ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 52(2) (2022), 250-260**
(Dedicated to Professor D. S. Hooda on His 80^{th} Birth Anniversary Celebrations)

A SEMI-ANALYTIC APPROACH FOR SOLVING FISHER'S REACTION-DIFFUSION EQUATION BY METHOD OF LINES USING REPRODUCING KERNEL HILBERT SPACE METHOD

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(Received: May 13, 2022; In format: August 24, 2022; Revised: November 19, 2022; Accepted: November 22, 2022)

DOI: https://doi.org/10.58250/jnanabha.2022.52229

Abstract

Many nonlinear systems are described with the nonlinear Fisher's reaction-diffusion equation. The purpose of this work is to propose the method of lines to find out the solution of the Fisher's reaction-diffusion equation in one dimension with quadratic and cubic nonlinearity using reproducing kernel Hilbert space method. In this method, the partial derivatives of the space variable are discretized to get a system of *ODEs* in the time variable and then solve the system of *ODEs* using reproducing kernel Hilbert space method. Four test examples are given to demonstrate the technique's efficacy and applicability. The results are compared with the exact and existing numerical solutions by calculating the error norms L_2 and L_∞ at various time levels. It has been discovered that the recommended approach is not only simple to use, but also produces superior outcomes.
2020 Mathematical Sciences Classification: 35G31, 46E22, 65M20.
Keywords and Phrases: Method of Lines, Reproducing kernel Hilbert space method, Fisher's reaction-diffusion equation.

1. Introduction

Partial differential equations (*PDEs*) are used to represent real-world issues in engineering, biology, chemistry, physics, ecology, and other related disciplines of research. Most physical models are nonlinear in nature and are represented by nonlinear *PDEs*, they are extremely significant. Fisher's reaction-diffusion equation is a one-dimensional parabolic nonlinear *PDE* developed by Fisher [18], which was first used to investigate the wave propagation of a advantageous gene in a population. This equation is used in a variety of chemical and biological processes, as well as engineering applications and so on [1, 13, 19]. As a result, studying Fisher's reaction-diffusion equation is an interesting and significant topic of study.

In this article, we consider the following one dimensional Fisher's reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + F(u), \quad (x,t) \in \Omega \times \Gamma, \tag{1.1}$$

where $u = u(x,t)$ represents the concentration of one substance and α is diffusion coefficients and F accounts for all local reactions, subject to the initial condition

$$u(x,0) = \Theta(x), \qquad x \in \overline{\Omega}, \tag{1.2}$$

the Dirichlet boundary conditions

$$u(a,t) = \Phi_1(t), \quad u(b,t) = \Phi_2(t), \qquad t \in \overline{\Gamma}, \tag{1.3}$$

where $\Omega = (a,b)$, $\Gamma = (0,T]$ with $0 < T < \infty$.

Many researchers have presented numerous approaches for solving the nonlinear Fisher's reaction-diffusion equation in the last few decades. Gazdag and Canosa [20] have used a pseudo-spectral method to explain one of the first numerical solutions. For a special wave speed, Ablowitz and Zepetella [2] have found an explicit solution to Fisher's equation. The numerical study of Fisher's equation was described by Parekh and Puri [33]. Carey and Shen [14] have proposed a least-squares finite element formulation for solving Fisher's equation. Mickens [27] has proposed a best finite difference scheme for Fisher's equation. Olmos and Shizgal [32] showed a pseudospectral method numerical examination of Fisher's equation. Mittal and Arora [28] have proposed an efficient numerical solution of Fisher's equation by using B-spline method. Burrage et al. [7] have presented an efficient implicit *FEM* scheme for fractional-in-space reaction-diffusion equations. The numerical solutions of nonlinear Fisher's reaction–diffusion equation with modified cubic B-spline collocation method was established by Mittal and Jain [29]. Shukla and Tamsir [38] have proposed an extended modified cubic B-spline algorithm for nonlinear Fisher's reaction-diffusion equation. Tamsir et. al. [40] presented cubic trigonometric B-spline differential quadrature method for numerical treatment of Fishers reaction-diffusion equations. Tamsir and Huntul [41] used a numerical approach for solving Fisher's reaction–diffusion equation via a new kind of spline functions.

In this paper we present a semi-analytical method of lines (*MOL*) solution. The *MOL* were applied to solve the *PDEs* by Schiesser et al. [39]. This Schiesser's method is a technique based on fully numerical scheme. In 2004, Koto [23] applied this technique to approximations of delay differential equations using Runge-Kutta method. Hamdi et al. [22] gave basic idea of *MOL*. The semi-analytic *MOL* is used actively for solving linear *PDEs*. For example see [34, 35, 36].

In this work a different approach is used, the usual finite difference scheme is employed for spatial discretization in the nonlinear initial boundary valued *PDE* to convert nonlinear initial valued system of Ordinary differential equations (*ODEs*) and then using reproducing kernel Hilbert space method (*RKHSM*) to find solution. The theory of reproducing kernels dates to the first half of the 20th century, and its roots go back to the pioneering papers by S. Zeremba [44], Mercer [30], and Bergman [8, 9, 10, 11, 12]. In 1950, N. Aronszajn [3] outlined the past works and gave a systematic reproducing kernel theory and laid a good foundation for the research of each special case and greatly simplified the proof. This theory has been successfully applied on linear and nonlinear application with different type conditions by many authors [4, 5, 6, 15, 16, 21, 24, 25, 43]. The main idea is to construct the reproducing kernel space absorb the conditions for determining solution of the nonlinear system of *ODEs* [37]. The analytic solution is represented in the form of series. The *RKHSM* is easily implemented, grid - free and without time discretization. Also, we can evaluate the solution for finite number of points and use it often.

The paper is laid out as follows. In the next section, we show how we use *MOL* to solve the Fisher's reaction-diffusion equation. The results of numerical experiments are presented in Section 3. Final Section is dedicated to a brief conclusion. Finally some references are introduced at the end.

2. Method of Lines

In this section, we derive *MOL* to solve the Fisher's reaction-diffusion equation using *RKHSM*. To do this, we divides this section in to the two subsections. In first subsection, we discretize the spatial derivatives in the Fisher's reaction-diffusion equation to obtain system of *ODEs* in time variable. In second subsection, we explain *RKHSM* to solve system of *ODEs*.

2.1. Discretization

To use *MOL* for solving (1.1)-(1.3), we discretize the spatial coordinate x with $m-1$ grid points $x_i = x_{i-1} + h, h = (b-a)/m, x_0 = a, x_m = b, i = 1, 2, \ldots, m-1$. We apply a second order difference approximation for the second derivative in x in grid points x_1, x_{m-1} and a fourth order difference approximation to the second derivative in x in grid points $x_i, i = 2, 3, \ldots, m-2$.

Let us consider $u_i(t)$ approximate $u(x_i, t)$. Here, we are using the central difference approximations in second order and forth order for the second derivative in x of (1.1), we get

$$\begin{aligned}\frac{du_1}{dt} &= \alpha\frac{u_0 - 2u_1 + u_2}{h^2} + F(u_1),\\ \frac{du_{m-1}}{dt} &= \alpha\frac{u_{m-2} - 2u_{m-1} + u_m}{h^2} + F(u_{m-1}),\\ \frac{du_i}{dt} &= \alpha\frac{-u_{i-2} + 16u_{i-1} - 30u_i + 16u_{i+1} - u_{i+2}}{12h^2} + F(u_i),\\ &\quad i = 2, 3, \ldots, m-2.\end{aligned} \tag{2.1}$$

Further the conditions (1.2) and (1.3) becomes

$$\begin{aligned}&u_i(0) = \Theta(x_i), \quad i = 1, 2, 3, \ldots, m-1,\\ &u_0 = \Phi_1(t), \quad u_m = \Phi_2(t).\end{aligned} \tag{2.2}$$

In the next section, we will discuss the *RKHSM* to solve the first order nonlinear system of *ODEs* with homogeneous initial conditions.

2.2. The Reproducing Kernel Hilbert Space Method

In this section, firstly we construct homogeneous initial values system of *ODEs* from the equations (2.1) and (2.2). To do this, we take $u_i = \Theta(x_i)(1-t) + w_i$, $i = 1, 2, \ldots, m-1$, we get

$$\frac{dw_1}{dt} = \Theta(x_1) + \alpha\frac{\Phi_1(t) - 2(\Theta(x_1)(1-t) + w_1) + \Theta(x_2)(1-t) + w_2}{h^2}$$

$$
\begin{aligned}
&+F(\Theta(x_1)(1-t)+w_1),\\
\frac{dw_2}{dt} =&\Theta(x_2)+\alpha\frac{-\Phi_1(t)+16(\Theta(x_1)(1-t)+w_1)-30(\Theta(x_2)(1-t)+w_2)}{12h^2}\\
&+\alpha\frac{16(\Theta(x_3)(1-t)+w_3)-(\Theta(x_4)(1-t)+w_4)}{12h^2}+F(\Theta(x_2)(1-t)+w_2),\\
\frac{dw_i}{dt} =&\Theta(x_i)+\alpha\frac{-(\Theta(x_{i-2})(1-t)+w_{i-2})+16(\Theta(x_{i-1})(1-t)+w_{i-1})}{12h^2}\\
&+\alpha\frac{-30(\Theta(x_i)(1-t)+w_i)+16(\Theta(x_{i+1})(1-t)+w_{i+1})-(\Theta(x_{i+2})(1-t)+w_{i+2})}{12h^2}\\
&+F(\Theta(x_i)(1-t)+w_i), i=3,4,\dots,m-3, \qquad (2.3)\\
\frac{dw_{m-2}}{dt} =&\Theta(x_{m-2})+\alpha\frac{-(\Theta(x_{m-4})(1-t)+w_{m-4})+16(\Theta(x_{m-3})(1-t)+w_{m-3})}{12h^2}\\
&+\alpha\frac{-30(\Theta(x_{m-2})(1-t)+w_{m-2})+16\Theta m-1-\Phi_2(t)}{12h^2}\\
&+F(\Theta(x_{m-2})(1-t)+w_{m-2}),\\
\frac{dw_{m-1}}{dt} =&\Theta(x_{m-1})+\alpha\frac{\Theta(x_{m-2})(1-t)+w_{m-2}-2(\Theta(x_{m-1})(1-t)+w_{m-1})+\Phi_2(t)}{h^2}\\
&+F(\Theta(x_{m-1})(1-t)+w_{m-1}),
\end{aligned}
$$

with homogeneous initial conditions

$$w_i(0)=0, i=1,2,\dots,m-1. \qquad (2.4)$$

Now, we introduce the reproducing kernel Hilbert spaces $\mathcal{W}_2^2[0,T]$ and $\mathcal{W}_2^1[0,T]$ with corresponding reproducing kernel functions $\mathcal{R}(t,s)$ and $\mathcal{G}(t,s)$, respectively, to generate the algorithm of the method to solve the system.

Definition 2.1 ([17])**.** *Consider $\mathcal{H}=\{f(t): f(t)\in\mathbb{R}$ or $f(t)\in\mathbb{C}$, t is in abstract set$\}$ is endowed with $\langle f(t),g(t)\rangle_{\mathcal{H}}$, with respect to which $\mathcal{H}$ is a Hilbert space.*
For an abstract set X, a function $\mathcal{R}(t,s): X\times X\to\mathbb{F}$ ($\mathbb{F}$ denotes $\mathbb{R}$ or $\mathbb{C}$) is called the reproducing kernel of Hilbert space $\mathcal{H}$ if its satisfies,

$$\langle f(t),\mathcal{R}(t,s)\rangle_{\mathcal{H}}=f(s), \qquad (2.5)$$

for each fixed $s\in X$.

The equation (2.5) is known as "the reproducing property".

Definition 2.2 ([31])**.** *The inner product space $\mathcal{W}_2^2[0,T]$ is defined as $\mathcal{W}_2^2[0,T]=\{w: w, w'$ are absolutely continuous real valued functions on $[0,T]$, $w''\in\mathcal{L}^2[0,T]$, and $w(0)=0\}$ with the inner product and the norm of $\mathcal{W}_2^2[0,T]$ are defined, respectively, by*

$$\langle w(t),y(t)\rangle_{\mathcal{W}_2^2}=\sum_{i=0}^{1}\frac{d^iw(0)}{dt^i}\frac{d^iy(0)}{dt^i}+\int_0^T\frac{d^2w(t)}{dt^2}\frac{d^2y(t)}{dt^2}dt,$$

$$\|w\|_{\mathcal{W}_2^2}=\sqrt{\langle w(t),w(t)\rangle_{\mathcal{W}_2^2}}.$$

Theorem 2.1 ([31])**.** *The Hilbert space $\mathcal{W}_2^2[0,T]$ is a complete reproducing kernel and its reproducing kernel function $\mathcal{R}(t,s)$ can be written as*

$$\mathcal{R}(t,s)=\begin{cases}\frac{s}{6}(6t+3ts-s^2), & s\le t,\\ \frac{t}{6}(6s+3ts-t^2), & s>t.\end{cases}$$

Definition 2.3 ([26])**.** *The inner product space $\mathcal{W}_2^1[0,T]$ is defined as $\mathcal{W}_2^1[0,T]=\{w: w$ are absolutely continuous real valued functions on $[0,T]$, $w'\in\mathcal{L}^2[0,T]\}$ with the inner product and the norm of $\mathcal{W}_2^1[0,T]$ are defined, respectively, by*

$$\langle w(t),y(t)\rangle_{\mathcal{W}_2^1}=w(0)y(0)+\int_0^T\frac{dw(t)}{dt}\frac{dy(t)}{dt}dt,$$

$$\|w\|_{\mathcal{W}_2^1}=\sqrt{\langle w(t),w(t)\rangle_{\mathcal{W}_2^1}}.$$

Theorem 2.2 ([26])**.** *The Hilbert space* $\mathcal{W}_2^1[0,T]$ *is a complete reproducing kernel and its reproducing kernel function* $\mathcal{G}(t,s)$ *can be written as*

$$\mathcal{G}(t,s) = \begin{cases} 1+s, & s \le t, \\ 1+t, & s > t. \end{cases}$$

Now, we develop the differential linear operator in the space $\mathcal{W}_2^2[0,T]$ for the system (2.3). We interpret a differential operator for $Lw_i(t) = \frac{dw_i}{dt}$, $i = 1,2,\dots,m-1$ as

$$L : \mathcal{W}_2^2[0,T] \to \mathcal{W}_2^1[0,T],$$

such that, we converted (2.3) and (2.4) into the form,

$$Lw_i = f_i(t, w_1(t), w_2(t), \dots, w_{m-1}(t)), 0 < t < T,$$

subject to the initial conditions,

$$w_i(0) = 0, i = 1,2,\dots,m-1,$$

where $w_i(t) \in \mathcal{W}_2^2[0,T]$ and $f_i(t, w_1(t), w_2(t), \dots, w_{m-1}(t)) \in \mathcal{W}_2^1[0,T]$.

Theorem 2.3. *The operator* $L : \mathcal{W}_2^2[0,T] \to \mathcal{W}_2^1[0,T]$ *is bounded and linear.*

Proof. For the proof, we refer to [17]. Now, we consider orthogonal function in the form $\psi_j(t) = L^*\mathcal{G}(t,t_j)$, $j = 1,2,3,\dots$, where $\{t_j\}_{j=1}^{\infty}$ is dense on $[0,T]$ and L^* is the adjoint operator of L. Like, $\langle w_i(t), \psi_j(t)\rangle_{\mathcal{W}_2^2} = \langle w_i(t), L^*\mathcal{G}(t,t_j)\rangle_{\mathcal{W}_2^2} = \langle Lw_i(t), \mathcal{G}(t,t_j)\rangle_{\mathcal{W}_2^1} = Lw_i(t_j)$, $j = 1,2,3,\dots$, i=1,2,...,m-1. Since, $\psi_j(t) = L^*\mathcal{G}(t,t_j) = \langle L^*\mathcal{G}(t,t_j), \mathcal{R}(t,s)\rangle_{\mathcal{W}_2^2} = \langle \mathcal{G}(t,t_j), L_s\mathcal{R}(t,s)\rangle_{\mathcal{W}_2^1} = \langle L_s\mathcal{R}(t,s), \mathcal{G}(t,t_j)\rangle_{\mathcal{W}_2^1} = L_s\mathcal{R}(t,s)|_{s=t_j}$, $j = 1,2,3,\dots$.. Thus, $\psi_j(t)$ can be evaluated by $\psi_j(t) = L_s\mathcal{R}(t,s)|_{s=t_j}$, $j = 1,2,3,\dots$.

Theorem 2.4. *If* $\{t_j\}_{j=1}^{\infty}$ *is dense on* $[0,T]$, *then* $\{\psi_j(t)\}_{j=1}^{\infty}$ *is a complete system of the space* $\mathcal{W}_2^2[0,T]$.

Proof. For the proof, we refer to [24].

Now, we will derive the method of analytical solution of the equations (2.3) and (2.4) in the reproducing kernel Hilbert space $\mathcal{W}_2^2[0,T]$.

Since, $\psi_j(t) = L^*\mathcal{G}(t,t_j)$, where L^* is the adjoint operator of L. The orthonormal system $\overline{\psi}_j(t)$ of $\mathcal{W}_2^2[0,T]$, $j = 1,2,3,\dots$ can be constructed from Gram-Schmidt orthogonalization process of $\psi_j(t)$, $j = 1,2,3,\dots$ with orthogonal coefficients β_{jk}, as,

$$\overline{\psi}_j(t) = \sum_{k=1}^{j} \beta_{jk}\psi_k(t), j = 1,2,3,\dots . \tag{2.6}$$

Using these orthogonal vectors $\overline{\psi}_j(t)$, $j = 1,2,3,\dots$, we shall derive the analytic solution of (2.3) and (2.4) in infinite series form as following theorem.

Theorem 2.5. *If* $\{t_j\}_{j=1}^{\infty}$ *is dense on* $[0,T]$, *then the analytic solution of* (2.3) *and* (2.4) *represented by*

$$w_i(t) = \sum_{j=1}^{\infty}\sum_{k=1}^{j} \beta_{jk} f_i(t, w_1(t), w_2(t), \dots, w_{m-1}(t))\overline{\psi}_j(t), i = 1,2,\dots,m-1. \tag{2.7}$$

Proof. Let $w_i(t)$, $i = 1,2,\dots,m-1$ be the solution of (2.3) and (2.4) in $\mathcal{W}_2^2[0,T]$. Therefore, $\sum_{j=1}^{\infty}\langle w_i(t), \overline{\psi}_j(t)\rangle_{\mathcal{W}_2^2}\overline{\psi}_j(t)$, $i = 1,2,\dots,m-1$ are the expansion of about orthonormal system $\overline{\psi}_j(t)$, and $\mathcal{W}_2^2[0,T]$ is the Hilbert space, then the series $\sum_{j=1}^{\infty}\langle w_i(t), \overline{\psi}_j(t)\rangle_{\mathcal{W}_2^2}\overline{\psi}_j(t)$, $i = 1,2,\dots,m-1$ are convergent in the sense of $\|.\|_{\mathcal{W}_2^2}$. On the other hand, using (2.6), yields

$$\begin{aligned} w_i(t) &= \sum_{j=1}^{\infty}\langle w_i(t), \overline{\psi}_j(t)\rangle_{\mathcal{W}_2^2}\overline{\psi}_j(t) \\ &= \sum_{j=1}^{\infty}\langle w_i(t), \sum_{k=1}^{j}\beta_{jk}\psi_k(t)\rangle_{\mathcal{W}_2^2}\overline{\psi}_j(t) \\ &= \sum_{j=1}^{\infty}\sum_{k=1}^{j}\beta_{jk}\langle w_i(t), \psi_k(t)\rangle_{\mathcal{W}_2^2}\overline{\psi}_j(t) \end{aligned}$$

$$= \sum_{j=1}^{\infty}\sum_{k=1}^{j}\beta_{jk}\langle w_i(t), L^*\mathcal{G}(t,t_k)\rangle_{W_2^2}\overline{\psi}_j(t)$$
$$= \sum_{j=1}^{\infty}\sum_{k=1}^{j}\beta_{jk}\langle Lw_i(t), \mathcal{G}(t,t_k)\rangle_{W_2^1}\overline{\psi}_j(t)$$
$$= \sum_{j=1}^{\infty}\sum_{k=1}^{j}\beta_{jk}\langle f_i(t, w_1(t), w_2(t), \ldots, w_{m-1}(t)), \mathcal{G}(t,t_k)\rangle_{W_2^1}\overline{\psi}_j(t)$$
$$= \sum_{j=1}^{\infty}\sum_{k=1}^{j}\beta_{jk} f_i(t_k, w_1(t_k), w_2(t_k), \ldots, w_{m-1}(t_k))\overline{\psi}_j(\varrho), i = 1,2,\ldots,m-1.$$

Thus, (2.7) is the analytical solution of (2.3) and (2.4).

Hence, the proof is complete.

Here, equation (2.7) is the analytic solution of (2.3) and (2.4). Now, for the solution, we can define initial conditions as $w_{i,0}(t_1) = w_i(t_1)$ and set n-term truncation to $w_i(t), i = 1,2,\ldots,m-1$ by

$$w_{i,n}(t) = \sum_{j=1}^{n} A_j^{\{i\}}\overline{\psi}_j(t), i = 1,2,\ldots,m-1,$$

where the coefficients $A_j^{\{i\}}$ of $\overline{\psi}_j(t)$ are given as

$A_j^{\{i\}} = \sum_{k=1}^{j}\beta_{jk} f_i(t_k, w_{1,k-1}(t_k), w_{2,k-1}(t_k), \ldots, w_{m-1,k-1}(t_k)), i = 1,2,\ldots,m-1.$

3. Numerical Experiments

In this section, we contemplate four nonlinear time-dependent Fisher's reaction-diffusion equation on finite interval are implemented to demonstrate the accuracy and capability of the proposed algorithm, and all of them were performed on the computer using a program written in Matlab. To show the efficiency of the presented scheme we calculate the error norms L_2 and L_∞ as

$$L_2 = \|u_{exact} - u_{MOL}\|_2 = \sqrt{\sum_{i=1}^{m}\left|(u_{exact})_i - (u_{MOL})_i\right|^2},$$
$$L_\infty = \|u_{exact} - u_{MOL}\|_\infty = \max_i\left|(u_{exact})_i - (u_{MOL})_i\right|.$$

Where u_{exact} and u_{MOL} represent exact and MOL solutions respectively.

Example 3.1. To test the MOL in the domain $[-0.2, 0.8]$, we consider the equation (1.1) with $F(u) = -10000u(u-1)$ and constant $\alpha = 1$. Also, the conditions (1.2) and (1.3) are given as

$$\Theta(x) = \frac{1}{\left[1 + e^{\left(\sqrt{\frac{5000}{3}}x\right)}\right]^2}, \Phi_1(t) = \frac{1}{\left[1 + e^{\left(-\sqrt{\frac{5000}{3}}0.2 - \frac{25000}{3}t\right)}\right]^2}, \Phi_2(t) = \frac{1}{\left[1 + e^{\left(\sqrt{\frac{5000}{3}}0.8 - \frac{25000}{3}t\right)}\right]^2}.$$

The analytic solution is given as

$$u(x,t) = \frac{1}{\left[1 + e^{\left(\sqrt{\frac{5000}{3}}x - \frac{25000}{3}t\right)}\right]^2}.$$

***Table* 3.1:** Solution $u(x,t)$ at $t = 0.003$ for Example 3.1.

x	Exact solution	Proposed method	Tamsir and Huntul [41]	Mittal and Jain [29]
-0.1	1.00000	1.00000	1.00000	1.00000
0.1	1.00000	1.00000	1.00000	1.00000
0.2	1.00000	1.00000	1.00000	1.00000
0.3	0.99999	0.99999	0.99999	0.99999
0.4	0.99966	0.99964	0.99963	0.99964
0.5	0.97995	0.97889	0.97816	0.97839
0.6	0.38895	0.36920	0.34762	0.36191
0.7	0.00074	0.00059	0.00048	0.00057

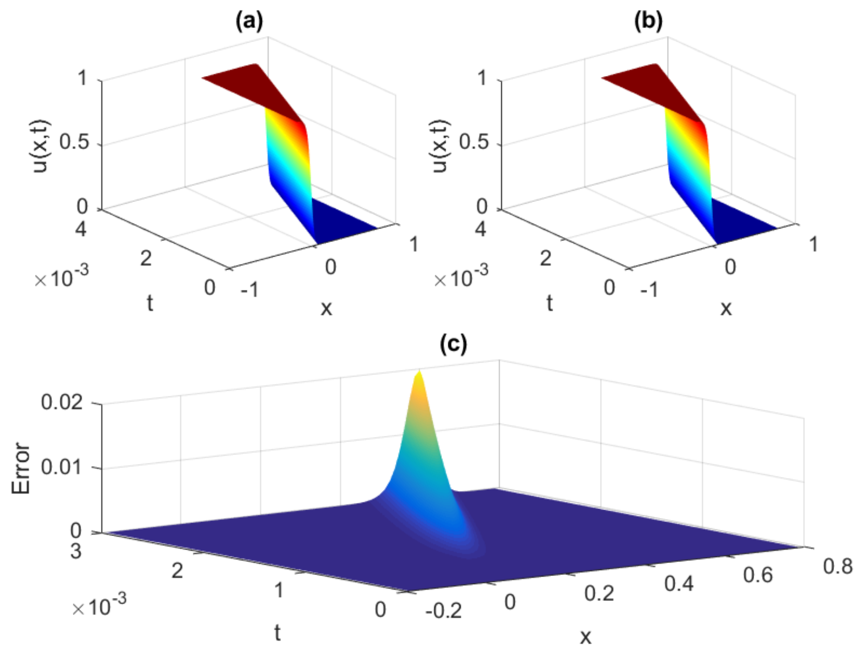

***Figure* 3.1:** The graph of the solution for Example 3.1 with $h = 0.0125$. (a) The analytic solution. (b) The *MOL* solution. (c) The absolute error.

We compare *MOL* solution with some past works in the Table 3.1 with $h = 0.0125$ and $\Delta t = 0.00001$ for the Example 3.1, which gives that proposed method is the most accurate one. The analytic solution, the approximate solution and the absolute errors are displayed in Figures 3.1(a), 3.1(b) and 3.1 (c), respectively.

Example 3.2. In this problem the *MOL* solution of (1.1) is calculated in the computational domain $[-30, 30]$ with $\alpha = 1$ and $F(u) = -u^2 + 0.5u$. Also, the conditions (1.2) and (1.3) are given as

$$\Theta(x) = \frac{-1}{8}\left[\operatorname{sech}^2\left(-\sqrt{\frac{1}{48}}x\right) - 2\tanh\left(-\sqrt{\frac{1}{48}}x\right) - 2\right],$$

$$\Phi_1(t) = \frac{-1}{8}\left[\operatorname{sech}^2\left(30\sqrt{\frac{1}{48}} + \frac{5}{24}t\right) - 2\tanh\left(30\sqrt{\frac{1}{48}} + \frac{5}{24}t\right) - 2\right],$$

$$\Phi_2(t) = \frac{-1}{8}\left[\operatorname{sech}^2\left(-30\sqrt{\frac{1}{48}} + \frac{5}{24}t\right) - 2\tanh\left(-30\sqrt{\frac{1}{48}} + \frac{5}{24}t\right) - 2\right].$$

The analytic solution is given as

$$u(x,t) = \frac{-1}{8}\left[\operatorname{sech}^2\left(-\sqrt{\frac{1}{48}}x + \frac{5}{24}t\right) - 2\tanh\left(-\sqrt{\frac{1}{48}}x + \frac{5}{24}t\right) - 2\right].$$

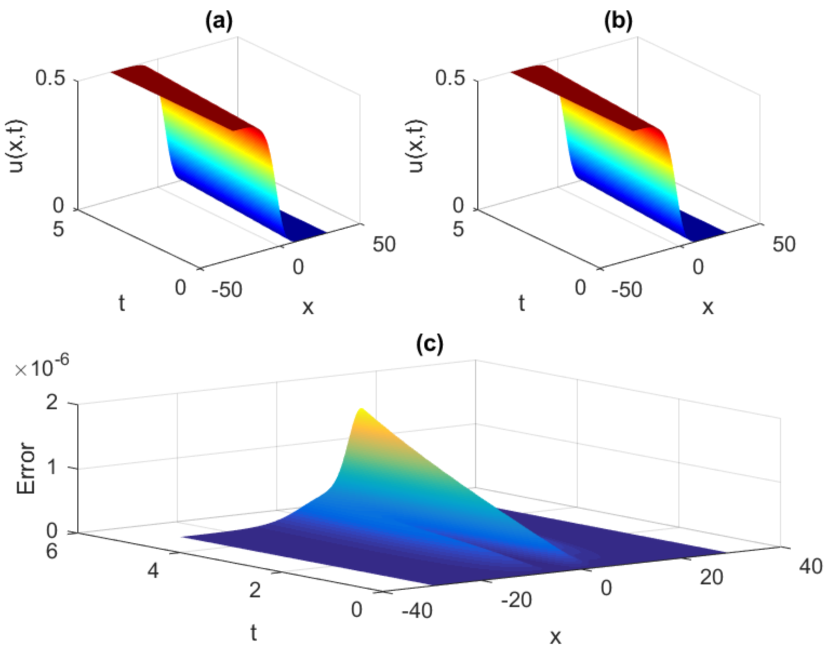

***Figure* 3.2:** The graph of the solution for Example 3.2 with $h = 0.5$. (a) The analytic solution. (b) The *MOL* solution. (c) The absolute error.

***Table* 3.2:** Solution $u(x, t)$ at $t = 2$ for Example 3.2.

x	Exact solution	Proposed method	Tamsir and Huntul [41]	Mittal and Jain [29]
-20	0.498652	0.498652	0.498652	0.498652
-16	0.495740	0.495740	0.495740	0.495741
-12	0.486669	0.486669	0.486669	0.486670
-8	0.459478	0.459477	0.459478	0.459477
-4	0.386791	0.386791	0.386789	0.386787
2	0.158850	0.158849	0.158852	0.158859
6	0.041851	0.041851	0.041851	0.041852
10	0.006465	0.006465	0.006464	0.006462
14	0.000755	0.000755	0.000754	0.000754
18	7.882E-05	7.915E-05	7.915E-05	7.900E-05

The Fisher equation in Example 3.2 is solved with $h = 0.5$ and $\Delta t = 0.01$. The *MOL* results compare with solution given by Tamsir and Huntul [41] ($h = 0.25$) and Mittal and Jain [29] ($h = 0.5$) are shown in Table 3.2. Also, plot of the analytical solution, the *MOL* solution and the absolute error are depicted in Figure 3.2.

Example 3.3. In this problem the *MOL* solution of (1.1) are calculated in the domain $[0, 1]$ with $\alpha = 1$ and $F(u) = u^2 - u^3$. Also, the conditions (1.2) and (1.3) are given as

$$\Theta(x) = 0.5\left(1 - \tanh\left(1 + 0.25\sqrt{2}x\right)\right),$$
$$\Phi_1(t) = 0.5\left(1 - \tanh\left(1 - 0.25t\right)\right), \Phi_2(t) = 0.5\left(1 - \tanh\left(1 + 0.25\sqrt{2} - 0.25t\right)\right).$$

The analytic solution is given as

$$u(x, t) = 0.5\left(1 - \tanh\left(1 + 0.25\sqrt{2}x - 0.25t\right)\right).$$

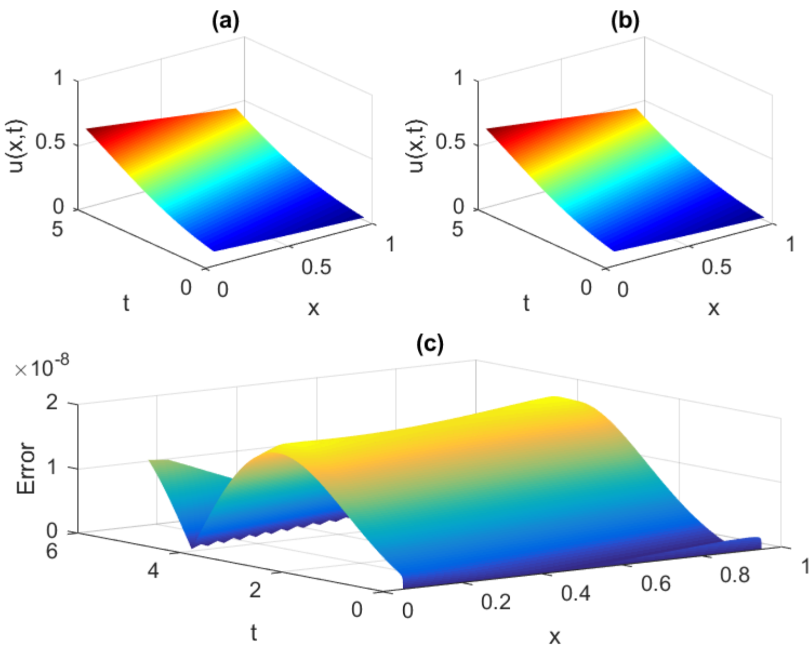

***Figure* 3.3:** The graph of the solution for Example 3.3 with $h = 0.05$. (a) The analytic solution. (b) The *MOL* solution. (c) The absolute error.

***Table* 3.3:** L_2 and L_∞ errors of Example 3.3 with at different time.

t	Proposed method $h = 0.05$		Tamsir and Huntul [41] $h = 0.02$		Verma et. al. [42] $h = 0.02$	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.2	7.185E-09	3.221E-09	0.912E-07	6.823E-07	2.105E-05	4.475E-03
0.5	1.226E-08	5.239E-09	1.004E-06	8.230E-07	1.736E-05	4.444E-03
1.0	2.512E-08	9.248E-09	1.002E-06	9.298E-07	1.103E-05	2.690E-03
3.0	7.480E-08	1.762E-08	1.017E-06	9.064E-07	3.788E-05	2.277E-05
5.0	2.862E-08	1.257E-08	9.987E-08	9.977E-08	2.777E-07	1.823E-07

The problem is solved using the proposed method with $h = 0.05$ and $\Delta t = 0.0001$. The results in terms of L_2 and L_∞ in Table 3.3 demonstrate the accuracy of the proposed method. Also, the graphs of the analytical solution, the *MOL* solution and the absolute error are depicted in Figures 3.3(a), 3.3(b) and 3.3(c), respectively.

Example 3.4. In this problem the *MOL* solution of (1.1) is calculated in the computational domain $[0, 1]$ with $\alpha = 1$ and $F(u) = -u^3 + (1 + c)u^2 - cu$. Also, the conditions (1.2) and (1.3) are given as

$$\begin{aligned}\Theta(x) &= 0.5 + 0.5c + (0.5 - 0.5c)\tanh\left(1 + 0.25\sqrt{2}(-1 + c)x\right),\\ \Phi_1(t) &= 0.5 + 0.5c + (0.5 - 0.5c)\tanh\left(1 - 0.25(c^2 - 1)t\right),\\ \Phi_2(t) &= 0.5 + 0.5c + (0.5 - 0.5c)\tanh\left(1 + 0.25\sqrt{2}(-1 + c)\left(1 - \frac{(1 + c)t}{\sqrt{2}}\right)\right).\end{aligned}$$

The analytic solution is given as

$$u(x, t) = 0.5 + 0.5c + (0.5 - 0.5c)\tanh\left(1 + 0.25\sqrt{2}(-1 + c)\left(x - \frac{(1 + c)t}{\sqrt{2}}\right)\right).$$

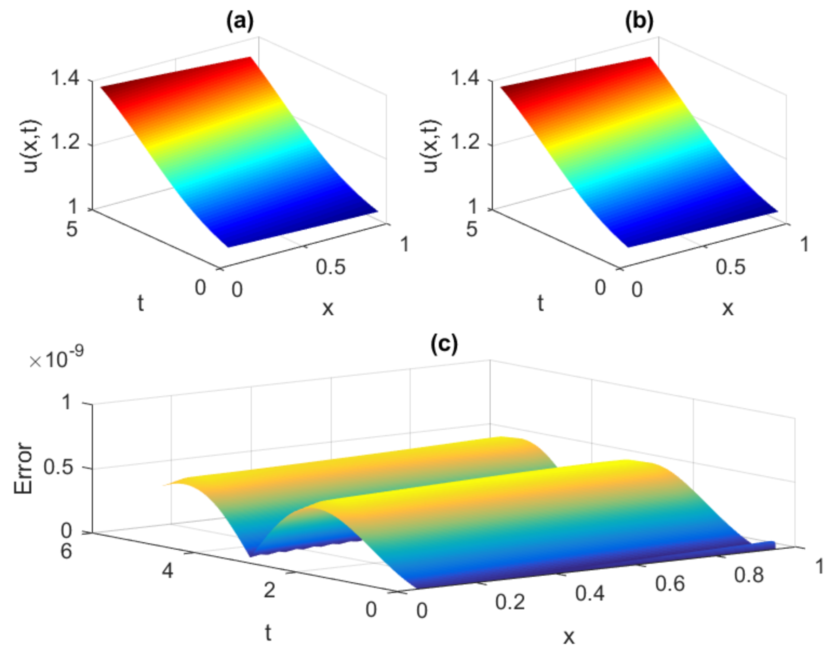

***Figure* 3.4:** The graph of the solution for Example 3.4 with $h = 0.05$. (a) The analytic solution. (b) The *MOL* solution. (c) The absolute error.

***Table* 3.4:** L_2 and L_∞ errors of Example 3.4 with at different time.

t	Proposed method $h = 0.05$		Tamsir and Huntul [41] $h = 0.02$		Verma et. al. [42] $h = 0.02$	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.2	7.185E-09	3.221E-09	0.912E-07	6.823E-07	2.105E-05	4.475E-03
0.5	1.226E-08	5.239E-09	1.004E-06	8.230E-07	1.736E-05	4.444E-03
1.0	2.512E-08	9.248E-09	1.002E-06	9.298E-07	1.103E-05	2.690E-03
3.0	7.480E-08	1.762E-08	1.017E-06	9.064E-07	3.788E-05	2.277E-05
5.0	2.862E-08	1.257E-08	9.987E-08	9.977E-08	2.777E-07	1.823E-07

In Table 3.4, we introduce the L_2 and L_∞ errors between the *MOL* and analytic solutions with $c = 0.5$, $h = 0.05$ and $\Delta t = 0.0001$ for the Example 3.4. Also, a comparison of some past works is given in the table, which gives that proposed method is accurate. The analytic solution, the *MOL* solution and the absolute error of Example 3.4 with $h = 0.05$ are displayed in Figures 3.4(a), 3.4(b) and 3.4(c), respectively.

4. Conclusion

In this study, we proposed an efficient algorithm to solve nonlinear time-dependent Fisher's reaction-diffusion equation. It is analyzed that the proposed method is well suited for use in solution of nonlinear time dependent Fisher's reaction-diffusion equation. The numerical examples presented in this paper show good performance in using *MOL*. High solution accuracy is observed in all problems. It is worth noting that the *MOL* with *RKHSM* is appropriate for solving various nonlinear *PDEs* and can be used to do so.

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