzian metric g.
Furthermore, from [5] on a Lorentzian para-Kenmotsu manifold M, the following relations holds:
$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.6}$$
$$R(\xi, X)Y = -R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.7}$$
$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.8}$$
$$S(X, \xi) = (n-1)\eta(X), \tag{2.9}$$
$$Q\xi = (n-1)\xi, \tag{2.10}$$
$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.11}$$
for any vector fields X,Y and Z on M.
The Ricci Tensor S of the manifold M is defined as
$$S(X, Y) = \sum_{i=1}^{n} \epsilon_i g(R(e_i, X)Y, e_i),$$
and the scalar curvature r is defined as
$$r = \sum_{i=1}^{n} \epsilon_i S(e_i, e_i),$$
where $\{e_1, e_2, ...e_n = \xi\}$ is a frame of orthonormal basis of the tangent space at any point of the manifold M. Furthemore, we also have
$$g(X, Y) = \sum_{i=1}^{n} \epsilon_i g(X, e_i)g(Y, e_i)$$
where X and Y are vector field on M and $\epsilon_i = g(e_i, e_i) = \pm 1$.

Definition 2.2. *A Lorentzian para-Kenmotsu manifold M is said to be an η-Einstein manifold if its Ricci tensor S is of the form*
$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$
where a and b are scalar functions on M. In particular the manifold M is called Einstein manifold if $b = 0$.

Definition 2.3. *The sectional curvature of a manifold is defined as*
$$k(X, Y) = -\frac{R(X, Y, X, Y)}{g(X, X)g(Y, Y) - g(X, Y)^2}, \tag{2.12}$$
where $R(X, Y, Z, W)$ is associated curvature tensor.

3. Curvature properties of a *LP*-Kenmotsu manifold with respect to the Schouten-van Kampen connection

For an almost contact metric manifold M, the Schouten-van Kampen connection $\bar{\nabla}$ is given by [15]

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi, \tag{3.1}$$

using (2.4) and (2.5), (3.1) reduced to

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi. \tag{3.2}$$

We define the curvature tensor $\bar{R}$ of a *LP*-Kenmotsu manifold with respect to Schouten-van Kampen connection $\bar{\nabla}$ by

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{3.3}$$

In view of (3.2) and (3.3) we have

$$\begin{aligned}\bar{R}(X, Y)Z = R(X, Y)Z &+ 3g(Y, Z)X - 3g(X, Z)Y + 2g(Y, Z)\eta(X)\xi \\ &-2g(X, Z)\eta(Y)\xi + 2\eta(Y)\eta(Z)X - 2\eta(X)\eta(Z)Y.\end{aligned} \tag{3.4}$$

Taking inner product in both sides of (3.4) with W, we have

$$\begin{aligned}\bar{R}(X, Y, Z, W) = R(X, Y, Z, W) &+ 3g(Y, Z)g(X, W) - 3g(X, Z)g(Y, W) \\ &+2g(Y, Z)\eta(X)\eta(W) - 2g(X, Z)\eta(Y)\eta(W) \\ &+2\eta(Y)\eta(Z)g(X, W) - 2\eta(X)\eta(Z)g(Y, W),\end{aligned} \tag{3.5}$$

where $\bar{R}(X, Y, Z, W) = g(\bar{R}(X, Y)Z, W)$.

Putting $X = W = e_i$, and taking sum over i, we have

$$\bar{S}(Y, Z) = S(Y, Z) + (3n - 7)g(Y, Z) + 2n\eta(Y)\eta(Z). \tag{3.6}$$

From which we can obtain

$$\bar{Q}Y = QY + (3n - 7)Y + 2n\eta(Y)\xi. \tag{3.7}$$

Again putting $Y = Z = e_i$, in (3.6) and taking sum over i, we have

$$\bar{r} = r + (3n^2 - 11n + 14), \tag{3.8}$$

where $\bar{r}$ and r, are the scalar curvatures with respect to the Schouten-van Kampen connection $\bar{\nabla}$ and Levi-Civita connection ∇ respectively.

Further, from (3.4) and using Binachi's first identity $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$, we get

$$\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = 0.$$

Again from (3.5), we get

$$\bar{R}(X, Y, Z, W) = -\bar{R}(Y, X, Z, W),$$
$$\bar{R}(X, Y, Z, W) = -\bar{R}(X, Y, W, Z),$$
$$\bar{R}(X, Y, Z, W) = \bar{R}(Z, W, X, Y).$$

Thus, in view of above discussion we state the following theorem:

Theorem 3.1. *In an n-dimensional LP-Kenmotsu manifold with respect to the Schouten-van Kampen connection the following hold:*

1. *The curvature tensor $\bar{R}$, given by* (3.4)*,*
2. *The curvature tensor $\bar{R}$, is symmetric in pair of slots, and skew-symmertic in first two and last two slots,*
3. *The Ricci tensor $\bar{S}$, is symmetric and is given by* (3.6)*,*
4. *The scalar curvature $\bar{r}$ is given by* (3.8)*.*

Now we give a Lemma which can be proved directly from (3.4), (3.6) and (3.7).

Lemma 3.1. *Let M be an n-dimensional LP-Kenmotsu manifolds with respect to Schouten-van kampen connection, then we have the followings:*

$$\bar{R}(X, Y)\xi = 2(\eta(Y)X - \eta(X)Y) = 2R(X, Y)\xi, \tag{3.9}$$

$$\bar{R}(\xi, X)Y = 2(g(X, Y)\xi - \eta(Y)X) = 2R(\xi, X)Y, \tag{3.10}$$

$$\bar{R}(X, \xi)Y = 2(-g(X, Y)\xi + \eta(Y)X) = 2R(X, \xi)Y = -\bar{R}(\xi, X)Y, \tag{3.11}$$

$$\bar{R}(\xi, X)\xi = 2(\eta(X)\xi + X) = 2R(\xi, X)\xi, \tag{3.12}$$

$$\bar{S}(X, \xi) = (2n - 8)\eta(X), \tag{3.13}$$

$$\bar{Q}\xi = (2n - 8)\xi, \tag{3.14}$$

$$\eta(\bar{R}(X, Y)Z = 2\eta(R(X, Y)Z), \tag{3.15}$$

for all X,Y on M.

Now consider a LP-Kenmotsu manifold with respect to the Schouten-van Kampen connection $\bar{\nabla}$ satisfying the condition

$$\bar{R}(X,Y)\cdot\bar{S}=0.$$

Then we have

$$\bar{S}(\bar{R}(X,Y)U,V)+\bar{S}(U,\bar{R}(X,Y)V)=0, \tag{3.16}$$

for any vector fields X,Y,U and V on M. Putting $X=\xi$ in (3.16), we obtain

$$\bar{S}(\bar{R}(\xi,Y)U,V)+\bar{S}(U,\bar{R}(\xi,Y)V)=0. \tag{3.17}$$

Using (3.10), we get from (3.17),

$$\bar{S}(2g(Y,U)\xi-\eta(U)Y,V)+\bar{S}(U,2g(Y,V)\xi-\eta(V)Y)=0.$$

Which implies,

$$2g(Y,U)\bar{S}(\xi,V)-\eta(U)\bar{S}(Y,V)+2g(Y,V)\bar{S}(U,\xi)-\eta(V)\bar{S}(U,Y)=0.$$

Replacing U by ξ gives,

$$\bar{S}(Y,V)=2(2n-8)g(Y,V)-(2n-8)\eta(Y)\eta(V). \tag{3.18}$$

Using (3.6), (3.18) becomes

$$S(Y,V)=(n-9)g(Y,V)+8\eta(Y)\eta(V).$$

Thus we have the following theorem:

Theorem 3.2. *If a LP-Kenmotsu manifold satisfy the condition* $\bar{R}\cdot\bar{S}=0$, *then the manifold is an* η*- Einstein manifold.*

We now consider a LP -Kenmotsu manifold admitting Schouten-van Kampen connection $\bar{\nabla}$ satisfying the condition

$$(\bar{S}(X,Y)\cdot\bar{R})(U,V)Z=0,$$

for any vector fields X,Y,Z,U and U on M.

This implies that

$$(X\wedge_{\bar{S}}Y)\bar{R}(U,V)Z+\bar{R}((X\wedge_{\bar{S}}Y)U,V)Z+\bar{R}(U,(X\wedge_{\bar{S}}Y)V)Z$$
$$+\bar{R}(U,V)(X\wedge_{\bar{S}}Y)Z=0, \tag{3.19}$$

where the endomorphism $X\wedge_{\bar{S}}Y$ is defined by

$$(X\wedge_{\bar{S}}Y)Z=\bar{S}(Y,Z)X-\bar{S}(X,Z)Y. \tag{3.20}$$

Taking $Y=\xi$ in (3.19), we have

$$(X\wedge_{\bar{S}}\xi)\bar{R}(U,V)Z+\bar{R}((X\wedge_{\bar{S}}\xi)U,V)Z+\bar{R}(U,(X\wedge_{\bar{S}}\xi)V)Z$$
$$+\bar{R}(U,V)(X\wedge_{\bar{S}}\xi)Z=0. \tag{3.21}$$

From (3.20), (3.21) and (3.13), we get

$$(2n-8)[\eta(\bar{R}(U,V)Z)X+\eta(U)\bar{R}(X,V)Z+\eta(V)\bar{R}(U,X)Z$$
$$+\eta(Z)\bar{R}(U,V)X]-\bar{S}(X,\bar{R}(U,V)Z)\xi-\bar{S}(X,U)\bar{R}(\xi,V)Z$$
$$-\bar{S}(X,V)\bar{R}(U,\xi)Z-\bar{S}(X,Z)\bar{R}(U,V)\xi=0.$$

Taking inner product with ξ, we have

$$(2n-8)[\eta(\bar{R}(U,V)Z)\eta(X)+\eta(U)\eta(\bar{R}(X,V)Z)+\eta(V)\eta(\bar{R}(U,X)Z)$$
$$+\eta(Z)\eta(\bar{R}(U,V)X)]+\bar{S}(X,\bar{R}(U,V)Z)-\bar{S}(X,U)\eta(\bar{R}(\xi,V)Z)$$
$$-\bar{S}(X,V)\eta(\bar{R}(U,\xi)Z-\bar{S}(X,Z)\eta(\bar{R}(U,V)\xi)=0. \tag{3.22}$$

By taking $U=Z=\xi$, in (3.22) and using Lemma 3.1, we get

$$2(2n-8)g(V,X)+2\bar{S}(X,V)+4(2n-8)\eta(X)\eta(V)=0. \tag{3.23}$$

with the help of (3.6), (3.23) becomes

$$S(X,V)=-(5n-15)g(X,V)-(5n-16)\eta(X)\eta(V).$$

Therefore we state the following theorem:

Theorem 3.3. *If a LP-Kenmotsu manifold satisfy the condition* $\bar{S}\cdot\bar{R}=0$, *then the manifold is an* η*- Einstein manifold.*

Now we consider Ricci-flat manifold with respect to Schouten-van Kampen connection.

Definition 3.1. *A LP-Kenmotsu manifold M is Ricci-flat with respect to Schouten-van Kampen connection* $\bar{\nabla}$ *if* $\bar{S}(Y,Z) = 0$.

We now have the following theorem:

Theorem 3.4. *A LP-Kenmotsu manifold M is Ricci-flat with respect to Schouten-van Kampen connection* $\bar{\nabla}$*, iff it is* η*-Einstein Manifold with Ricci tensor S of the form*

$$S(Y,Z) = -(3n-7)g(Y,Z) - 2n\eta(Y)\eta(Z).$$

Proof. If M is Ricci-flat with respect to Schouten-van Kampen connection then by virtue of (3.6), we get $S(Y,Z) = -(3n-7)g(Y,Z)-2n\eta(Y)\eta(Z)$. Conversely if, $S(Y,Z) = -(3n-7)g(Y,Z)-2n\eta(Y)\eta(Z)$, then again by (3.6), $\bar{S}(Y,Z) = 0$. This completes the proof of the Theorem.

Next, let us suppose that, $\bar{R}(X,Y)Z = 0$, on M. Let $\xi^{\perp}$ denote the $(n-1)$-dimensional distribution orthogonal to ξ, then for any $X \in \xi^{\perp}$, $g(X,\xi) = \eta(X) = 0$. from (3.5), we have

$$\begin{aligned}\bar{R}(X,Y,X,Y) = R(X,Y,X,Y) &+ 3g(X,Y)g(X,Y) - 3g(X,X)g(Y,Y)\\ &+2g(X,Y)\eta(X)\eta(Y) - 2g(X,X)\eta(Y)\eta(Y)\\ &+ 2\eta(Y)\eta(X)g(X,Y) - 2\eta(X)\eta(X)g(Y,Y).\end{aligned}$$

from (2.12), we get

$$k(X,Y) = -3.$$

Thus we can state the following:

Theorem 3.5. *If* $\bar{R}(X,Y)Z = 0$*, in a LP-Kenmotsu manifold, then the sectional curvature of the plane section determined by* $X, Y \in \xi^{\perp}$*, is -3.*

Now, we consider locally ϕ-symmetric *LP*-Kenmotsu manifold with respect to the Schouten-van Kampen connection. We begin with the following definition.

Definition 3.2. *A LP-Kenmotsu manifold is said to be locally* ϕ*-symmetric with respect to the Schouten-van Kampen connection* $\bar{\nabla}$ *if its curvature tensor* $\bar{R}$ *with respect to the connection* $\bar{\nabla}$ *satisfies the condition*

$$\phi^2((\bar{\nabla}_W\bar{R})(X,Y)U) = 0.$$

for any vector fields X, Y, U, W orthogonal to ξ*.*

By the help of (3.2), we have

$$(\bar{\nabla}_W\bar{R})(X,Y)U = (\nabla_W\bar{R})(X,Y)U + \eta(\bar{R}(X,Y)U)W - g(W,\bar{R}(X,Y)U)\xi, \tag{3.24}$$

by virtue of (3.15), (3.24) reduces to

$$(\bar{\nabla}_W\bar{R})(X,Y)U = (\nabla_W\bar{R})(X,Y)U + 2\eta(R(X,Y)U)W - g(W,\bar{R}(X,Y)U)\xi. \tag{3.25}$$

Covarient differentiation of (3.4) with respect to W gives

$$\begin{aligned}(\nabla_W\bar{R})(X,Y)U = (\nabla_W R)(X,Y)U &+ 2g(Y,U)[-g(W,X)\xi - 2\eta(X)\eta(W)\xi -\\ &\eta(X)W] - 2g(X,U)[-g(W,Y)\xi - 2\eta(W)\eta(Y)\xi - \eta(Y)W]\\ &- 2g(W,Y)\eta(U)X - 4\eta(W)\eta(Y)\eta(U)X - 2\eta(Y)g(W,U)X\\ &+2g(W,X)\eta(U)Y + 4\eta(W)\eta(X)\eta(U)Y + 2\eta(X)g(W,U)Y.\end{aligned} \tag{3.26}$$

By virtue of (3.26),(3.25) becomes

$$\begin{aligned}(\bar{\nabla}_W\bar{R})(X,Y)U = (\nabla_W R)(X,Y)U &- 5g(Y,U)g(W,X)\xi - 6g(Y,U)\eta(X)\eta(W)\xi\\ &+ 5g(X,U)g(W,Y)\xi + 4g(X,U)\eta(W)\eta(Y)\xi + 2g(X,U)\eta(Y)\xi\\ &+ 2g(X,U)\eta(Y)\eta(W)\xi - 2g(W,Y)\eta(U)X - 4\eta(W)\eta(Y)\eta(U)X\\ &- 2\eta(Y)g(W,U)X + 2g(W,X)\eta(U)Y + 4\eta(W)\eta(X)\eta(U)Y\\ &+ 2\eta(X)g(W,U)Y - 2g(X,U)\eta(Y)W - R(X,Y,U,W)\xi\\ &-2\eta(Y)\eta(V)g(X,W)\xi + 2\eta(X)\eta(V)g(Y,W)\xi.\end{aligned} \tag{3.27}$$

Applying ϕ^2 on both side of (3.27), and using (2.1), we have

$$\begin{aligned}\phi^2((\bar{\nabla}_W\bar{R})(X,Y)U) = \phi^2((\nabla_W R)(X,Y)U) &- 2g(Y,W)\eta(U)X - 2g(Y,W)\eta(X)\eta(U)\xi\\ &- 4\eta(W)\eta(Y)\eta(U)X - 4\eta(W)\eta(Y)\eta(U)\eta(X)\xi - 2\eta(Y)g(W,U)X\\ &- 2\eta(Y)g(W,U)\eta(X)\xi + 2g(W,X)\eta(U)Y + 2g(W,X)\eta(U)\eta(Y)\xi\\ &+ 4\eta(X)\eta(W)\eta(U)Y + 4\eta(X)\eta(W)\eta(U)\eta(Y)\xi + 2g(W,U)\eta(X)Y\\ &+2g(W,U)\eta(X)\eta(Y)\xi - 2g(X,U)\eta(Y)W - 2g(X,U)\eta(Y)\eta(W)\xi.\end{aligned}$$

Now taking X, Y, U and W orthogonal to ξ we get

$$\phi^2((\bar{\nabla}_W\bar{R})(X,Y)U) = \phi^2((\nabla_W R)(X,Y)U).$$

Thus we can state the following:

Theorem 3.6. *A LP-Kenmotsu manifold is locally ϕ-symmetric with respect to the Schouten-van Kampen connection $\bar{\nabla}$ if and only if it is so with respect to the Levi-Civita connection ∇.*

Now, we study concircular curvature tensor with respect to the Schouten-van Kampen connection on the *LP*-Kenmotsu manifold.

Definition 3.3. *For an n-dimensional LP-Kenmotsu manifold the concircular curvature tensor $\bar{C}$ with respect to the Schouten-van Kampen connection is defined by*

$$\bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \left(\frac{\bar{r}}{n(n-1)}\right)[g(Y,Z)X - g(X,Z)Y]. \tag{3.28}$$

Using (3.4) and (3.8), (3.28) becomes

$$\begin{aligned}\bar{C}(X,Y)Z = C(X,Y)Z + \left(\frac{8n-14}{n(n-1)}\right)&[g(Y,Z)X - g(X,Z)Y] + 2g(Y,Z)\eta(X)\xi\\ &-2g(X,Z)\eta(Y)\xi + 2\eta(Y)\eta(Z)X - 2\eta(X)\eta(Z)Y.\end{aligned}$$

Using $R(X,Y)Z + R(Y,Z)X = 0$, we get

$$\bar{C}(X,Y)Z + \bar{C}(Y,X)Z = 0.$$

Further, one can easily verify that

$$\bar{C}(X,Y)Z + \bar{C}(Y,Z)X + \bar{C}(Z,X)Y = 0.$$

Thus we can say that the concircular curvature tensor $\bar{C}$ with respect to the Schouten-van Kampen connection is skew-symmetric and cyclic.

Now suppose that, a *LP*-Kenmotsu manifold M is concircularly flat with respect to Schouten-van Kampen connection, then we have

$$\bar{C}(X,Y)Z = 0. \tag{3.29}$$

By virtue of (3.29), (3.28) becomes

$$\bar{R}(X,Y)Z = \frac{\bar{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{3.30}$$

Taking inner product in both side of (3.30) with ξ, we get

$$g(\bar{R}(X,Y)Z,\xi) = \frac{\bar{r}}{n(n-1)}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)].$$

Now, using (3.4), (3.8) and (2.6), we get

$$\frac{-n^2+9n-r-14}{n(n-1)}[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] = 0.$$

Which implies either the scalar curvature of M is $r = -n^2 + 9n - 14$, or

$$g(Y,Z)\eta(X) - g(X,Z)\eta(Y) = 0.$$

Replacing Y by ξ, X by QX and using (2.10), we get

$$S(X,Z) = (1-n)\eta(X)\eta(Z).$$

Thus we can state the following:

Theorem 3.7. *For a concircularly flat LP-Kenmotsu manifold with respect to the Schouten-van Kampen connection, either the scalar curvature is $r = -n^2 + 9n - 14$ or the manifold is a special type of η- Einstein manifold.*

We now consider locally concircular ϕ-symmetric *LP*-Kenmotsu manifold with respect to the Schouten-van Kampen connection $\bar{\nabla}$.

Definition 3.4. *A LP-Kenmotsu manifold is said to be locally concircular ϕ-symmetric with respect to the Schouten-van Kampen connection $\bar{\nabla}$, if its concircular curvature tensor $\bar{C}$ with respect to the connection $\bar{\nabla}$ satisfies the condition*

$$\phi^2((\bar{\nabla}_W \bar{C})(X,Y)Z) = 0.$$

We now give a theorem, whose proof runs on similar lines as of Theorem 3.6.

Theorem 3.8. *A LP-Kenmotsu manifold is locally concircular ϕ-symmetric with respect to the Schouten-van Kampen connection $\bar{\nabla}$ if and only if it is so with respect to the Levi-Civita connection ∇.*

Definition 3.5. *A LP-Kenmotsu manifold M with respect to the Schouten-van Kampen connection $\bar{\nabla}$ is said to be ξ-concircularly flat if $\bar{C}(X,Y)\xi = 0$.*

Now, we assume that the manifold M with respect to the Schouten-van Kampen connection is ξ-concircularly flat, that is $\bar{C}(X,Y)\xi = 0$. Then from (3.28), it follows that

$$\bar{R}(X,Y)\xi = \frac{\bar{r}}{n(n-1)}[\eta(Y)X - \eta(X)Y].$$

Using (3.8) and (3.9), we get

$$\frac{-n^2 + 9n - r - 14}{n(n-1)}[\eta(Y)X - \eta(X)Y] = 0.$$

Putting $Y = \xi$, we get

$$\frac{-n^2 + 9n - r - 14}{n(n-1)}[-\eta(X)\xi - X] = 0.$$

Taking inner product with U, we have

$$-\frac{-n^2 + 9n - r - 14}{n(n-1)}[\eta(X)\eta(U) + g(X,U)] = 0.$$

Which implies either the scalar curvature of M is $r = -n^2 + 9n - 14$, or

$$g(X,U) = -\eta(X)\eta(U).$$

Replacing X by QX and using (2.10), we get

$$S(X,U) = (1-n)\eta(X)\eta(U). \tag{3.31}$$

Thus we have the following:

Theorem 3.9. *For a ξ-concircularly flat LP-Kenmotsu manifold with respect to the Schouten-van Kampen connection, either the scalar curvature is $r = -n^2 + 9n - 14$ or the manifold is a special type of η- Einstein manifold.*

Corollary 3.1. *If a LP-Kenmotsu M, admitting Schouten-van Kampen connection is ξ- concircullarly flat, then M is of constant scalar curvature.*

Proof. The proof directly follows from (3.31), by putting $X = U = e_i$, and taking sum over i.

Definition 3.6. *A LP-Kenmotsu manifold is said to be pseudo-concircularly flat with respect to the Schouten-van Kampen connection $\bar{\nabla}$ if it satisfies,*

$$g(\bar{C}(\phi X, Y)Z, \phi W)) = 0, \tag{3.32}$$

for any vector fields X, Y, Z on M.

In view of (3.28) and (3.32) we get

$$g(\bar{R}(\phi X, Y)Z, \phi W)) - \frac{\bar{r}}{n(n-1)}[g(Y, Z)g(\phi X, \phi W) - g(\phi X, Z)g(Y, \phi W)] = 0.$$

Making use of (3.4) and (3.8), we get

$$g(R(\phi X, Y)Z, \phi W)) + 3g(Y, Z)g(\phi X, \phi W) - 3g(\phi X, Z)g(Y, \phi W) + 2\eta(Y)\eta(Z)g(\phi X, \phi W)$$
$$-\frac{r + 3n^2 - 11n + 14}{n(n-1)}[g(Y, Z)g(\phi X, \phi W) - g(\phi X, Z)g(Y, \phi W)] = 0.$$

Putting $Y = Z = e_i$ and summing for i, we get

$$S(\phi X, \phi Y) = \left[(-3n + 7) + \frac{(r + 3n^2 - 11n + 14)(n + 3)}{n(n-1)}\right] g(\phi X, \phi Y). \tag{3.33}$$

Again putting $\left[(-3n + 7) + \frac{(r + 3n^2 - 11n + 14)(n + 3)}{n(n-1)}\right] = \alpha$, and making use of (2.11) and (2.2), (3.33) becomes

$$S(X, Y) = \alpha g(X, Y) + (\alpha - n + 1)\eta(X)\eta(Y). \tag{3.34}$$

Hence, we state the following:

Theorem 3.10. *Let the LP-Kenmotsu manifold M is pseudo-concircularly flat with respect to Schouten-van Kampen connection* $\bar{\nabla}$*, then M is an* η*- Einstein manifold.*

Corollary 3.2. *If a LP-Kenmotsu M, admitting Schouten-van Kampen connection is pseudo-concircullarly flat, then M is of constant scalar curvature.*

Proof. The proof directly follows from (3.34), by putting $X = Y = e_i$, and taking sum over i.

Definition 3.7. *A LP-Kenmotsu manifold is said to be* ϕ*-concircularly semisymmetric with respect to Schouten-van Kampen connection* $\bar{\nabla}$*, if* $\bar{C}(X, Y) \cdot \phi = 0$ *holds on M.*

Now, we consider ϕ-concircularly semisymmetric *LP*-Kenmotsu manifold with respect to Schouten-van Kampen connection $\bar{\nabla}$. Then we have

$$(\bar{C}(X, Y) \cdot \phi)Z = \bar{C}(X, Y)\phi Z - \phi\bar{C}(X, Y)Z = 0.$$

Replacing Z by ξ, we get

$$\phi(\bar{C}(X, Y)\xi) = 0.$$

By virtue of (3.28) and (3.9), we get

$$\left(2 - \frac{\bar{r}}{n(n-1)}\right)[\eta(Y)\phi X - \eta(X)\phi Y] = 0.$$

Using (3.8), we get

$$\frac{-n^2 + 9n - r - 14}{n(n-1)}[\eta(Y)\phi X - \eta(X)\phi Y] = 0.$$

Replacing Y by ξ and X by ϕX, we have

$$\frac{-n^2 + 9n - r - 14}{n(n-1)}[X + \eta(X)\xi] = 0.$$

Taking inner product with ξ, we get

$$\frac{-n^2 + 9n - r - 14}{n(n-1)}[g(X, U) + \eta(X)\eta(U)] = 0.$$

Which implies, either the scalar curvature of the manifold M is $r = -n^2 + 9n - 14$, or

$$g(X, U) = -\eta(X)\eta(U)$$

replacing X by QX and using (2.10), we have

$$S(X, U) = (1 - n)\eta(X)\eta(U). \tag{3.35}$$

Thus we can state the following:

Theorem 3.11. *For a ϕ-concircularly semisymmetric LP-Kenmotsu manifold M with respect to the Schouten-van Kampen Connection $\bar{\nabla}$, either the scalar curvature is $r = -n^2 + 9n - 14$ or the manifold is a special type of η-Einstein Manifold.*

Corollary 3.3. *If a LP-Kenmotsu M, admitting Schouten-van Kampen connection is ϕ- concircullarly semisymmetric, then M is of constant scalar curvature.*

Proof. The proof directly follows from (3.35), by putting $X = U = e_i$, and taking sum over i.
Further, we also have

$$(\bar{R}(X,Y)\cdot\bar{C})(U,V,W) = \bar{R}(X,Y)\bar{C}(U,V)W - \bar{C}(\bar{R}(X,Y)U,V)W$$
$$-\bar{C}(U,\bar{R}(X,Y)V)W - \bar{C}(U,V)\bar{R}(X,Y)W. \quad (3.36)$$

By the help of (3.28), (3.36) becomes

$$(\bar{R}(X,Y)\cdot\bar{C})(U,V,W) = \bar{R}(X,Y)\bar{R}(U,V)W - \bar{R}(\bar{R}(X,Y)U,V)W$$
$$-\bar{R}(U,\bar{R}(X,Y)V)W - \bar{R}(U,V)\bar{R}(X,Y)W +$$
$$\frac{\bar{r}}{n(n-1)}[g(\bar{R}(X,Y)V,W)U + g(V,\bar{R}(X,Y)W)U$$
$$-g(\bar{R}(X,Y)U,W)V - g(U,\bar{R}(X,Y)W)V].$$

By the symmetricity of $\bar{R}(X,Y)Z$, we get

$$(\bar{R}(X,Y)\cdot\bar{C})(U,V,W) = \bar{R}(X,Y)\bar{R}(U,V)W - \bar{R}(\bar{R}(X,Y)U,V)W$$
$$-\bar{R}(U,\bar{R}(X,Y)V)W - \bar{R}(U,V)\bar{R}(X,Y)W.$$

Finally, we have

$$(\bar{R}(X,Y)\cdot\bar{C})(U,V,W) = (\bar{R}(X,Y)\cdot\bar{R})(U,V,W).$$

Thus we can state the following:

Theorem 3.12. *Let M be a LP-Kenmotsu manifold equipped with Schouten-van Kampen connection, then we have $\bar{R}\cdot\bar{C} = \bar{R}\cdot\bar{R}$.*

4. Example

Example 4.1. Consider the three dimensional manifold

$$M^3 = \{(x,y,z) \in R^3 : z > 0\},$$

where (x,y,z) are the standard coordinates in R^3. Let e_1, e_2 and e_3 be the vector fields on M^3 given by

$$e_1 = z\frac{\partial}{\partial x}, e_2 = z\frac{\partial}{\partial y}, e_3 = z\frac{\partial}{\partial z} = \xi,$$

which are linearly independent at each point of M^3 and hence form a basis of T_pM^3. Define a Lorentzian metric g on M^3 as,

$$g(e_1,e_1) = g(e_2,e_2) = 1, \quad g(e_3,e_3) = -1, \quad g(e_1,e_2) = g(e_1,e_3) = g(e_2,e_3) = 0.$$

The 1-form η is defined by $\eta(X) = g(X,e_3)$ for all $X \in \chi(M)$. Further we define the (1,1)-tensor field ϕ by,

$$\phi e_1 = -e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0.$$

It has been shown in [5] that this manifold is a *LP-Kenmotsu* manifold.
Using (3.2) we calculate the following

$$\bar{\nabla}_{e_1}e_1 = -2e_3, \quad \bar{\nabla}_{e_1}e_2 = 0, \quad \bar{\nabla}_{e_1}e_3 = -e_1,$$
$$\bar{\nabla}_{e_2}e_1 = 0, \quad \bar{\nabla}_{e_2}e_2 = -2e_3, \quad \bar{\nabla}_{e_2}e_3 = -e_2$$
$$\bar{\nabla}_{e_3}e_1 = 0, \quad \bar{\nabla}_{e_3}e_2 = 0, \quad \bar{\nabla}_{e_3}e_3 = 0.$$

Further using (3.4) we can calculate

$$\bar{R}(e_1,e_2)e_1 = -4e_2, \quad \bar{R}(e_1,e_3)e_1 = -4e_3, \quad \bar{R}(e_2,e_3)e_1 = 0,$$
$$\bar{R}(e_1,e_2)e_2 = 4e_1, \quad \bar{R}(e_1,e_3)e_2 = 0, \quad \bar{R}(e_2,e_3)e_2 = -2e_3,$$
$$\bar{R}(e_1,e_2)e_3 = 0, \quad \bar{R}(e_1,e_3)e_3 = -2e_1, \quad \bar{R}(e_2,e_3)e_3 = -2e_2.$$

similarly using (3.6) we get

$$\bar{S}(e_1,e_1) = 4, \quad \bar{S}(e_2,e_2) = 4, \quad \bar{S}(e_3,e_3) = 2.$$

which implies $\bar{r} = \sum_{i=1}^{3}\bar{S}(e_i,e_i) = 10$, which can also be verified from (3.8).

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