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(Dedicated to Professor D. S. Hooda on His 80^{th} Birth Anniversary Celebrations)

GENERALIZED HERMITE-FEJÉR INTERPOLATION BY THE FUNCTION WITH NON-UNIFORM NODES ON THE UNIT CIRCLE

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Abstract

In this research paper, we consider generalized Hermite-Fejér interpolation on the nodes, which are obtained by vertically projected zeros of the $(1+x)P_n^{(\alpha,\beta)}(x)$ on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial. Explicitly representing the interpolatory polynomial as well as establishment of convergence theorem are the key highlights of this paper.

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1. Introduction

Hermite-Fejér interpolation is a classic topic that has been investigated by several researchers over the last few decades. Numerous considerations have been made on the Hermite-Fejr and Hermite interpolation polynomials' convergence characteristics on the Jacobi polynomials' zeros. Hermite-Fejér interpolation finds its importance in various field of mathematics.The Jacobi polynomial-related nodal system plays an important part in the theory of Hermite interpolation. In paper [6], analytical functions defined on open sets including the unit disc are examined for convergence of the Hermite-Fejr interpolation polynomial on the roots of the unity, then results extended to the case of the Hermite interpolation and deal with the convergence both inside and outside the unit disc. The convergence of the HermiteFejr interpolation polynomials for continuous functions defined on the unit circle by using Laurent polynomials has been studied by Daruis and González-vera [8]. Some interpolatory polynomials and their behaviour on different nodal system were studied in [1, 2, 3, 7, 9, 10, 11, 12, 13]. Hermite interpolation with equally spaced nodes on the unit circle was studied by authors [4] and then the authors extended their work in which authors showed analogous related result can be obtained using method given in [5] and [7].

By projecting the zeros of $(1+x)P_n^{(\alpha,\beta)}(x)$ on the unit circle, nodal points are being obtained in the present paper. Hermite-Fejér data are prescribed on zeros of Jacobi polynomial and generalized Hermite-Fejér data are prescribed on $z=-1$. In present paper an explicit representation of interpolation polynomial is given and convergence of it is studied for analytic function within the unit disc.

2. Preliminaries

In this section, we write the following well known results, which will be used in our further investigations.

The differential equation satisfied by $P_n^{(\alpha,\beta)}(x)$ is,

$$(1-x^2)P_n^{(\alpha,\beta)''}(x)+[\beta-\alpha-(\alpha+\beta+2)x]P_n^{(\alpha,\beta)'}(x)+n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)=0, \tag{2.1}$$

where $x=\frac{1+z^2}{2z}$.

$$\mathscr{W}(z)=\prod_{k=1}^{2n}(z-z_k)=K_nP_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right)z^n, \tag{2.2}$$

where

$$K_n=2^{2n}n!\frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)}.$$

$$\mathscr{L}_k(z)=\frac{\mathscr{W}(z)}{(z-z_k)\mathscr{W}'(z_k)} \qquad ,k=1(1)2n. \tag{2.3}$$

For $-1\le x\le 1$ and $\alpha\ge\beta$

$$(1-x^2)^{1/2}\mid P_n^{(\alpha,\beta)}(x)\mid=O(n^{\alpha-1}), \tag{2.4}$$

$$| P_n^{(\alpha,\beta)}(x) |= O(n^{\alpha}). \tag{2.5}$$

Let $x_k = cos\theta_k$, $k = 1(1)n$ be the zeros of $P_n^{(\alpha,\beta)}(x)$, then

$$(1 - x_k^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2}, \tag{2.6}$$

$$| P_n^{(\alpha,\beta)'}(x_k) |\sim k^{-\alpha-\frac{3}{2}} n^{\alpha+2}, \tag{2.7}$$

$$|z_k - 1| = \sqrt{2(1 - x_k)}, \tag{2.8}$$

$$\frac{1}{|z - z_k|} \leq \frac{1 - xx_k}{x - x_k}, \tag{2.9}$$

$$\mathscr{W}'(z_k) \sim K_n k^{-\alpha-\frac{1}{2}} n^{\alpha+1}. \tag{2.10}$$

For more information see [14].

3. The problem and explicit representation of interpolatory polynomial

Let $\{z_k\}_{k=0}^{2n}$ be set of nodes obtained by projecting vertically the zeros of $(1+x)P_n^{(\alpha,\beta)}(x)$ on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial of degree n. Here, we are interested in determining the convergence of interpolatory polynomial $\mathscr{H}_n(z)$ on the above said nodes and satisfying the conditions.

$$\begin{cases} \mathscr{H}_n(z_k) = \alpha_k, & k = 1(1)2n, \\ \mathscr{H}_n'(z_k) = 0, & k = 1(1)2n, \\ \mathscr{H}_n^{(r)}(-1) = 0, & r = 0, 1, 2, ..., m, \end{cases} \tag{3.1}$$

where α_k are arbitrary complex constants and $m < \infty$.

We shall write $\mathscr{H}_n(z)$ satisfying (3.1)

$$\mathscr{H}_n(z) = \sum_{k=1}^{2n} \alpha_k \mathscr{A}_k(z), \tag{3.2}$$

where $\mathscr{A}_k(z)$ is unique polynomial of degree $\leq 4n + m$.

For $k = 1(1)2n$

$$\begin{cases} \mathscr{A}_k(z_j) = \delta_{kj}, & j = 1(1)2n, \\ \mathscr{A}_k'(z_j) = 0 & j = 1(1)2n, \\ \mathscr{A}_k^{(r)}(-1) = 0 & r = 0, 1, 2, ..., m. \end{cases} \tag{3.3}$$

Theorem 3.1. *For $k = 1(1)2n$,*

$$\mathscr{A}_k(z) = (z + 1)^{m+1}\Big[a_k \mathscr{W}(z)\mathscr{L}_k(z) + \frac{\mathscr{L}_k^2(z)}{(z_k + 1)^{m+1}}\Big], \tag{3.4}$$

where

$$a_k(z) = -\frac{1}{(z_k + 1)\mathscr{W}'(z_k)}\Big[2\mathscr{L}_k'(z_k) + \frac{m + 1}{z_k + 1}\Big].$$

Proof. *For $k = 1(1)2n$, on differentiating* (3.4) *equation and putting $z = z_k$, we get*

$$\mathscr{A}_k'(z_k) =(z_k + 1)^{m+1} a_k \mathscr{W}'(z_k) + 2\mathscr{L}_k'(z_k) + \frac{m + 1}{z_k + 1},$$

$$a_k = -\frac{1}{(z_k + 1)\mathscr{W}'(z_k)}\Big[2\mathscr{L}_k'(z_k) + \frac{m + 1}{z_k + 1}\Big].$$

Hence, the theorem follows.

4. Estimate of fundamental polynomial

We need to calculate estimates in order to obtain the rate of convergence of interpolatory polynomials.

Lemma 4.1. *Let* $\mathscr{A}_k(z)$ *be given by* (3.4)*, then*

$$\sum_{k=1}^{2n} |\mathscr{A}_k(z)| = \begin{cases} \boldsymbol{O}(\log n), & -1 < \alpha \leq -1/2, \\ \boldsymbol{O}(n^{1+2\alpha} \log n), & -1/2 < \alpha < 0, \\ \boldsymbol{O}(n^{1+2\alpha}), & \alpha \geq 0. \end{cases} \tag{4.1}$$

Proof. For $k = 1(1)2n$, consider (3.4)

$$|\mathscr{A}_k(z)| = |(z+1)^{m+1}|[I_1 + I_2], \tag{4.2}$$

where

$$I_1 = |\, a_k\, \omega(z) \mathscr{L}_k(z)\, | \quad \text{and} \quad I_2 = \left| \frac{\mathscr{L}_k^2(z)}{(z_k+1)^{m+1}} \right|.$$

Using (2.1), (2.2) and (2.3), we get

$$I_1 \leq \frac{|w(z)||\mathscr{L}_k(z)|}{|z_k+1|||\omega'(z_k)|} \left\{ \left| \frac{\omega''(z_k)}{\omega'(z_k)} \right| + \left| \frac{m+1}{z_k+1} \right| \right\}. \tag{4.3}$$

Using (2.2), (2.3), (2.7), (2.8) and (2.9), we get

$$I_1 \leq \frac{\sqrt{1-xx_k}}{2\sqrt{1+x_k}(x-x_k)\sqrt{1-x_k^2}\,k^{-2\alpha-2}n^3} \left\{ 2n + \frac{3+2|\alpha|+2|\beta|}{\sqrt{1-x_k^2}} + \frac{|m+1|}{\sqrt{2}\sqrt{1+x_k}} \right\}. \tag{4.4}$$

Using (2.6) and condition $|x - x_k| \geq \sqrt{1-x_k^2}$

$$I_1 \leq \frac{\sqrt{2}}{\delta k^{-2\alpha} n} \left\{ 2n + \frac{(3+2|\alpha|+2|\beta|)n}{k} + \frac{|m+1|}{\sqrt{2}\delta} \right\}, \tag{4.5}$$

where $0 < \delta < \sqrt{1+x_k}$.

Now,

$$I_2 = \left| \frac{\mathscr{L}_k^2(z)}{(z_k+1)^{m+1}} \right|.$$

Using (2.2), (2.3), (2.8), (2.9) and (2.11) we get,

$$I_2 \leq \left(\frac{\sqrt{1-xx_k}}{(x-x_k)nk^{-\alpha-1/2}} \right)^2 \left(\frac{1}{2(x_k+1)} \right)^{\frac{m+1}{2}}. \tag{4.6}$$

Using (2.6) and condition $|x - x_k| \geq \sqrt{1-x_k^2}$

$$I_2 \leq \frac{1}{2^{\frac{m-1}{2}} k^{-2\alpha+1} \delta^{\frac{m+1}{2}}}\,. \tag{4.7}$$

From (4.5) and (4.7), we get

$$|\mathscr{A}_k(z)| \leq 2^{m+1} \left[\frac{\sqrt{2}}{\delta k^{-2\alpha} n} \left\{ 2n + \frac{(3+2|\alpha|+2|\beta|)n}{k} + \frac{|m+1|}{\sqrt{2}\delta} \right\} + \frac{1}{2^{\frac{m-1}{2}} k^{-2\alpha+1} \delta^{\frac{m+1}{2}}} \right]. \tag{4.8}$$

After some computations for different ranges of α, the lemma follows.

The estimate remains the same in the case, where $|x - x_k| > \sqrt{1-x_k^2}$.

5. Convergence

Let $f(z)$ be continuous for $|z| \leq 1$, analytic for $|z| < 1$ and $f^{(r+1)} \in Lip\, \nu, \nu > 0$, then

$$\omega_r\Big(f, \frac{1}{n}\Big) = \mathbf{O}(n^{-r-\nu+1}) \quad \{\nu > 2 + 2\alpha - r\}, \tag{5.1}$$

where $\omega_r(f, n^{-1})$ be the r^{th} modulus of continuity of $f(z)$.

For the convergence of interpolatory polynomial, we shall use the following

Remark 5.1. Let $f(z)$ be continous for closed unit disc and analytic for open unit disc. Then, there exists a polynomial $F_n(z)$ of degree $\leq 4n + m$ satisfying **Jackson's** inequality

$$|f(z) - F_n(z)| \leq \begin{cases} \mathbf{O}\Big(\omega\Big(f, \frac{1}{n}\Big)\Big), & -1 < \alpha \leq -1/2, \\ \mathbf{O}\Big(\omega_2\Big(f, \frac{1}{n}\Big)\Big), & -1/2 < \alpha < 0, \\ \mathbf{O}\Big(\omega_r\Big(f, \frac{1}{n}\Big)\Big), & \alpha \geq 0. \end{cases} \tag{5.2}$$

Theorem 5.1. *Let $f(z)$ be continous function on closed unit disc and analytic in open unit disc and let $\mathscr{H}_n(z)$ be polynomial of degree atmost $4n + m$ defined in (3.2), then*

$$|\mathscr{H}_n(z) - f(z)| = \begin{cases} \boldsymbol{O}\Big(\omega\Big(f, \frac{1}{n}\Big)\log n\Big), & -1 < \alpha \leq -1/2, \\ \boldsymbol{O}\Big(n^{1+2\alpha}\,\omega_2\Big(f, \frac{1}{n}\Big)\log n\Big), & -1/2 < \alpha < 0, \\ \boldsymbol{O}\Big(n^{2\alpha+1}\omega_r\Big(f, \frac{1}{n}\Big)\Big), & \alpha \geq 0. \end{cases} \tag{5.3}$$

Proof. Since $F_n(z)$ be the uniquely determined polynomial of degree $\leq 4n + m$ and the polynomial $F_n(z)$ can be expressed as

$$F_n(z) = \sum_{k=0}^{2n-2} F_n(z_k)\mathscr{A}_k(z). \tag{5.4}$$

Then

$$|\mathscr{H}_n(z) - f(z)| \leq |\mathscr{H}_n(z) - F_n(z)| + |F_n(z) - f(z)|,$$

$$\begin{aligned} |\mathscr{H}_n(z) - f(z)| &\leq |\mathscr{H}_n(z) - F_n(z)| + |F_n(z) - f(z)|, \\ &\leq \sum_{k=0}^{2n} |f(z_k) - F_n(z_k)||\mathscr{A}_k(z)| + |F_n(z) - f(z)|. \end{aligned}$$

Using equation (5.1), (5.2) and Lemma 4.1, we have Theorem 5.1.

6. Conclusion

This research article involves an interpolation problem by introducing the generalized Hermite-Fejér boundary condition at point $z = -1$. This additional node increases the degree of interpolatory polynomial. Order of convergence of function depends on the degree of interpolatory polynomial, so we require the r^{th} modulus of continuity for convergence of function. From (5.3), we can analyze that for different values of α, we require different order of modulus of continuity. Since, first modulus of continuity is used for $-1 < \alpha \leq -\frac{1}{2}$, we get better convergence for this range of α as compare to other ranges. Authors suggest an open problem by considering the generalised Hermite-Fejér boundary condition at $z = 1$ as well as $z = \pm 1$. This will offer a much wider perspective on convergence and comparison to the present problem.

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