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(Dedicated to Professor D. S. Hooda on His 80^{th} Birth Anniversary Celebrations)

ESSENTIALLY λ-RATIONALIZED TOEPLITZ HANKEL OPERATORS

Ruchika Batra
Ramjas College, University of Delhi, Delhi - 110007, India.
Email: ruchikaverma@ramjas.du.ac.in

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Abstract

We introduce the notion of Essentially λ-Rationalized Toeplitz Hankel operator on the space L^2 for a general complex number λ. Precisely, we define such operators via operator equation $\lambda^{k_2} M_{z^{k_2}} X - X M_{z^{k_1}} = K_1$, where k_1 and k_2 are non zero integers and K_1 is a compact operator on L^2. We investigate some properties of the set λ-$ERTHO(L^2)$, the set of all Essentially λ-Rationalized Toeplitz Hankel operators of order (k_1, k_2).

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1. Introduction

Toeplitz operators and Hankel operators is a subject of investigations for the researchers around the globe. The notion of Toeplitz operator was introduced in the year 1911 by O.Toeplitz [12], however subsequently many mathematicians worked on these operators on different spaces. Along with Toeplitz and Hankel operators, many generalizations have been studied by different mathematicians. Motivated by these Ho [9] in the year 1995 introduced a new class of operators called slant Toeplitz operators on the space $L^2(\mathbb{T})$, $\mathbb{T}$ being the unit circle in the complex plane. Arora and Batra [2] have introduced slant Hankel operators and also generalized the notion of slant Toeplitz operator[1] as k^{th} order slant Toeplitz operator.

After that so much work has been done on these kinds of operators on different spaces. Sun [11] solved completely the operator equation $U^* X U = \lambda X$ for *a* general complex number λ, where U denotes the unilateral shift and, the solutions of this equation were referred as λ-Toeplitz operators. Later the study of Toeplitz and Hankel operators was also done on the calkin algebra [6], [7] and thereby introducing the notion of essentially Toeplitz and essentially Hankel operators

We introduced the k^{th} order slant Toeplitz operator on the space L^2 [1] which is characterized as the solution of the operator equation $M_z X = X M_{z^k}, k \geq 2$. Then a lot of generalizations of these equations have been studied by researchers. Inspired by all these various variants of Toeplitz and Hankel operators, recently the author [3] has generalized the notion of all kinds of Toeplitz, Hankel, slant Toeplitz and slant Hankel operators and a Rationalization of all such kinds of operators is introduced as the Rationalized Toeplitz Hankel operator $RTHO(L^2)$ on the space L^2. Precisely, we define [3] for $\varphi \in L^\infty$, a Rationalized Toeplitz Hankel operator on the space L^2 of order (k_1, k_2) as $R_\varphi : L^2 \to L^2$, $R_\varphi(f) = W_{k_1} M_\varphi W^*_{k_2}(f)\ \forall\ f \in L^2$
where k_1 and k_2 are non zero integers and

$$W_k(z^i) = \begin{cases} z^{i/k} & if\ i\ is\ divisible\ by\ k \\ 0 & otherwise\ . \end{cases}$$

It is also proved in [3] that if k_1 and k_2 are relatively prime then a bounded linear operator on L^2 is a Rationalized Toeplitz Hankel operator if and only if $M_{z^{k_2}} R = R M_{z^{k_1}}$. Further if k_1 and k_2 are not relatively prime and if $d = g.c.d(k_1, k_2)$. i.e. $k_1 = dn$ and $k_2 = dm$ then a bounded operator R on L^2 is Rationalized Toeplitz Hankel operator if and only if

$$R|\widetilde{N}_i = W_m M_{\tilde{\varphi}_i} W^*_n |\tilde{N}_i\ ,$$

for some $\tilde{\varphi}_i$ in L^∞ and for $i = 1, 2, ..., d-1$

$$L^2 = \tilde{N}_0 \oplus \tilde{N}_1 \oplus ... \oplus \tilde{N}_{d-1}\ ,$$

where

$$\begin{aligned}\tilde{N}_0 &= \tilde{N}_0 \oplus \tilde{N}_1 \oplus ...\tilde{N}_{m-1} \ ,\\ \tilde{N}_1 &= \tilde{N}_m \oplus \tilde{N}_{m+1}...\tilde{N}_{2m-1} \ ,\\ \vdots &= \vdots \\ \tilde{N}_{d-1} &= \tilde{N}_{(d-1)m} \oplus \tilde{N}_{(d-1)m+1} \oplus ...\tilde{N}_{dm-1} \ .\end{aligned}$$

N_i = The closed linear span of $\{z^{k_1 t+1} : t \in \mathbb{Z}\}$.
Further in [4], we introduced a class of operators on L^2 as Essentially Rationalized Toeplitz Hankel operators of order (k_1, k_2) satisfying the operator equation $M_{z^{k_2}}X - XM_{z^{k_1}} = K_0$, for some compact operator K_0 on L^2.
Also λ-Rationalized Toeplitz Hankel operators [5] were introduced and studied for a fixed complex number λ, $\lambda \neq 0$ as the class of operators on L^2 satisfying $\lambda^{k_2} M_{z^{k_2}} X = XM_{z^{k_1}}$.
Our focus in this paper is to introduce and study the class of operators on the space L^2 satisfying the operator equation $\lambda^{k_2} M_{z^{k_2}} X - XM_{z^{k_1}} \in K$ for a fixed complex number $\lambda \neq 0$ and K, denote the set of all compact operators on L^2.

2. Essentially λ-Rationalized Toeplitz Hankel operators

It is known [3] that a bounded linear operator R on the space L^2 is the Rationalized Toeplitz Hankel operator of order (k_1, k_2) if and only if $M_{z^{k_2}}R = RM_{z^{k_1}}$.
Also we have seen [5] that for a fixed complex number λ and non zero integers k_1 and k_2, a λ-Rationalized Toeplitz Hankel operator is solution of the equation

$$\lambda^{k_2} M_{z^{k_2}} X = XM_{z^{k_1}} \ .$$

We introduce the following
Definition 2.1. *A bounded linear operator R on the space L^2 is said to be an Essentially λ-Rationalized Toeplitz Hankel operator of order (k_1, k_2) if*

$$\lambda^{k_2} M_{z^{k_2}} R - RM_{z^{k_1}} \in K \ .$$

That is,$\lambda^{k_2} M_{z^{k_2}} R - RM_{z^{k_1}} = K_0$ for some compact operater K_0 on L^2. The class of all Essentially λ-Rationalized Toeplitz Hankel operators on L^2 of order (k_1, k_2) is denoted as *λ-ERTHO*(L^2).
We observe that if $\lambda = 1$ then the set coincides with *ERTHO*(L^2) the set of all essentially Rationalized Toeplitz Hankel operaters on L^2.
Observation 2.1. If R is a λ-Rationalized Toeplitz Hankel operator of order (k_1, k_2) then it must satisfy $\lambda^{k_2} M_{z^{k_2}} R = RM_{z^{k_1}}$ and therefore $\lambda^{k_2} M_{z^{k_2}} R - RM_{z^{k_1}} = 0$ As zero operator is compact, so we conclude that R is Essentially λ-Rationalized Toeplitz Hankel operator. Thus, every λ-Rationalized Toeplitz Hankel operator is a member of *λ-ERTHO*(L^2). Infact any compact perturbation of a λ-Rationalized Toeplitz Hankel operator on L^2 is in *λ-ERTHO*(L^2).
Observation 2.2. As every compact operator is in *λ-ERTHO*(L^2), therefore we observe that *λ-ERTHO*$(L^2) \cap K = K$.
As we have observed earlier that every λ-Rationalized Toeplitz Hankel operator is a member of *λ-ERTHO*(L^2), however the following example shows that the converse need not be true.
Example 2.1. For a complex number λ with unit modulus, let $D_{\bar{\lambda}}$ be the composition operator on L^2 defined as $D_{\bar{\lambda}} f(z) = f(\lambda z)$ for all $f \in L^2$ as

$$T = D_{\bar{\lambda}} W_{k_1} M_{z^{k_2}} + K'$$

where $W_k : L^2 \longrightarrow L^2$ be defined as

$$W_k(z^i) = \begin{cases} z^{i/k} & if\ i\ is\ divisible\ by\ k \\ 0 & otherwise \end{cases}$$

and k_1 and k_2 be non zero integers and K is defined on L^2 as

$$K' e_n = \begin{cases} e_1\ if n = 0 \\ 0\ otherwise\ . \end{cases}$$

Then T is Essentially λ-Rationalized Toeplitz Hankel operator however it is not λ-Rationalized Toeplitz Hankel operator. For this, we consider
$\lambda^{k_2} M_{z^{k_2}} T - TM_{z^{k_1}}$
$= \lambda^{k_2} \bar{\lambda}^{k_2} D_{\bar{\lambda}} M_{z^{k_2}} W_{k_1} M_{z^{k_2}} - D_{\bar{\lambda}} W_{k_1} M_{z^{k_2+k_1}} + K_1$
$= |\lambda|^{2k} D_{\bar{\lambda}} (M_{z^{k_2}} W_{k_1}) M_{z^{k_2}} - D_{\bar{\lambda}} W_{k_1} M_{z^{k_1+k_2}} + K_1$
$= K_1$, where K_1 is a non zero compact operator.
Hence $\lambda^{k_2} M_{z^{k_2}} T - TM_{z^{k_2}} \in K$. Therefore T is an essentially λ-Rationalized Toeplitz Hankel operator on L^2 which is

not a λ-Rationalized Toeplitz Hankel operator.
Further as we have seen the following results in [5].
1) X is a solution of the equation $\lambda^{k_2} M_{z^{k_2}} X = X M_{z^{k_1}}, |\lambda| \neq 1$ Then $X = 0$
2) For $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, the equation $\lambda^{k_2} M_{z^{k_2}} X = X M_{z^{k_1}}$ admits of a non zero solution and each non zero solution is of the form $X = D_{\bar{\lambda}} R_{\varphi}$, where $D_{\bar{\lambda}}$ is the composition operator on L^2 defined as $D_{\bar{\lambda}} f(z) = f(\lambda z)$ for all $f \in L^2$ and $R_{\varphi}, \varphi \in L^{\infty}$ is the Rationalized Toeplitz Hankel operator.
we begin with the following

Theorem 2.1. *If $B(L^2)$ is the set of all bounded linear operators on the space L^2. Then λ-ERTHO(L^2) is a norm closed vector subspace of $B(L^2)$ for a fixed complex number λ.*

Proof. Let T_1, T_2 be in λ-*ERTHO*(L^2) and let α, β are complex numbers Then consider $\alpha T_1 + \beta T_2$ we get

$$\lambda^{k_2} M_{z^{k_2}} (\alpha T_1 + \beta T_2) - (\alpha T_1 + \beta T_2) M_{z^{k_1}}$$
$$= \alpha(\lambda^{k_2} M_{z^{k_2}} T_1 - T_1 M_{z^{k_1}}) + \beta(\lambda^{k_2} M_{z^{k_2}} T_2 - T_2 M_{z^{k_1}}) \in K$$

$\Rightarrow \alpha T_1 + \beta T_2 \in K$ Now for each n if T_n is in λ-*ERTHO*(L^2) then

$$\lambda^{k_2} M_{z^{k_2}} T_n - T_n M_{z^{k_1}} \to \lambda^{k_2} M_{z^{k_2}} T - T M_{z^{k_1}}$$

uniformly in $B(L^2)$. This implies that $T \in \lambda$-*ERTHO*(L^2) as K is uniformly closed. Hence λ-*ERTHO*(L^2) is a norm closed vector subspace of $B(L^2)$.

As example 2.1 ensures that the set of λ-*RTHO*(L^2) is properly contained in the set λ-*ERTHO*(L^2). As T is a non compact operator on L^2, so we can assert that λ-*ERTHO*(L^2) is a proper superset of λ-*RTHO*(L^2) . The following theorem determines the intersection of two classes of essentially λ-Rationalized Toeplitz Hankel operators.

Theorem 2.2. *If λ and μ are two complex numbers such that $\lambda \neq \mu$ then the intersection λ-ERTHO(L^2) $\cap$ μ-ERTHO(L^2) $= K$.*

Proof. Let $T \in \lambda$-*ERTHO*(L^2) $\cap$ μ-*ERTHO*(L^2) Then $\lambda^{k_2} M_{z^{k_2}} T - T M_{z^{k_1}}$ and $\lambda^{k_2} M_{z^{k_2}} T - T M_{z^{k_1}}$ are both compact operators on L^2. Therefore $(\lambda^{k_2} - \mu^{k_2}) M_{z^{k_2}} T$ is a compact operator. Since $\lambda \neq \mu$, so this implies that T is a compact operator. The Converse inclusion is trivial. Thus we get λ-*ERTHO*(L^2) $\cap$ μ-*ERTHO*(L^2) $= K$.

One can also show with trivial computations that if $|\lambda| \neq |\mu|$ and if λ-*ERTHO*(L^2) is the set of all such operators of order (k_1, k_2) and μ-*ERTHO*(L^2) is the set of all such operators of order (m, n) where $(k_1, k_2) \neq (m, n)$ then λ-*ERTHO*(L^2) $\cap$ μ-*ERTHO*(L^2) $= K$.

Remark. However, we can see that in general the product of two essentially λ-Rationalized Toeplitz Hankel opeators is not an Essentially λ-Rationalized Toeplitz Hankel operator, therefore for λ-*ERTHO*(L^2) is not an algebra of operators on L^2 and also it is not a self-adjoint set.

Example 2.2. Let λ be a complex number with $|\lambda| = 1$. Consider the operator T on the space L^2 defined as

$$T = D_{\bar{\lambda}} W_{k_1} M_{z^{k_2}} + K' \ ,$$

where $D_{\bar{\lambda}}, W_K$ and K' are defined in Example 2.1. We have proved in example 1 that $T \in \lambda$-*ERTHO*(L^2). Here we claim that $T^2 \notin \lambda$-*ERTHO*(L^2). As if $T^2 \in \lambda$-*ERTHO*(L^2) then the operator $\lambda^{k_2} M_{z^{k_2}} T^2 - T^2 M_{z^{k_1}}$, must be a compact opreator on L^2. This implies that $\lambda^{k_2} M_{z^{k_2}} (D_{\bar{\lambda}} W_{k_1} M_{z^{k_2}})^2 - (D_{\bar{\lambda}} W_{k_1}, M_{z^{k_2}})^2 M_{z^{k_2}}$ must be a compact operator. But we can check that it in not so. Hence λ-ERTHO(L^2), $|\lambda| = 1$ is not an algebra.
Further this set is also not self adjoint since $T^* \notin \lambda$-*ERTHO*(L^2). Here, we have $T^* = M_{\bar{z}^{k_2}} W_{k_1}^* D_{\lambda}$ and we can check that $\lambda^{k_2} M_{z^{k_2}} T^* - T^* M_{z^{k_1}}$ is not a compact operator on L^2.

As in general, the product of two λ-*ERTHO*(L^2) is not a member of λ-*ERTHO*(L^2) so we attempt to investigate the condition which ensures this to happen in the following theorem.

Theorem 2.3. *If $T_1, T_2 \in \lambda$-ERTHO(L^2) then the product $T_1 T_2 \in \lambda$-ERTHO(L^2) if and only if*

$$T_1 M_{z^{k_1}} T_2 - \lambda^{k_2} T_1 M_{z^{k_2}} T_2 \in K \ .$$

Proof. Let $T_1, T_2 \in \lambda$-*ERTHO*(L^2). Then for the product $T_1 T_2$, consider

$$\begin{aligned} &\lambda^{k_2} M_{z^{k_2}} (T_1 T_2) - (T_1 T_2) M_{z^{k_1}} \\ &\qquad = \lambda^{k_2} M_{z^{k_2}} T_1 T_2 - T_1 T_2 M_{z^{k_1}} \\ &\qquad = T_1 M_{z^{k_1}} T_2 - T_1 T_2 M_{z^{k_1}} (mod\ K) \\ &as\ T_1 \in \lambda - ERTHO(L^2) \\ &\qquad = T_1 M_{z^{k_1}} T_2 - \lambda^{k_2} T_1 M_{z^{k_2}} T_1 (mod\ K) \\ &as\ T_1 \in \lambda - ERTHO(L^2)\ . \end{aligned}$$

This shows that $T_1T_2 \in \lambda$-$ERTHO(L^2)$ if and only if $T_1M_{z^{k_1}}T_2 - \lambda^{k_2}T_1M_{z^{k_2}}T_2 \in K$..
Now if we put $\lambda = 1$, then from here we can draw the conclusion that the product of two Essentially Rationalized Toeplitz Hankel operators is an Esssentially Rationalized Toeplitz Hankel operator if and only if $T_1M_{z^{k_1}}T_2 = T_1M_{z^{k_2}}T_2(mod\ K)$ which is also proved in [4]. In general, we can prove the following.
Theorem 2.4. *Let $T \in \lambda$-$ERTHO(L^2)$ and $r \in \mathbb{N}$, $r > 1$, If $n(r)$ denotes the number of partitions of r as the sum of two natural numbers $r = s_i + t_i$.*

$$(s_i, t_i \in \mathbb{N}, i = 1, 2, ...n(r)).$$

The T^{s_i}, $T^{t_i} \in \lambda$-$ERTHO(L^2)$ Then the following are equivalent
i) $T^r \in \lambda$-$ERTHO(L^2)$,
ii) $T^{s_i}M_{z^{k_1}}T^{t_i} = \lambda^{k_2}T^{s_i}M_{z^{k_2}}T^{t_i}(mod\ K)$,
iii) $T^{t_i}M_{z^{k_1}}T^{s_i} = \lambda^{k_2}T^{t_i}M_{z^{k_2}}T^{s_i}(mod\ K)$.
The following theorem provides a sufficient condition so that the product of any two bounded operators on L^2 lies in the set λ-$ERTHO(L^2)$.
Theorem 2.5. *Let $T_1, T_2 \in B(L^2)$, then the product $T_1T_2 \in \lambda$-$ERTHO(L^2)$ if any one of the following conditions holds*
i) T_1 is in essential commutant of $M_{z^{k_2}}$ and $T_2 \in \lambda$-$ERTHO(L^2)$
ii) $T_1 \in \lambda$-$ERTHO(L^2)$ and T_2 is in essential commutant of $M_{z^{k_1}}$.
Proof. Let $T_1, T_2 \in B(L^2)$ such that $T_1M_{z^{k_1}} - M_{z^{k2}}T_1 \in K$ an $T_2 \in \lambda$-$ERTHO(L^2)$. Consider for the product T_1T_2, the equation

$$\begin{aligned}\lambda^{k_2}M_{z^{k_2}}T_1T_2 - T_1T_2M_{z^{k_1}}& \\ &= \lambda^{k_2}M_{z^{k_2}}T_1T_2 - \lambda^{k_2}T_1M_{z^{k_1}}T_2(mod\ K)\\ &= \lambda^{k_2}M_{z^{k_2}}T_1T_2 - \lambda^{k_2}M_{z^{k_2}}T_1T_2(mod\ K)\\ &= 0(\mathrm{mod} K)\ .\end{aligned}$$

Hence $T_1T_2 \in \lambda$-$ERTHO(L^2)$.
Similarly for the (ii) part
If $T_1 \in \lambda$-$ERTHO(L^2)$ and $T_2M_{z^{k_1}} - M_{z^{k_1}}T_2 \in K$.
Then

$$\begin{aligned}\lambda^{k_2}M_{z^{k_2}}T_1T_2 - T_1T_2M_{z^{k_1}}& \\ &= T_1M_{z^{k_1}}T_2 - T_1T_2M_{z^{k_1}}\\ &= T_1M_{z^{k_1}}T_2 - T_1M_{z^{k_1}}T_2(mod\ K)\\ &= 0(mod\ K)\ .\end{aligned}$$

which gives $T_1T_2 \in \lambda$-$ERTHO(L^2)$.
Corollary 2.1. *For φ in L^∞, if M_φ is the Multiplication operator an L^2 induced by φ and $T \in \lambda$-$ERTHO(L^2)$ then both TM_φ and $M_\varphi T$ are in λ - ERTHO (L^2).*
Finally, we obtain a necessary condition for an essentially λ-Rationalized Toeplitz Hankel operator to be self adjoint.
Theorem 2.6. *If $T, T^* \in \lambda$-$ERTHO(L^2)$ then $ST^* = T^*S^*(mod\ K)$ where $S = \lambda^{k_2}M_{z^{k_2}} + M_{\bar{z}^{k_1}}$.*
Proof. It is given that both T and T^* are members of λ-$ERTHO(L^2)$. So by definition

$$\lambda^{k_2}M_{z^{k_2}}T - TM_{z^{k_1}} = K_1 \tag{2.1}$$

$$\lambda^{k_2}M_{z^{k_2}}T^* - T^*M_{z^{k_1}} = K_2 \tag{2.2}$$

where K_1 and K_2 are compact operaters (ie $K_1, K_2 \in K$). From equation (1) and (2) we get
$(\lambda^{k_2}M_{z^{k_2}}T - TM_{z^{k_1}})^* - (\lambda^{k_2}M_{z^{k_2}}T^* - T^*M_{z^{k_1}}) = K_0$ for some $K_0 \in K$. This yields that if $T, T^* \in \lambda$-$ERTHO(L^2)$,then $ST^* - T^*S^* \in K$, where $S = \lambda^{k_2}M_{z^{k_2}} + M_{\bar{z}^{k_1}}$. With this we have following theorem.
Theorem 2.7. *A necessary condition for any operator $T \in \lambda$-$ERTHO$ (L^2) to be self adjoint is that ST is essentially self adjoint where $S = \lambda^{k_2}M_{z^{k_2}} + M_{\bar{z}^{k_1}}$.*

3. Conclusion

In this paper, we have ensured that the set of all λ-Rationalized Toeplitz Hankel operators is properly contained in the set $\lambda - ERTHO(L^2)$ which is a proper subset of K (the set of all compact operators on L^2). We have also got the conditions which ensure the product of two Essentially λ-Rationalized Toeplitz Hankel operators is again an Essentially λ-Rationalized Toeplitz Hankel operator. In the end, we have obtained a necessary condition for an Essentially λ-Rationalized Toeplitz Hankel operator to be self adjoint.

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