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*(Dedicated to Professor D. S. Hooda on His $80^{th}$ Birth Anniversary Celebrations)*

# PPF AEPENDENCE AIXED AOINT AESULTS AVER $A^*$-ALGEBRA-VALUED AETRIC APACE

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**Abstract**

In this paper, we prove the existence of *PPF*(Past,Present,Future) dependent fixed point results via using $C^*$-algebra-valued metric spaces. Also we use Banach contraction to find a *PPF* dependent fixed point in the setting of $C^*$-algebra-valued metric spaces.

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**Keywords and Phrases:** *PPF* dependent fixed point, $C^*$-Algebra-valued metric space, Razumikhin class, Algebraically closed.

## 1. Introduction

One of the most fundamental results in fixed point theory is the Banach contraction principle. This approach is used to determine a unique metric fixed point. The result of Banach contraction is significant in developing the fixed point theorem, as well as the uniqueness and existence of fixed points.

Bernfeld et al. [5] introduced the *PPF* (past, present, future) concept, which they defined with several domains and ranges. In the Razumikhin class, they also proposed the concept of Banach type contraction for non-self mappings of *PPF* dependent fixed points. *PPF* dependent fixed point in partially ordered metric spaces was established by Dirci et al [8]. Several mathematicians have induced a different type of mapping and have also produced significant results in the search for a fixed point in *PPF* dependency and also in $C^*$-algebra-valued metric spaces; for more information, see [1, 2, 3, 4, 6, 7, 9, 10, 11, 12, 13, 14, 15].

The idea of *PPF* dependent fixed point in the setting of $C^*$-algebra-valued metric spaces is discussed in this work.

## 2. Preliminaries

Let $X$ be a non-empty set. Suppose that $\mathcal{B}$ is a $C^*$ -algebra valued metric space with the $\|\|_{\mathcal{B}}$ and $[a, b]$ in $\mathbb{R}$. Let $E_0$ be the set of all continuous $C^*$-algebra-valued metric space on $[a, b]$ equipped with the supremum norm $\|.\|_{E_0}$, defined by

$$\|\phi\|_{E_0} = sup_{t\in I}\|\phi(t)\|_{\mathcal{B}}, \tag{2.1}$$

for all $\phi \in E_0$. For a fixed element $c \in [a, b]$, the Razumikhin class of mappings in $E_0$ is defined by

$$R_c = \{\phi \in E_0 : \|\phi\|_{E_0} = \|\phi(c)\|_{\mathcal{B}}\}. \tag{2.2}$$

**Definition 2.1** ([5]). *Let $\mathcal{T} : E_0 \to \mathcal{B}$ be a mapping . A function $\phi \in E_0$ is said to be PPF dependent fixed point of $\mathcal{T}$, if $\mathcal{T}\phi = \phi(c)$ for some $c \in [a, b]$.*

**Definition 2.2** ([5]). *Let $R_c$ be Razumikhin class of continuous functions in $E_0$. we say that*

*(i) The class $R_c$ is algebraically closed with respect to the difference if $\phi - \psi \in R_c$, whenever $\phi, \psi \in R_c$,*
*(ii) the class $R_c$ is topologically closed if it is closed with respect to the topology on $E_0$ by the norm $\|.\|_{E_0}$.*

**Theorem 2.1** ([4]). *Let $R_c$ be the Razumikhin class of functions in $E_0$ and $(\mathcal{B}, \|.\|_{\mathcal{B}})$ is a real banach space it simply $\mathcal{B}$. Then*

*(i) $E_0 = \cup_{c\in[a,b]} R_c$*
*(ii) for any $\phi \in R_c$ and $\alpha \in \mathbb{R}$, we have $\alpha\phi \in R_c$,*
*(iii) the Razumikhin class $R_c$ is topologically closed with respect to the norm defined on $E_0$,*
*(iv) $\cap_{c\in[a,b]} R_c = \{\phi \in E_0/\phi : I \to \mathcal{B} is\ constant\}$.*

*Clearly every constant function from $I$ to $\mathcal{B}$ belongs to $R_c$.*

**Definition 2.3** ([5]). *Let $\mathcal{T} : E_0 \to \mathcal{B}$ be a mapping and if it is called Banach type contraction if there exist $s \in [0,1)$ such that $\|\mathcal{T}\phi - \mathcal{T}\psi\|_{\mathcal{B}} \le s\|\phi - \psi\|_{E_0}$.*

**Definition 2.4** ([14]). *Let $\mathtt{X}$ be a nonempty set. Suppose the mapping $d : \mathtt{X} \times \mathtt{X} \to \mathcal{B}$ satisfies*

*(i) $\theta \preceq d(x,y)$ for all $x, y \in X$ and $d(x,y) = \theta \iff x = y$;*
*(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;*
*(iii) $d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.*

*Then $(\mathtt{X}, \mathcal{B}, d)$ is called $C^*$-algebra -valued metric space.*

**Theorem 2.2** ([5]). *Let $\mathcal{T} : E_0 \to \mathcal{B}$ be a Banach type contraction. Let $R_c$ be algebraically closed with respect to the difference and topologically closed. Then $\mathcal{T}$ has a unique PPF dependent fixed point in $R_c$.*

## 3. Main Results

**Theorem 3.1.** *Let $(\mathtt{X}, \mathcal{B}, d)$ be complete $C^*$-algebra valued metric space and and $E_0$ is set of all continuous $C^*$-valued mapping equipped with $[a,b]$, $a, b \in \mathbb{R}$. A mapping $T : E_0 \to \mathcal{B}$ is non-decreasing . We make the following assumptions:*

*(i) for any $\phi, \psi \in E_0, \|A\|^2 < 1$ and $\phi \preceq \psi$ such that*
*$d(\mathcal{T}\phi, \mathcal{T}\psi) \preceq A^* d(\phi, \psi) A$;*
*(ii) there exists a lower solution $\phi_0$ such that $\phi_0(c) \preceq T\phi_0$ for some $c \in [a,b]$.*

*Then there exists a fixed point $\hat{\lambda}$ such that $\hat{\lambda}(c) = \mathcal{T}\hat{\lambda}, c \in [a,b]$.*

*Proof.* Let us take $\mathcal{T}\phi_0 = x_1$, where $x_1 \in \mathcal{B}$. Choose $\phi_1 \in E_0$ such that $x_1 = \phi_1(c), c \in [a,b]$ and $\phi_1 \succeq \phi_0$. Since $\phi_0$ is a lower solution

$$\implies d(\phi_1(c), \phi_0(c)) = d(\phi_1, \phi_0).$$

Then $\mathcal{T}\phi_0 = \phi_1(c) \preceq \mathcal{T}\phi_1$, since $\mathcal{T}$ is non-decreasing. Again $\mathcal{T}\phi_1 = x_2, x_2 \in B$. Also choose $\phi_1 \in E_0$ such that $x_2 = \phi_2(c), \phi_2 \succeq \phi_1$.

$$\implies d(\phi_2(c), \phi_1(c)) = d(\phi_2, \phi_1).$$

Then $\mathcal{T}\phi_0 = \phi_1(c) \preceq \mathcal{T}\phi_1 = \phi_2(c) \preceq \mathcal{T}\phi_2$, continuing this process, we have a sequence

$$\mathcal{T}\phi_0 = \phi_1(c) \preceq \mathcal{T}\phi_1 = \phi_2(c) \preceq \mathcal{T}\phi_2 \preceq ... \preceq \mathcal{T}\phi_n = \phi_{n+1}(c) \preceq \mathcal{T}\phi_{n+1},$$

$$\phi_0 \preceq \phi_1 \preceq ... \preceq \phi_n \preceq \phi_{n+1} \preceq ... .$$

Also, $d(\phi_n(c), \phi_{n-1}(c)) = d(\phi_n, \phi_{n-1})$.

Assume that $R_c$ is algebraically closed with respect to the difference.

$$\begin{aligned}
d(\phi_{n+1}, \phi_n) &= d(\phi_{n+1}(c), \phi_n(c)), c \in [a,b] \\
&= d(\mathcal{T}\phi_n, \mathcal{T}\phi_{n-1}) \\
&\preceq A^* d(\phi_n . \phi_{n-1}) A \\
&= A^* d(\phi_n(c) . \phi_{n-1}(c)) A \\
&= A^* d(\mathcal{T}\phi_{n-1}, \mathcal{T}\phi_{n-2}) A \\
&\preceq (A^*)^2 d(\phi_{n-1}, \phi_{n-2}) (A)^2 \\
&\ \ \vdots \\
d(\phi_{n+1}, \phi_n) &\preceq (A^*)^n d(\phi_1, \phi_0)(A)^n \\
&= (A^*)^n D(A)^n.
\end{aligned}$$

where $D = d(\phi_1, \phi_0)$. Next, we prove $\mathcal{T}\phi_n$ is Cauchy in $\mathcal{B}$. Let $m > n$

$$\begin{aligned}
d(\phi_m, \phi_n) &= d(\phi_m(c), \phi_n(c)) \\
&= d(\mathcal{T}\phi_m, \mathcal{T}\phi_n) \\
&\preceq d(\mathcal{T}\phi_m, \mathcal{T}\phi_{m-1}) + d(\mathcal{T}\phi_{m-1}, \mathcal{T}\phi_{m-2}) + d(\mathcal{T}\phi_{m-2}, \mathcal{T}\phi_{m-3}) + \ldots + d(\mathcal{T}\phi_{n+1}, \mathcal{T}\phi_n) \\
&\preceq (A^*)^{m-1} D(A)^{m-1} + (A^*)^{m-2} D(A)^{m-2} + (A^*)^{m-3} D(A)^{m-3} + \ldots + (A^*)^n D(A)^n
\end{aligned}$$

$$= \sum_{i=n}^{m-1} (A^*)^i D A^i$$
$$= \sum_{i=n}^{m-1} (A^*)^i D^{1/2} D^{1/2} A^i$$
$$= \sum_{i=n}^{m-1} (A^i D^{1/2})^* (A^i D^{1/2})$$
$$= \sum_{i=n}^{m-1} |A^i D^{1/2}|^2$$
$$\leq \| \sum_{i=n}^{m-1} |A^i D^{1/2}|^2 \| I$$
$$\leq \sum_{i=n}^{m-1} \|D^{1/2}\|^2 \|A^i\|^2 I$$
$$\leq \|D^{1/2}\|^2 \sum_{i=n}^{m-1} \|A^i\|^2 I$$
$$= \|D^{1/2}\|^2 \left[ \|A\|^{2n} + \|A\|^{2(n+1)} + \ldots + \|A\|^{2(m-1)} \right] I$$
$$= \|D^{1/2}\|^2 \left[ \lambda^n + \lambda^{n+1} + \ldots + \lambda^{2(m-1)} \right] \quad \text{where} \quad \lambda = \|A\|^2 I$$
$$= \|D^{1/2}\|^2 \left[ \frac{\lambda^n}{1-\lambda} \right] I \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, it concludes that $\mathcal{T}\phi_n$ is Cauchy sequence. By the completeness of $(\mathtt{X}, B, d)$, we get $\lim_{n\to\infty} \phi_n = \hat{\lambda}$ for some $\phi \in E_0$ and
$$\lim_{n\to\infty} \mathcal{T}\phi_n = \mathcal{T}\hat{\lambda} = \lim_{n\to\infty} \phi_{n+1}(c), c \in [a, b].$$

To prove $\hat{\lambda}$ is a fixed point of $\mathcal{T}$. Also $\mathcal{T}$ is continuous at $\hat{\lambda}$.Given $\frac{\epsilon}{2} > 0$, there exists $\delta > 0$ such that $d(\mathcal{T}\phi_{n+1}, \mathcal{T}\hat{\lambda}) \prec \frac{\epsilon}{2}$ whenever $d\ (\phi_{n+1}, \hat{\lambda}) \prec \delta$. Since $\mathcal{T}\phi_n \to \hat{\lambda}(c)$ for any $r = min\{\frac{\epsilon}{2}, \delta\}$ and there exists $n_0 \in \mathbb{N}$ such that
$$d(\mathcal{T}\phi_n, \hat{\lambda}(c)) \prec r \quad \text{for} \quad n \geq n_0.$$

Thus $d(\mathcal{T}\hat{\lambda}, \hat{\lambda}(c)) \preceq d(\mathcal{T}\hat{\lambda}, \mathcal{T}\phi_n) + d(\mathcal{T}\phi_n, \hat{\lambda}(c)) \prec \frac{\epsilon}{2} + r \prec \epsilon$.

As $\epsilon$ is arbitrary, we get
$$d(\mathcal{T}\hat{\lambda}, \hat{\lambda}(c)) = \theta.$$

Hence $\mathcal{T}\hat{\lambda} = \hat{\lambda}(c)$, $\hat{\lambda}$ is a fixed point of $\mathcal{T}$.

Now suppose $\hat{\gamma} \neq \hat{\lambda}$ another fixed point of $\mathcal{T}$.

Therefore,
$$\begin{aligned} 0 \leq \|d(\hat{\lambda}(c), \hat{\gamma}(c))\| &= \|d(\mathcal{T}\hat{\lambda}, \mathcal{T}\hat{\gamma})\| \\ &\leq \|A^* d(\hat{\lambda}, \hat{\gamma})\| \\ &\leq \|A^*\| \|d(\hat{\lambda}, \hat{\gamma})\| \|A\| \\ &= \|A\|^2 \|d(\hat{\lambda}, \hat{\gamma})\|, \text{where} \|A\|^2 \in (0, 1) \\ &< \|d(\hat{\lambda}, \hat{\gamma})\| \\ \|d(\hat{\lambda}(c), \hat{\gamma}(c))\| &< \|d(\hat{\lambda}, \hat{\gamma})\|. \end{aligned}$$

This is imppossible, since $\hat{\lambda}, \hat{\gamma} \in E_0$ and $R_c$ is Razumikhin class such that $\hat{\lambda} = \hat{\gamma}$.

**Example 3.1.** Consider $\mathtt{X} = \mathbb{R}, E_0 = C([0, 1], \mathbb{R}), \mathcal{B} = M_2(\mathbb{R})$ and $A \in \mathcal{B}^+, \|A\| \leq 1$. Define $\mathcal{T}\phi = \phi(1/2)$ and $\phi(t) = t - \frac{1}{4}, \mathcal{B}^+ = \{\phi \in \mathbb{R}; \phi \geq 0\}$, the metric defined as $d(\phi, \psi) = diag(|\phi - \psi|, \alpha|\phi - \psi|), \alpha > 0$. Therefore $(\mathtt{X}, \mathcal{B}, d)$ is complete.

$$
\begin{aligned}
d(\mathcal{T}\phi,\mathcal{T}\psi) &= A^*\begin{bmatrix}|\mathcal{T}\phi-\mathcal{T}\psi| & 0\\ 0 & \alpha|\mathcal{T}\phi-\mathcal{T}\psi|\end{bmatrix}A\\
&\leq A^*\begin{bmatrix}||\phi(1/2)-\psi(1/2)||_E & 0\\ 0 & \alpha||\phi(1/2)-\psi(1/2)||_E\end{bmatrix}A\\
&\leq A^*\begin{bmatrix}||\phi-\psi||_{E_0} & 0\\ 0 & \alpha||\phi-\psi||_{E_0}\end{bmatrix}A\\
&= A^*d(\phi,\psi)A.
\end{aligned}
$$

Therefore, theorem 3.1 is satisfied and $\dfrac{1}{2}$ is a unique *PPF* dependent fixed point.

**Theorem 3.2.** *Let $(\mathtt{X},\mathcal{B},d)$ be a complete $C^*$-algebra valued metric space and $E_0$ is set of all continuous $C^*$-valued mapping equipped with $[a,b]$, where $a,b\in\mathbb{R}$. A non decreasing mapping $\mathcal{T}:E_0\to\mathcal{B}$ and We make the following assumptions:*

*(i) For any $\phi,\psi\in E_0, ||A||<\frac{1}{2}$ and $\phi\leq\psi$ such that $d(\mathcal{T}\phi,\mathcal{T}\psi)\leq A(d(\phi,\mathcal{T}\psi)+d(\psi,\mathcal{T}\phi))$,*
*(ii) There exist a lower solution $\phi_0$ such that $\phi_0(c)\leq\mathcal{T}\phi_0$ for some $c\in[a,b]$.*

*Then there exist a fixed point $\hat{\lambda}$ such that $\hat{\lambda}(c)=\mathcal{T}\hat{\lambda}, c\in[a,b]$*

*Proof.* Without loss of generality $A\neq 0$. Note that $A\in\mathcal{B}^+$ and $A(d(\phi,\mathcal{T}\psi)+d(\psi,\mathcal{T}\phi))$ is a positive element. Let us take $\mathcal{T}\phi_0=x_1$, where $x_1\in\mathcal{B}$. Choose $\phi_1\in E_0$ such that $x_1=\phi_1(c), c\in[a,b]$ and $\phi_1\geq\phi_0$. Since $\phi_0$ is a lower solution

$$\implies d(\phi_1(c),\phi_0(c))=d(\phi_1,\phi_0).$$

Then $\mathcal{T}\phi_0=\phi_1(c)\leq\mathcal{T}\phi_1$, since $\mathcal{T}$ is non-decreasing. Again $\mathcal{T}\phi_1=x_2, x_2\in B$. Also choose $\phi_1\in E_0$ such that $x_2=\phi_2(c),\phi_2\geq\phi_1$.

Then $\mathcal{T}\phi_0=\phi_1(c)\leq\mathcal{T}\phi_1=\phi_2(c)\leq\mathcal{T}\phi_2$, continuing this process, we get

$$\mathcal{T}\phi_0=\phi_1(c)\leq\mathcal{T}\phi_1=\phi_2(c)\leq\mathcal{T}\phi_2\leq...\leq\mathcal{T}\phi_n=\phi_{n+1}(c)\leq\mathcal{T}\phi_{n+1},$$
$$\phi_0\leq\phi_1\leq...\leq\phi_n\leq\phi_{n+1}\leq... .$$

Therefore $d(\phi_n(c),\phi_{n-1}(c))=d(\phi_n,\phi_{n-1})$.

Assume that $R_c$ is algebraically closed with respect to the difference

$$
\begin{aligned}
d(\phi_{n+1},\phi_n) &= d(\phi_{n+1}(c),\phi_n(c)), c\in[a,b]\\
&= d(\mathcal{T}\phi_n,\mathcal{T}\phi_{n-1})\\
&\leq A(d(\mathcal{T}\phi_n,\phi_n)+d(\mathcal{T}\phi_{n-1},\phi_{n-1}))\\
&\leq\left(\frac{A}{1-A}\right)d(\phi_n,\phi_{n-1}).
\end{aligned}
$$

Since $A\in\mathcal{B}^+$ with $||A||<\dfrac{1}{2}$. Also $\dfrac{A}{1-A}\in\mathcal{B}^+$ and is lessthan 1.

$$d(\phi_{n+1},\phi_n)\leq Hd(\phi_n,\phi_{n-1}),\text{ where } H=\frac{A}{1-A}.$$

On continuing this process we get $d(\phi_{n+1},\phi_n)\leq H^n d(\phi_1,\phi_0)$. To prove $\mathcal{T}\phi_n$ is cauchy sequence in $\mathcal{B}$. Let $n+1>m$. Here $D=d(\mathcal{T}\phi_(c),\mathcal{T}\phi_0(c))$ .

Therefore,

$$
\begin{aligned}
d(\phi_{n+1},\phi_m) &= d(\phi_{n+1}(c),\phi_m(c))\\
&= d(\mathcal{T}\phi_n,\mathcal{T}\phi_{m-1})\\
&\leq (H^{n-1}+H^{n-2}+\cdots+H^m)D\\
&= \sum_{k=m}^{n-1}(H^{k/2}D^{1/2})^*(H^{k/2}D^{1/2})
\end{aligned}
$$

$$= \sum_{k=m}^{n-1} |H^{k/2} D^{1/2}|^2$$

$$\leq \|D^{1/2}\|^2 \sum_{k=m}^{n-1} \frac{\|H\|^m}{1 - \|H\|} I \to 0 \text{ as } m \to \infty.$$

By the completeness of $(\mathtt{X}, B, d)$, then we get $\lim_{n\to\infty} \phi_n = \hat{\lambda}$ for some $\phi \in E_0$

$$\lim_{n\to\infty} \mathcal{T}\phi_n = \mathcal{T}\hat{\lambda} = \lim_{n\to\infty} \phi_{n+1}(c), c \in [a, b].$$

To prove $\hat{\lambda}$ is a fixed point of $\mathcal{T}$. Also $\mathcal{T}$ is continuous at $\hat{\lambda}$.Given $\frac{\epsilon}{2} > 0$, there exist $\delta > 0$ such that $d(\mathcal{T}\phi_{n+1}, \mathcal{T}\hat{\lambda}) \prec \frac{\epsilon}{2}$ whenever $d\ (\phi_{n+1}, \hat{\lambda}) \prec \delta$. Since $\mathcal{T}\phi_n \to \hat{\lambda}(c)$ for any $r = min\{\frac{\epsilon}{2}, \delta\}$ and there exist $n_0 \in \mathbb{N}$ such that

$$d(\mathcal{T}\phi_n, \hat{\lambda}(c)) \prec r \quad \text{for} \quad n \geq n_0.$$

Thus

$$d(\mathcal{T}\hat{\lambda}, \hat{\lambda}(c)) \prec \epsilon.$$

As $\epsilon$ is arbitrary, we get

$$d(\mathcal{T}\hat{\lambda}, \hat{\lambda}(c)) = \theta.$$

Hence $\mathcal{T}\hat{\lambda} = \hat{\lambda}(c)$, $\hat{\lambda}$ is a fixed point of $\mathcal{T}$.

**Corollary 3.1.** *Let $(X, \mathcal{B}, d)$ be a $C^*$-algebra valued metric space and $E_0$ is set of all continuous $C^*$-valued mapping equipped with $[a, b]$, where $a, b \in \mathbb{R}$. A non decreasing mapping $\mathcal{T} : E_0 \to \mathcal{B}$ and We make the following assumptions:*

*(i) For any $\phi, \psi \in E_0, \|A\| < \frac{1}{2}$ and $\phi \leq \psi$ such that*

$d(\mathcal{T}\phi, \mathcal{T}\psi) \leq A(d(\phi, \mathcal{T}\psi) + d(\psi, \mathcal{T}\phi))$

*(ii) There exists a lower solution $\phi_0$ such that $\phi_0(c) \leq \mathcal{T}\phi_0$ for some $c \in [a, b]$.*

*Then there exists a fixed point $\hat{\lambda}$ such that $\hat{\lambda}(c) = \mathcal{T}\hat{\lambda}, c \in [a, b]$.*

## 4. Conclusion

In this paper we concluded that the *PPF* (past, present, future) dependent fixed point results in $C^*$-algebra valued metric space using different contraction. This paper can be extended to Banach algebra instead of $C^*$-algebra valued metric space.

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## References

[1] M. Asim and M. Imdad, $C^*$- algebra valued symmetric spaces and fixed point results with an application, *The Korean Journal of Mathematics*, **28**(1) (2020), 17-30.

[2] M. Asim and M. Imdad, $C^*$- algebra valued extended $b$-metric spaces and fixed point results with an application, *University Polytechnica of Bucharest Scientific Bulletin*, *Series A* **82**(1) (2020), 207-218.

[3] H. H. Alsulami, R. P. Agarwal, E. Karapinar and F. Khojasteh, A short note on $C^*$-valued contraction mappings,*Journal of Inequalities and Applications*, **50** (2016), 1-3.

[4] G. V. R. Babu, G. Satyanarayana and M. Vinod Kumar, Properties of Razumikhin class of functions and PPF dependent fixed point of Weakly contractive type mappings, *Bulletin of the international mathematical Virtual Institute*, **9**(1) (2019), 65-72.

[5] S. R. Bernfeld, V. Lakshmikantham and Y. M. Reddy, Fixed point theorems of operators with PPF dependence in Banach spaces, *Applicable Analysis*, **6**(4) (1977), 271280.

[6] B. C. Dhage, On some common fixed point theorems with PPF dependence in Banach spaces, *Journal of Nonlinear Science and Its Applications*, **5**(3) (2012), 220232.

[7] B. C. Dhage, Fixed point theorems with PPF dependence and functional differential equations, *Fixed Point Theory*, **13**(2) (2012), 439452.

[8] Z. Drici, F. A. McRae and J. Vasundhara Devi, Fixed-point theorems in partially ordered metric spaces for operators with PPF dependence, *Nonlinear Analysis: Theory, Methods and Applications*, **67**(2) (2007), 641647.

[9] Z. Drici, F. A.McRae, and J.VasundharaDevi, Fixed point theorems for mixed monotone operators with PPF dependence, *Nonlinear Analysis: Theory, Methods and Applications*, **69**(2) (2008), 632636.

[10] A. Farajzadeh, A. Kaewcharoen and Anchalee. (2014). On fixed point theorems for mappings with PPF dependence, *Journal of Inequalities and Applications*, **372** (2014), 1-14.

[11] A. Kaewcharoen, PPF dependent common fixed point theorems for mappings in Banach spaces, *Journal of Inequalities and Applications*, **2013**(287) (2013), 1-14.

[12] Z. Kadelburg and S. Radenovi, Fixed point results in $C^*$ -algebra-valued metric spaces are direct consequences of their standard metric counterparts, *Fixed Point Theory and Applications*, **53** (2016), 1-6.

[13] W. Sintunavarat and P. Kumam, PPF dependent fixed point theorems for rational type contraction mappings in Banach spaces, *Journal of Nonlinear Analysis and Optimization*, **4**(2) (2013), 157162.

[14] Z. Ma, J. Lining and H. Sun, $C^*$-algebra-valued metric spaces and related fixed point theorems, *Fixed Point Theory and Applications*, **206** (1) (2014), 1-11.

[15] N Mlaiki, M Asim and M Imdad, $C^*$-algebra-valued partial b-metric spaces and fixed point results with an application, *Mathematics*, **8**(1381) (2020), 1-11.