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(Dedicated to Professor D. S. Hooda on His 80^{th} Birth Anniversary Celebrations)

FUZZY OPTIMIZATION AND *gH*- SYMMETRICALLY DERIVATIVE OF FUZZY FUNCTIONS

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Abstract

In this paper, we introduce a new concept called Algebra of generalized Hukuhara symmetrically (*gHs*) differentiable fuzzy function. We specifically state the prerequisites for the *gHs* differentiability of the product and composition of a differentiable real function and a *gHs* differentiable fuzzy function, as well as the *gHs* differentiability of the sum of two *gHs* differentiable fuzzy functions.

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1. Introduction

In optimization, there are two components namely, objective functions and constraints. Practically objective functions rarely holds real number as coefficients. Most of the time, they have uncertainty. These values may not be accurate also. The disadvantages of uncertainty or inaccuracy can be tackled by the use of fuzzy programming approach.

Works of Rommelfanger [10] and Delgado et al. [6] viewed this from 90s onwards. Lodwick [9] gives a detailed literature review on this topic. Paper of Slowinski and Teghem [11] compares optimization problems with multiple objectives. Inuiguchi [8] has done a similar comparison but for problems to solve portfolio selection.

Generalization of hukuhara differentiability(*HD*) of set valued functions will give *HD* of fuzzy valued functions where the differentiability is based upon Hukuhara difference. Hukuhara [7] developed the subtraction of two sets. Hukuhara derivatives introduced in [7] is widely used by researchers in the field of set and fuzzy valued functions due to its importance in fuzzy differential equations as well as optimization problems.

It is found from the works of [1], [2],[3] and [5], compared to *H* differentiable functions *gH* differentiable fuzzy functions are relatively general. H.C.Wu [13] studied the *KKT* optimality conditions for fuzzy function and also for multiobjective fuzzy function.[14]

Here we propose a new idea known as generalized Hukuhara symmetrically(*gHs*) differentiable fuzzy functions. We can see that *gHs* derivative of fuzzy functions is more general than *gH* derivative.

Section 2 contains preliminaries. we define our main definition, *gHs* differentiable fuzzy function and some theorems related to it in section 3. Section 4 deals with fuzzy optimization of *gHs* differentiable functions. In last section, we obtain the optimality conditions of non-dominated solution applying *gHs* derivative to fuzzy optimization.

2. Preliminaries

Assume $\mathcal{I}_C$ represents the family of all intervals belongs to $\mathbb{R}$ which are bounded .
ie $\mathcal{I}_C$={$[\underline{k},\overline{k}] | \underline{k},\overline{k} \in \mathbb{R}$ and $\underline{k} <= \overline{k}$ }.
Suppose $A = [\underline{a},\overline{a}]$ and $B = [\underline{b},\overline{b}]$ denote the two fuzzy intervals. Now we explain the HausdorffPompeiu distance(H_p) from A to B as

$$H_p(A, B) = max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}\}. \tag{2.1}$$

Clearly $(\mathcal{I}_C, H_p)$ denotes a complete metric space .
Let $\mathbb{R}^n$ denotes a mapping $l : \mathbb{R}^n \to [0,1]$. We represent the α level set, $[l]^\alpha$ ={$t \in \mathbb{R}^n | l(t) >= \alpha$ } for any $\alpha \in (0,1]$.
Now we recall the definition of support as: supp(l) = $\{t \in \mathbb{R}^n | l(t) > 0\}$.

Definition 2.1. *Suppose l denotes fuzzy set on* $\mathbb{R}$*and l becomes a fuzzy interval only when the following conditions are hold:*

1. *l is normal and upper semi continuous,*

2. *The value of $l(\lambda x + (1-\lambda)y)$ should be greater than or equal to $min\{l(x), l(y)\}, x, y \in \mathbb{R}, \lambda \in [0,1]$,*
3. *$[l]^0$ should be compact.*

Assume $\mathcal{F}_C$ stands for family of all fuzzy intervals. The α levels of fuzzy intervals are defined as, $[l]^\alpha = [\underline{l}_\alpha, \bar{l}_\alpha]$, where $\underline{l}_\alpha, \bar{l}_\alpha \in \mathbb{R}, \forall\ \alpha \in [0,1]$ and $[l]^\alpha \in \mathcal{I}_C, \forall\ \alpha \in [0,1]$.
Now we define the arithmetic operations such as addition and scalar multiplication of fuzzy intervals $l, m \in \mathcal{F}_C$ as follows:

$$(l+m)(t) = \sup_{y+z=t} min\{l(y), m(z)\},$$
$$(\lambda l)(t) = \begin{cases} l(\frac{t}{\lambda}) & if \lambda \neq 0 \\ 0 & if \lambda = 0. \end{cases} \tag{2.2}$$

Clearly $\forall\ \alpha \in$[0,1],

$$[l+m]^\alpha = [(\underline{l+m})_\alpha, (\overline{l+m})_\alpha] = [\underline{l}_\alpha + \underline{m}_\alpha, \bar{l}_\alpha + \overline{m}_\alpha] \tag{2.3}$$

and

$$[(\lambda l)]^\alpha = [(\underline{\lambda l})_\alpha, (\overline{\lambda l})_\alpha] = [min\{\lambda \underline{l}_\alpha, \lambda \bar{l}_\alpha\}, max\{\lambda \underline{l}_\alpha, \lambda \bar{l}_\alpha\}]. \tag{2.4}$$

Definition 2.2. *A p dimensional fuzzy number l on $\mathbb{R}$ is defined as a mapping, $l : \mathbb{R} \to [0,1]^p, l = (l_1, l_2, ..., l_p)$ where each l_i is a fuzzy number.*

Assume $\mathcal{F}_C{}^p$ stands for family of all p dimensional fuzzy numbers.

Definition 2.3. *(*Stefanini [12]*) $l \ominus_{gH} m = p \Leftrightarrow \begin{cases} (1) l = m + p, \\ or (2) m = l + (-1)p. \end{cases}$*

$[l \ominus_{gH} m]^\alpha = [l]^\alpha \ominus_{gH} [m]^\alpha = [min\{\underline{l}_\alpha - \underline{m}_\alpha, \bar{l}_\alpha - \overline{m}_\alpha\}, max\{\underline{l}_\alpha - \underline{m}_\alpha, \bar{l}_\alpha - \overline{m}_\alpha\}]$, $\forall\ \alpha \in$[0,1], where $[l]^\alpha \ominus_{gH} [m]^\alpha$ represents gH- difference from l to m. Let $l, m \in \mathcal{F}_C$, we can define distance from l to m by

$$D(l,m) = \sup_{\alpha \in [0,1]} H([l]^\alpha, [m]^\alpha) = \sup_{\alpha \in [0,1]} max\{|\underline{l}_\alpha - \underline{m}_\alpha|, |\bar{l}_\alpha - \overline{m}_\alpha|\}.$$

So $(\mathcal{F}_C, D)$ denotes complete metric space.

Proposition 2.1. *Suppose the length of the α cuts of l amd m be*

$$[l]^\alpha \ominus_{gH} [m]^\alpha = \begin{cases} [\underline{l}_\alpha - \underline{m}_\alpha, \bar{l}_\alpha - \overline{m}_\alpha], if & len[l]^\alpha \leq len[m]^\alpha \\ [\bar{l}_\alpha - \overline{m}_\alpha, \underline{l}_\alpha - \underline{m}_\alpha], if & len[l]^\alpha \geq len[m]^\alpha, \end{cases} \tag{2.5}$$

where $len[l]^\alpha = \bar{l}_\alpha - \underline{l}_\alpha$.

2.1. *gHs* derivative of fuzzy functions

Let E be an open subset of $\mathbb{R}^n$ and define $\mathcal{F} : E \to \mathcal{F}_C, \forall\ \alpha \in [0,1]$. The collection of all interval-valued fuzzy functions represented by, $\mathcal{F}_\alpha : E \to \mathcal{I}_C$ and is given by $\mathcal{F}_\alpha(t) = [\mathcal{F}(t)]^\alpha$. For any $\alpha \in [0,1]$, with lower function $\underline{f}_\alpha(t)$ and upper function $\bar{f}_\alpha(t)$ we denote $\mathcal{F}_\alpha(t) = [\underline{f}_\alpha(t), \bar{f}_\alpha(t)]$.

Definition 2.4. *Let $E \subset \mathbb{R}$ with $\mathcal{F} : E \to \mathcal{F}_C$ be a fuzzy function and $t_0 \in E$ and $t_0 + h, t_0 - h \in E$. Then the gHs-derivative of $\mathcal{F}$ at t_o is defined as*

$$\mathcal{F}^s(t_0) = \lim_{h \to 0} \frac{\mathcal{F}(t_0 + h) \ominus_{gH} \mathcal{F}(t_0 - h)}{2h}. \tag{2.6}$$

If $\mathcal{F}^s(t_0) \in \mathcal{I}_C$ satisfying (2.6) exists, we say that $\mathcal{F}$ is gHs differentiable at t_0.

Theorem 2.1. *If $\mathcal{F} : E \to \mathcal{F}_C$ is gHs differentiable then the fuzzy function defined on an interval $\mathcal{F}_\alpha : E \to \mathcal{I}_C$ is gHs differentiable, $\forall\ \alpha \in [0,1]$. Furthermore $[\mathcal{F}^s(t)]^\alpha = \mathcal{F}^s_\alpha(t)$.*

Proof. Obvious from the definition of *gHs* differentiability.

Theorem 2.2. *Assume $\mathcal{F} : E \to \mathcal{F}_C$ and if $\mathcal{F}$ is gHs differentiable at $t_0 \in E$ uniformly $\forall\ \alpha \in [0,1]$, then one of the following conditions is satisfied:*

1. $\bar{f}_\alpha$ *and* $\underline{f}_\alpha$ *are symmetrically differentiable at* t_0. *Also*

$$[\mathcal{F}^s(t_0)]^\alpha = [min\{(\underline{f}_\alpha)^s(t_0), (\bar{f}_\alpha)^s(t_0)\}, max\{(\underline{f}_\alpha)^s(t_0), (\bar{f}_\alpha)^s(t_0)\}], \tag{2.7}$$

2. $(\underline{f}_\alpha)^s_-(t_0), (\underline{f}_\alpha)^s_+(t_0), (\bar{f}_\alpha)^s_-(t_0), (\bar{f}_\alpha)^s_+(t_0)$ *exist and satisfy* $(\underline{f}_\alpha)^s_-(t_0) = (\bar{f}_\alpha)^s_+(t_0) and (\underline{f}_\alpha)^s_+(t_0) = (\bar{f}_\alpha)^s_-(t_0)$. *Moreover* $[\mathcal{F}^s(t_0)]^\alpha = [min\{(\underline{f}_\alpha)^s_-(t_0), (\bar{f}_\alpha)^s_-(t_0)\}, max\{(\underline{f}_\alpha)^s_-(t_0), (\bar{f}_\alpha)^s_-(t_0)\}]$ $= [min\{(\underline{f}_\alpha)^s_+(t_0), (\bar{f}_\alpha)^s_+(t_0)\}, max\{(\underline{f}_\alpha)^s_+(t_0), (\bar{f}_\alpha)^s_+(t_0)\}]$.

Proof. Assume that $\mathcal{F}$ is *gHs* differentiable at t_0 and $[\mathcal{F}^s(t_0)]^\alpha = [\underline{g}_\alpha(t_0), \bar{g}_\alpha(t_0)]$ exists. According to (2.1) and definition 2.3

$$\underline{g}_\alpha(t_0) = \lim_{h\to 0} min\Big\{\frac{\underline{f}_\alpha(t_0+h) - \underline{f}_\alpha(t_0-h)}{2h}, \frac{\bar{f}_\alpha(t_0+h) - \bar{f}_\alpha(t_0-h)}{2h}\Big\},$$

$$\bar{g}_\alpha(t_0) = \lim_{h\to 0} max\Big\{\frac{\underline{f}_\alpha(t_0+h) - \underline{f}_\alpha(t_0-h)}{2h}, \frac{\bar{f}_\alpha(t_0+h) - \bar{f}_\alpha(t_0-h)}{2h}\Big\},$$

exist. Therefore $\underline{f}^s_\alpha(t_0), \bar{f}^s_\alpha(t_0)$ must exist and (2.4) is satisfied.

Conversely suppose $\bar{f}_\alpha$ and $\underline{f}_\alpha$ are symmetrically differentiable at t_0. If $(\bar{f}_\alpha)^s(t_0) \geq (\underline{f}_\alpha)^s(t_0)$ then by definition (2.4) and proposition (2.1) we have

$$\begin{aligned}[(\underline{f}_\alpha)^s(t_0), (\bar{f}_\alpha)^s(t_0)] &= [\lim_{h\to 0}\frac{\underline{f}_\alpha(t_0+h) - \underline{f}_\alpha(t_0-h)}{2h}, \lim_{h\to 0}\frac{\bar{f}_\alpha(t_0+h) - \bar{f}_\alpha(t_0-h)}{2h}] \\ &= \lim_{h\to 0}\frac{\mathcal{F}_\alpha(t_0+h) \ominus_{gH} \mathcal{F}_\alpha(t_0-h)}{2h} \\ &= \mathcal{F}^s_\alpha(t_0) \\ &= [\mathcal{F}^s(t_0)]^\alpha.\end{aligned}$$

So $\mathcal{F}$ is *gHs* differentiable. Similarly, if $(\bar{f}_\alpha)^s(t_0) <= (\underline{f}_\alpha)^s(t_0)$ then $[F^s(t_0)]^\alpha = [(\bar{f}_\alpha)^s(t_0), (\underline{f}_\alpha)^s(t_0)]$.

Now we go through the explanation of a partial derivative of fuzzy function on $E \subset \mathbb{R}^n$. Let $\mathcal{F} : E \to \mathcal{F}_C$, the fuzzy interval , $\mathcal{F}(t) = [\underline{f}(t), \bar{f}(t)]\ \forall\ \alpha \in [0,1]$, is defined as

$$\mathcal{F}_\alpha(t) = [\underline{f}_\alpha(t), \bar{f}_\alpha(t)] = [\underline{f}(\alpha, t), \bar{f}(\alpha, t)].$$

Definition 2.5. *Let* $\mathcal{F}$ *on* $E \subset \mathbb{R}^n$ *and suppose* $t_0 = (t_1^{(0)}, ..., t_n^{(0)})$ *be a fixed element of E.* $k_i(t_i) = \mathcal{F}(t_1^{(0)}, ..., t_i - 1^{(0)}, t_i, t_i + 1^{(0)}, ..., t_n^{(0)})$. *If* k_i *is gHs differentiable at* $t_i^{(0)}$, *then clearly* $\mathcal{F}$ *has the* i^{th} *partial gHs derivative at* t_0 $\Big($ *represented as* $\Big(\frac{\partial^s \mathcal{F}}{\partial t_i}\Big)(t_0)\ \Big)$ *and* $\Big(\frac{\partial^s \mathcal{F}}{\partial t_i}\Big)(t_0) = (k_i)^s(t_i^{(0)})$.

Definition 2.6. *Suppose* $\mathcal{F}$ *is defined on E and assume that* $t_0 \in E$ *be fixed such that* $t_0 = (t_1^{(0)}, ..., t_n^{(0)})$. *Then* $\mathcal{F}$ *is gHs differentiable at* t_0 *if the entire partial gHs derivatives* $\Big(\frac{\partial^s F}{\partial t_1}\Big)(t_0), ..., \Big(\frac{\partial^s F}{\partial t_n}\Big)(t_0)$ *exist on some neighbourhood of* t_0. *Also they are continuous at* t_0.

If $\mathcal{F}$ is *gHs* differentiable at t_0, then $\Big(\frac{\partial^s F}{\partial t_i}\Big)(t_0)$ is a fuzzy interval. Now we define,

$$\Big[\frac{\partial^s F}{\partial t_i}(t_0)\Big]^\alpha = \frac{\partial^s \mathcal{F}}{\partial t_i}(t_0) = \Big[\frac{\partial^s \underline{\mathcal{F}}_\alpha}{\partial t_i}(t_0), \frac{\partial^s \overline{\mathcal{F}}_\alpha}{\partial t_i}(t_0)\Big], \forall\ \alpha \in [0,1].$$

Proposition 2.2. *If* $\mathcal{F} : E \to \mathcal{F}_C$ *is gHs differentiable at* $t_0 \in E$ *then,* $\forall\ \alpha \in [0,1]$, $\underline{f}_\alpha + \bar{f}_\alpha : E \to \mathbb{R}$ *is symmetrically differentiable at* t_0. *Moreover*

$$\frac{\partial^s \underline{\mathcal{F}_\alpha}}{\partial t_i}(t_0) + \frac{\partial^s \overline{\mathcal{F}_\alpha}}{\partial t_i}(t_0) = \frac{\partial^s(\underline{f}_\alpha + \bar{f}_\alpha)}{\partial t_i}(t_0) \tag{2.8}$$

Proof. Follows directly from Theorem 2.2.

Definition 2.7. *The symmetric gradient of* $\mathcal{F} : E \to \mathcal{F}_C$ *at* t_0, $\nabla^s \mathcal{F}(t_0)$, *is defined as*

$$\nabla^s \mathcal{F}(t_0) = \Big(\Big(\frac{\partial^s \mathcal{F}}{\partial t_1}\Big)(t_0), ..., \Big(\frac{\partial^s \mathcal{F}}{\partial t_n}\Big)(t_0)\Big), \tag{2.9}$$

where $(\frac{\partial^s F}{\partial t_j})(t_0)$ *denotes the* j^{th} *partial gHs derivative of* $\mathcal{F}$ *at* t_0.

3. Algebra of gH Symmetric Differentiable Fuzzy Function

Suppose $\mathcal{F}, \mathcal{G} : E \to \mathcal{F}_C$ be two fuzzy valued functions with $\mathcal{F}(t) = [\underline{f}(t), \overline{f}(t)]$ and $\mathcal{G}(t) = [\underline{g}(t), \overline{g}(t)]$. Let σ and η be two real valued functions so that $\sigma : E \to \mathbb{R}$ and $\eta : D \to E$, for some $D \subset \mathbb{R}$. the basic algebraic operations are defined by

$(\mathcal{F} + \mathcal{G})(t) = \mathcal{F}(t) + \mathcal{G}(t) = [\underline{f}(t) + \underline{g}(t), \overline{f}(t) + \overline{g}(t)]$,

$(\mathcal{F} \ominus_{gH} \mathcal{G})(t) = \mathcal{F}(t) \ominus_{gH} \mathcal{G}(t) = [min\{\underline{f}(t) - \underline{g}(t), \overline{f}(t) - \overline{g}(t)\}, max\{\underline{f}(t) - \underline{g}(t), \overline{f}(t) - \overline{g}(t)\}]$,

$(\sigma.\mathcal{F})(t) = \sigma(t).\mathcal{F}(t) = [min\{\sigma(t).\underline{f}(t), \sigma(t).\overline{f}(t)\}, max\{\sigma(t).\underline{f}(t), \sigma(t).\overline{f}(t)\}]$,

$(\mathcal{F} \circ \eta)(t) = [min\{\underline{f}(\eta(t)), \overline{f}(\eta(t))\}, max\{\underline{f}(\eta(t)), \overline{f}(\eta(t))\}]$.

This section examines the characteristics of the algebra of fuzzy functions that are *gHs* differentiable. We specifically investigate the *gHs*-differentiability of $\mathcal{F} + \mathcal{G}$, given that $\mathcal{F}$ and $\mathcal{G}$ are gHs-differentiable and that σ and η are differentiable. In this paper we discuss only the sum of *gHs* differentiable fuzzy function.

3.1. Sum of *gHs* differentiable fuzzy function

Theorem 3.1. *Suppose $\mathcal{F}, \mathcal{G} : E \to \mathcal{F}_C$ be two fuzzy valued functions. If $\mathcal{F}$ and $\mathcal{G}$ are gHs differentiable at t_0 then $\mathcal{F} + \mathcal{G}$ is gHs differentiable at t_0.*

Moreover $(\mathcal{F} + \mathcal{G})'(t) = \mathcal{F}'(t) + \mathcal{G}'(t)$.

Proof. Let $\mathcal{F}$ and $\mathcal{G}$ be two fuzzy valued functions such that $\mathcal{F}(t) = [\underline{f}(t), \overline{f}(t)]$ and $\mathcal{G}(t) = [\underline{g}(t), \overline{g}(t)]$. If $\mathcal{F}$ and $\mathcal{G}$ are *gHs* differentiable fuzzy functions at t_0 then by the properties of lateral derivatives

$(\underline{f} + \underline{g})'_-(t_0), (\underline{f} + \underline{g})'_+(t_0), (\overline{f} + \overline{g})'_-(t_0)$ and $(\overline{f} + \overline{g})'_+(t_0)$ exist and satisfy

$(\underline{f} + \underline{g})'_-(t_0) = (\overline{f} + \overline{g})'_+(t_0), (\underline{f} + \underline{g})'_+(t_0) = (\overline{f} + \overline{g})'_-(t_0)$. Also we have

$(\mathcal{F} + \mathcal{G})'(t_0) = [min\{(\underline{f+g})'_-(t_0), (\underline{f+g})'_+(t_0)\}, max\{(\underline{f+g})'_-(t_0), (\underline{f+g})'_+(t_0)\}]$

$=[(\underline{f+g})'_-(t_0), (\underline{f+g})'_+(t_0)]$.

Thus $\mathcal{F} + \mathcal{G}$ is *gHs* differentiable at t_0.

Theorem 3.2. *Suppose $\mathcal{F}, \mathcal{G} : E \to \mathcal{F}_C$ be two fuzzy valued functions.*
(i) If $\mathcal{F}$ is gHs differentiable at t_0 and $\mathcal{G}$ is gHs differentiable at t_0 then $\mathcal{F} + \mathcal{G}$ is gHs differentiable at t_0.

Proof. Since $\mathcal{F}$ and $\mathcal{G}$ are *gHs* differentiable at t_0, the end point functions $\underline{f}, \overline{f}, \underline{g}, \overline{g}$ are also differentiable at t_0. Thus from the previous results and theorems $\mathcal{F} + \mathcal{G}$ is *gHs* differentiable at t_0 and

$(\mathcal{F}+\mathcal{G})'(t_0) = [min\{(\underline{f})'(t_0)+(\underline{g})'(t_0), (\overline{f})'(t_0)+(\overline{g})'(t_0)\}, max\{(\underline{f})'(t_0)+(\underline{g})'(t_0), (\overline{f})'(t_0)+(\overline{g})'(t_0)\}]$. The *gHs* differentiability of the product for a real-valued function h is now examined. When $\mathcal{F}$ is *gHs* differentiable, the following theorem specifies the prerequisites for $h\mathcal{F}$ to be *gHs* differentiable.

3.2. Product of *gHs* differentiable fuzzy function

Theorem 3.3. *Suppose $\mathcal{F} : E \to \mathcal{F}_C$ be a fuzzy valued functions and h be a real valued function which is gHs differentiable at t_0*
(a) if $\mathcal{F}$ is gHs differentiable at t_0 and $h(t_0).h'(t_0) > 0$ then $h.\mathcal{F}$ is gHs differentiable at t_0
(b) if $\mathcal{F}$ is gHs differentiable at t_0 and $h(t_0).h'(t_0) < 0$ then $h.\mathcal{F}$ is gHs differentiable at t_0.
Moreover $(h.\mathcal{F}'(t_0)) = h(t_0).\mathcal{F}'(t_0) + h'(t_0).\mathcal{F}(t_0)$

Proof. Let $\mathcal{F}(t) = [\underline{f}(t), \overline{f}(t)]$, the product is given by

$$(h.\mathcal{F}(t) = h(t).\mathcal{F}(t)) = \begin{cases} [h(x)\underline{f}(t), h(x)\overline{f}(t)], & if \quad h(x) > 0, \\ [h(x)\overline{f}(t), h(x)\underline{f}(t)], & if \quad h(x) < 0. \end{cases}$$

If $h(t_0) > 0, h'(t_0) > 0$ and $\mathcal{F}$ is *gHs* differentiable then

$$(h.\mathcal{F})'(t_0) = [h'(t_0)\underline{f}(t_0) + h(t_0)(\underline{f})'(t_0), h'(t_0)\overline{f}(t_0) + h(t_0)(\overline{f})'(t_0)]$$

implying that $h.\mathcal{F}$ is *gHs* differentiable. Also we have

$$\begin{aligned} h'(t_0).\mathcal{F}(t_0) + h(t_0).\mathcal{F}'(t_0) &= h'(t_0)[\underline{f}(t_0), \overline{f}(t_0)] + h(t_0)[(\underline{f})'(t_0), (\overline{f})'(t_0)] \\ &= [h'(t_0)\underline{f}(t_0), h'(t_0)\overline{f}(t_0)] + [h(t_0)(\underline{f})'(t_0), h(t_0)(\overline{f})'(t_0)] \\ &= [h'(t_0)\underline{f}(t_0) + h(t_0)(\underline{f})'(t_0), h'(t_0)\overline{f}(t_0) + h(t_0)(\overline{f})'(t_0)] \\ &= (h.\mathcal{F})'(t_0). \end{aligned}$$

If $h(t_0) < 0, h^{'}(t_0) < 0$ and $\mathcal{F}$ is *gHs* differentiable then

$$(h.\mathcal{F})^{'}(t_0) = [h^{'}(t_0)\bar{f}(t_0) + h(t_0)(\bar{f})^{'}(t_0), h^{'}(t_0)\underline{f}(t_0) + h(t_0)(\underline{f})^{'}(t_0)]$$

implying that $h.\mathcal{F}$ is *gHs* differentiable. Also we have

$$\begin{aligned} h^{'}(t_0).\mathcal{F}(t_0) + h(t_0).\mathcal{F}^{'}(t_0) &= h^{'}(t_0)[\underline{f}(t_0), \bar{f}(t_0)] + h(t_0)[(\underline{f})^{'}(t_0), (\bar{f})^{'}(t_0)] \\ &= [h^{'}(t_0)\bar{f}(t_0), h^{'}(t_0)\underline{f}(t_0)] + [h(t_0)(\bar{f})^{'}(t_0), h(t_0)(\underline{f})^{'}(t_0)] \\ &= [h^{'}(t_0)\bar{f}(t_0) + h(t_0)(\bar{f})^{'}(t_0), h^{'}(t_0)\underline{f}(t_0) + h(t_0)(\underline{f})^{'}(t_0)] \\ &= (h.\mathcal{F})^{'}(t_0). \end{aligned}$$

3.3. Composition of *gHs* differentiable fuzzy function

In this section we derive the *gHs* differentiability of the composition of a *gHs* differentiable function and a real valued function.

Theorem 3.4. *Let $\mathcal{F} : E \to \mathcal{F}_C$ be a fuzzy valued functions at y_0, suppose $S \subset \mathbb{R}$ be an open set, Let h be a real valued function differentiable at t_0 so that $h(S) \subseteq E$ and $y_0 = h(t_0)$. Then the composite function $(\mathcal{F} \circ h) = \mathcal{F}(g(t))$ is gHs differentiable at t_0 and $(\mathcal{F} \circ h)^{'}(t_0) = \mathcal{F}^{'}(y_0).h^{'}(t_0)$*

Proof. We assume that $\mathcal{F}$ is *gHs* differentiable at y_0. Then $\underline{f} \circ h$ and $\bar{f} \circ h$ are differentiable at t_0. From the theorem above $(\mathcal{F} \circ h)$ is *gHs* differentiable and

$$\begin{aligned} (\mathcal{F} \circ h)^{'}(t_0) &= [min\{(\underline{f} \circ h)^{'}(t_0), (\bar{f} \circ h)^{'}(t_0)\}, max\{(\underline{f} \circ h)^{'}(t_0), (\bar{f} \circ h)^{'}(t_0)\}] \\ &= [min\{(\underline{f})^{'}(h(t_0)).h^{'}(t_0), (\bar{f})^{'}(h(t_0)).h^{'}(t_0)\}, max\{(\underline{f})^{'}(h(t_0)).h^{'}(t_0), (\bar{f})^{'}(h(t_0)).h^{'}(t_0)\}] \\ &= g^{'}(t_0).[min\{(\underline{f})^{'}(y_0), (\bar{f})^{'}(y_0)\}, max\{(\underline{f})^{'}(y_0), (\bar{f})^{'}(y_0)\}] \\ &= \mathcal{F}^{'}(y_0).h(t_0). \end{aligned}$$

Now we assume that $\mathcal{F}$ is *gHs* differentiable at y_0 and the lateral derivatives $(\underline{f} \circ h)^{'}_{-}(t_0), (\underline{f} \circ h)^{'}_{+}(t_0), (\bar{f} \circ h)^{'}_{-}(t_0), (\bar{f} \circ h)^{'}_{+}(t_0)$ exist. Also

$$(\underline{f} \circ h)^{'}_{-}(t_0) = (\underline{f})^{'}_{-}(h(t_0)).h^{'}(t_0) = (\bar{f})^{'}_{-}(h(t_0)).h^{'}(t_0) = (\bar{f} \circ h)^{'}_{+}(t_0)$$

and

$$(\underline{f} \circ h)^{'}_{+}(t_0) = (\underline{f})^{'}_{+}(h(t_0)).h^{'}(t_0) = (\bar{f})^{'}_{+}(h(t_0)).h^{'}(t_0) = (\bar{f} \circ h)^{'}_{-}(t_0).$$

Therefore $\mathcal{F} \circ h$ is *gHs* differentiable. In addition

$$\begin{aligned} (\mathcal{F} \circ h)^{'}(t_0) &= [min\{(\underline{f} \circ h)^{'}_{-}(t_0), (\bar{f} \circ h)^{'}_{-}(t_0)\}, max\{(\underline{f} \circ h)^{'}_{-}(t_0), (\bar{f} \circ h)^{'}_{-}(t_0)\}] \\ &= [min\{(\underline{f})^{'}_{-}(h(t_0)).h^{'}(t_0), (\bar{f})^{'}_{-}(h(t_0)).h^{'}(t_0)\}, max\{(\underline{f})^{'}_{-}(h(t_0)).h^{'}(t_0), (\bar{f})^{'}_{-}(h(t_0)).h^{'}(t_0)\}] \\ &= g^{'}(t_0).[min\{(\underline{f})^{'}_{-}(y_0), (\bar{f})^{'}_{-}(y_0)\}, max\{(\underline{f})^{'}_{-}(y_0), (\bar{f})^{'}_{-}(y_0)\}] \\ &= \mathcal{F}^{'}(y_0).h(t_0). \end{aligned}$$

Definition 3.1. *The symmetric gradient of $\mathcal{F} : E \to \mathcal{F}_C$ at t_0, $\nabla^s\mathcal{F}(t_0)$, became a p dimensional fuzzy number and is defined as*

$$\nabla^s\mathcal{F}(t_0) = \Big(\Big(\frac{\partial^s\mathcal{F}}{\partial t_1}\Big)(t_0), ..., \Big(\frac{\partial^s\mathcal{F}}{\partial t_n}\Big)(t_0)\Big), \tag{3.1}$$

where $(\frac{\partial^s \mathsf{F}}{\partial t_j})(t_0)$ denotes the j^{th} partial gHs derivative of $\mathcal{F}$ at t_0.

4. *gHs* differentiable functions in fuzzy optimization

The efficient solutions in the crisp multiobjective optimization problem are also stationary points, which can be discovered by reducing the gradient to zero. We can independently determine if each of these stationary positions is an efficient solution from these stationary points. However, we lack suitable definitions of stationary points for issues involving fuzzy multiobjective programming. Furthermore, it is yet to be demonstrated that all viable solutions to a multiobjective fuzzy optimization problem are stationary points. Now, we create a prerequisite for the resolution of p-dimensional fuzzy optimization issues. It is crucial to note that no finding of this kind has ever been obtained in prior study. We begin by defining the fuzzy p-dimensional stationary point:

Definition 4.1. *Let $\mathcal{F} : E \to \mathcal{F}_C{}^p$ be a p dimensional fuzzy function. It is said that $t \in E$ is*

1. *a strongly efficient solution if there exists no $t \in E$ such that $\mathcal{F}(t) \leqq \mathcal{F}(t^*)$ and $\mathcal{F}(t) \neq \mathcal{F}(t^*)$,*
2. *an efficient solution if there exists not $\in E$ such that $\mathcal{F}_j(t) \leqq \mathcal{F}_j(t^*), \forall j = 1, 2, ..., p$ and $\exists k$ such that $\mathcal{F}_k(t) < \mathcal{F}_k(t^*)$,*
3. *a midly weakly efficient solution if there exists no $t \in E$ such that $\mathcal{F}_j(t) \leq \mathcal{F}_j(t^*), \forall j = 1, 2, ..., p$,*
4. *a weakly efficient solution if there exists no $t \in E$ such that $\mathcal{F}(t) < \mathcal{F}(t^*)$.*

The following relations are immediate:

$$\begin{array}{ccc} \textit{efficient} & \Longleftarrow & \textit{strongly efficient} \\ \Downarrow & & \Downarrow \\ \textit{weakly efficient} & \Longleftarrow & \textit{midly weakly efficient.} \end{array}$$

Definition 4.2. *Let $\mathcal{F}$ be a p dimensional gHs differentiable function on E, $t \in E$ is said to be a fuzzy p dimensional stationary point for $\mathcal{F}$, if for every $i = 1, 2, ..., n$ there exist a non-negative matrix*

$$\lambda^i \times \Big[\frac{\partial \mathcal{F}}{\partial t_i}(t^*)\Big]^0 = 0.$$

Proposition 4.1. *Suppose $\mathcal{F}$ be a p dimensional gHs differentiable function on E. If t^* is a weakly efficient solution for $\mathcal{F}$, then the following system has no solution at $y \in \mathbb{R}$, for any $i = 1, 2, ..., n$*

$$y\Big(\frac{\partial \mathcal{F}}{\partial t_i}(t^*)\Big) < 0^p.$$

Theorem 4.1. *Let $\mathcal{F}$ be a p dimensional gHs differentiable function at $t^* \in E$. If t^* is a weakly efficient solution for $\mathcal{F}$, then t^* is a fuzzy p dimensional stationary point for $\mathcal{F}$.*

Proof. If t^* is a weakly efficient solution for $\mathcal{F}$, then

$$\lambda^i \times \Big[\frac{\partial \mathcal{F}}{\partial t_i}(t^*)\Big]^0 = 0$$

has no solution for any $i = 1, 2, ..., p$.

$$y\Big(\frac{\partial \mathcal{F}}{\partial t_i}(t^*)\Big) < 0^p \Leftrightarrow y\Big(\frac{\partial \mathcal{F}_j}{\partial t_i}(t^*)\Big)^\alpha < [0, 0], \forall \alpha \in [0, 1], \forall j = 1, 2, ..., p \Leftrightarrow y\Big(\frac{\partial \mathcal{F}_j}{\partial t_i}(t^*)\Big)^0 < [0, 0],$$
$$\forall j = 1, 2, ..., p.$$

Now, for every $i = 1, 2, ..., n$, let us consider the following linear system $yA_i < 0$ and $yB_i < 0$ where A_i and B_i are

$$A_i = \begin{bmatrix} \frac{\partial \mathcal{F}_{1_0}}{\partial t_i}(t^*)^L \\ . \\ . \\ . \\ \frac{\partial \mathcal{F}_{p_0}}{\partial t_i}(t^*)^L \end{bmatrix}, \ B_i = \begin{bmatrix} \frac{\partial \mathcal{F}_{1_0}}{\partial t_i}(t^*)^U \\ . \\ . \\ . \\ \frac{\partial \mathcal{F}_{p_0}}{\partial t_i}(t^*)^U \end{bmatrix}.$$

If the system $yA_i < 0$ and $yB_i < 0$ has a solution for some $i = 1, 2, ..., n$, then the system

$$\lambda^i \times \Big[\frac{\partial \mathcal{F}}{\partial t_i}(t^*)\Big]^0 = 0$$

has a solution for some i. This is impossible from proposition 4.1. Since $yA_i < 0$ and $yB_i < 0$ is a system of linear inequalities and it has no solution for any i, from known theorem, for

$$A_i^T \alpha_i + B_i^T \beta_i = 0 \Leftrightarrow \sum_{j=1}^{p} [\alpha_{ij}(\frac{\partial \mathsf{F}_{j_0}}{\partial t_i}(t^*)^L) + \beta_{ij}(\frac{\partial \mathsf{F}_{j_0}}{\partial t_i}(t^*)^U)] = 0.$$

By redefining $\Lambda^i = (\alpha_{ij}, \beta_{ij})$, it can be stated that, for every i, there exists $\Lambda^i \in \mathcal{M}^{p \times 2}$ such that $\Lambda^i \times \Big[\frac{\partial \mathsf{F}}{\partial t_i}(t^*)\Big]^0 = 0$. Hence the proof.

Now consider the fuzzy optimization problem, $\forall\ \alpha \in [0, 1]$

$$\min\Big(\underline{f}_\alpha(t), \bar{f}_\alpha(t)\Big),$$
$$\text{subject } \ to \ \ t \in T. \tag{4.1}$$

Lemma 4.1. *Suppose t^* denotes a pareto efficient result for*

$$\begin{aligned} &\min \mathcal{F}(x) \\ subject\ \ to\ \ &g_i(x) \leq 0, i = 1, 2, ..., m \\ &x \in X \subset \mathbb{R}^n \end{aligned} \tag{4.2}$$

$\forall\ \alpha \in [0, 1]$, *then t^* is a non-dominated result for the multiobjective fuzzy optimization problem* (4.1).

Proof. We prove the result by assuming the converse of the statement ie. we assume that t^* is a dominated solution. Then $\exists\ \tilde{t} \in T \mid \mathcal{F}(\tilde{t}) \preceq \mathcal{F}(t^*)$. In otherwords $\exists\ \alpha^*$ such that

$$\underline{f}_{\alpha^*}(\tilde{t}) \leq \underline{f}_{\alpha^*}(t^*),$$
$$\bar{f}_{\alpha^*}(\tilde{t}) \leq \bar{f}_{\alpha^*}(t^*).$$

Definition 4.3. *Let $E \subset \mathbb{R}^n$ and assume that E is convex. Consider the fuzzy function $\mathcal{F}$ on E and $\mathcal{F}$ become convex when*

$$\mathcal{F}(\lambda t^* + (1 - \lambda)x) \leq \lambda\mathcal{F}(t^*) + (1 - \lambda)\mathcal{F}(t)$$

$\forall\ \lambda \in (0, 1)$ *and each $t, t^* \in E$.*

Definition 4.4. *We assume that the constraint function of* (4.2) be fuzzy. Then (4.2) becomes a fuzzy pseudoinvex 2 problem if it satisfies following conditions:

1. $\mathcal{F}$ is *gHs* differentiable.
2. g is symmetrically differentiable on E.

Furthermore $\forall t,\ t^* \in T$, $\exists\ \eta(t^*, t) \in \mathbb{R}^n$ such that

$$\mathcal{F}(t) \leq \mathcal{F}(t^*) \Rightarrow \tilde{\nabla}^s\mathcal{F}(t^*).\eta(t, t^*) \leq 0,$$
$$-\nabla^s g_i(t^*)\eta(t, t^*) \leq 0 \ \ i \in I(t^*),$$

where $I(t^*)$ represents index set of constraints.

Result 4.1. Suppose that optimization problem (4.2) be a fuzzy pseudoinvex 2 on E. Let $\forall\alpha\ \exists$ non negative numbers $\mu_j(\alpha)$, $j = 1, ..., m$, which satisfy the following conditions $\forall\ \alpha \in [0, 1]$

1. $\nabla^s\big(\underline{f}_\alpha + \bar{f}_\alpha\big)(t^*) + \sum_{j=1}^{m} \mu_j(\alpha)\nabla^s g_j(t^*) = 0 \ \ \forall\alpha \in [0, 1],$
2. $\mu_j(\alpha)g_j(t^*) = 0 \ \ \forall\alpha \ \ j = 1, ..., m.$

Then t^* becomes a non-dominated result of optimization problem (4.2).

5. Conclusion

In this paper we defined a new concept called algebra of *gHs* derivative of fuzzy valued functions. We specifically gave conditions for the *gHs* differentiability of the sum, product and the composition of a *gHs* differentiable fuzzy function. Moreover the necessary efficiency criteria are found using a new notion of a *p*-dimensional fuzzy stationary point.

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