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(Dedicated to Professor D. S. Hooda on His 80^{th} Birth Anniversary Celebrations)

HYBRID CONTRACTION IN WEAK PARTIAL METRIC SPACES

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Abstract

In this paper, a fixed point theorem is established for hybrid contraction in weak partial metric space. Our result is supported by examples.

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Keywords and Phrases: Hybrid contraction mapping, Weak Partial metric space, Partial Hausdroff metric, Coincidence Point.

1. Introduction and Preliminaries

The theory of non-linear analysis has emerged as a fascinating field. Many authors have generalized and extended Banach contraction principle. In 1969, Nadler [7] initiated the study of fixed points for multi-valued contraction mappings using Hausdorff metric.

Let (X, d) be a non-empty metric space and $CB(X)$, the class of all nonempty closed and bounded subsets of X. The Hausdorff metric [3] induced by d on $CB(X)$ is

$$H(A, B) = max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\},$$

for every $A, B \in CB(X)$, where $d(a, B) = inf\{d(a, b); b \in B\}$ is the distance from a to $B \subseteq X$.

Let $f : X \to X$ be a single-valued mapping and $U : X \to CB(X)$ be a multi-valued mapping.

(i) A point $w \in X$ is a fixed point of f (resp. U) if $fw = w$(resp. $w \in U_x$).
The set of all fixed points of f (resp. U) is denoted by $Fix(f)$(resp. $Fix(U)$).

(ii) A point $w \in X$ is a coincidence point of f and U if $fw \in Uw$.
The set of all coincidence points of f and U is denoted by $C(f, U)$.

(iii) A point $w \in X$ is a common fixed point of f and U if $w = fw \in Uw$.
The set of all common fixed points of f and U is denoted by $Fix(f, U)$.

Nadler [7] proved the following

Theorem 1.1 ([7]). *Let (X, d) be a complete metric space and $U : X \to CB(X)$ be a multi-valued mapping satisfying*

$$H(Ux, Uy) \leq kd(x, y), \quad \forall x, y \in X$$

where $k \in [0, 1)$ then $\exists \; x \in X$ such that $x \in Ux$.

Afterward, a rapid progress has been observed using weak and generalized contraction mappings. Multi-valued contraction mapping has many applications in differential equations, control theory and economics.

Singh and Mishra [9] introduced the concept of (IT)- commutativity for a hybrid pair of single-valued and multivalued mappings. Further, in 2004, Kamran [12] introduced the notion of $T-$ weak commutativity for a hybrid pair of single-valued and multivalued maps which is weaker than (IT)- commutativity. The definitions of (IT)-commutativity and T- weak commutativity are as follows ([9]). *A mapping $f : X \longrightarrow X$ and $U : X \longrightarrow CB(X)$ are said to be (IT)- commuting at $w \in X$ if $fUw \subseteq Ufw$.*

Definition 1.1 ([12]). *Let $f : X \longrightarrow X$ and $U : X \longrightarrow CB(X)$, the map f is said to be $T-$ weakly commuting at $w \in X$ if $ffw \in Ufw$.*

On the other hand, the distance notion in the metric fixed point theory has been introduced and generalized in several different ways by many authors. In 1992, Mathews [8] introduced the notion of partial metric space as a part of the study of denotational semantics of data flow networks. He presented a modified version of Banach contraction principle. Several authors have done work in this direction ([4], [2], [6]) .

Definition 1.2. *Let X be a non empty set. Then a mapping $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on X if for all $x, y, z \in X$,*
(P1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;
(P2) $p(x, x) \le p(x, y)$;
(P3) $p(x, y) = p(y, x)$;
(P4) $p(x, y) \le p(x, z) + p(z, x) - p(z, z)$.
The pair (X, p) is called a partial metric space.

Recently, a weaker form of partial metric space is introduced by Ismat Beg and H. K. Pathak [5] known as Weak Partial Metric Space and defined as:
Definition 1.4 ([5]). *Let X be a non empty set. A function $q : X \times X \to \mathbb{R}^+$ is called a weak partial metric on X if for all $x, y, z \in X$, the following conditions hold*:
(WP1) $q(x, x) = q(x, y) \Leftrightarrow x = y$;
(WP2) $q(x, x) \le q(x, y)$;
(WP3) $q(x, y) = q(y, x)$;
(WP4) $q(x, y) \le q(x, z) + q(z, x)$.
The pair (X, q) is a weak partial metric space. Further, many authors have worked on weak partial metric space ([1], [10], [11]).

Example 1.1.

(i) $(\mathbb{R}^+, q)$, where $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ defines as

$$q(x, y) = e^{|x-y|} \ \forall \ \ x, y \in \mathbb{R}^+.$$

(ii) $(\mathbb{R}^+, q)$, where $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ defines as

$$q(x, y) = |x - y| + \frac{1}{3} max\{x, y\} \ \ \forall x, y \in \mathbb{R}^+.$$

Observe that
- ⋄ *If* $q(x, y) = 0$, *then (WP1) and (WP2)* $\Rightarrow x = y$. *But the converse need not be true.*
- ⋄ *(P1) ⇒(WP1), but the converse need not be true.*
- ⋄ *(P4) ⇒ (WP4), but the converse need not be true.*

Each weak partial metric q on X generates a T_0 topology τ_q on X. Topology τ_q has as a base the family of open q-balls $\{B_q(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_q(x, \epsilon) = \{y \in X : q(x, y) < q(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.
If q is weak partial metric on X, then the function $q^s : X \times X \to \mathbb{R}^+$ given by

$$q^s(x, y) = q(x, y) - \frac{1}{2}[q(x, x) + q(y, y)]$$

defines a metric on X.

Definition 1.5 ([5]). *Let (X, q) be a weak partial metric space. Then*
(i) *P is said to be a bounded subset in (X, q) if $\exists \ \ x \in X$ and $L \ge 0$ such that $\forall \ \ p \in P$, we have $p \in B_q(x_0, L)$ that is*

$$q(x_0, p) < q(p, p) + L.$$

(ii) *A sequence $\{x_n\}$ in (X, q) converges to a point $x \in X$, w.r.t. τ_q iff $q(x, x) = \lim_{n\to\infty} q(x, x_n)$. Moreover, a sequence $\{x_n\}$ converges in (X, q^s) to a point $x \in X$ iff*

$$\lim_{n\to\infty m\to\infty} q(x_n, x_m) = \lim_{n\to\infty} q(x_n, x) = q(x, x)$$

(iii) *A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n,m\to\infty} q(x_n, x_m)$ exists and is finite.*
(iv) *(X, q) is called complete if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$ with respect to topology τ_q.*

Lemma 1.1 ([5]). *Let (X, q) be a weak partial metric space. Then*
(a) *A sequence $\{x_n\}$ in X is Cauchy sequence in (X, q) if and only if it is a Cauchy sequence in the metric space (X, q^s).*
(b) *(X, q) is called complete iff the metric space (X, q^s) is complete.*

For $L, M \in CB^q(X)$ *and* $x \in X$ *define* $q(x, L) = inf\{q(x, l) : l \in L\}, \delta_q(L, M) = sup\{q(l, M) : l \in L\}$ *and* $\delta_q(M, L) = sup\{q(m, L) : m \in M\}$.
Clearly $q(x, L) = 0 \Rightarrow q^s(x, L) = 0$ *where* $q^s(x, L) = inf\{q^s(x, l) : l \in L\}$.

Remark 1.1 ([4]). *Let* (X, q) *be a weak partial metric space and L be any non empty set in* (X, q), *then*

$$l \in \bar{L} \Leftrightarrow q(l, L) = q(l, l)$$

where $\bar{L}$ *denotes the closure of L with respect to weak partial metric q. Observe that L is closed in* (X, q) *iff* $L = \bar{L}$.

Now, we study the following properties of the mapping $\delta_q : CB^q(X) \times CB^q(X) \to [0, \infty)$.

Proposition 1.1 ([5]). *Let* (X, q) *be a weak partial metric space. For all* $L, M, N \in CB^q(X)$, *we have the following*:
(a) $\delta_q(L, L) = sup\{q(l, l) : l \in L\}$,
(b) $\delta_q(L, L) \leq \delta_q(L, M)$,
(c) $\delta_q(L, M) = 0 \Rightarrow L \subseteq M$,
(c) $\delta_q(L, M) \leq \delta_q(L, N) + \delta_q(N, M)$.

Proposition 1.2 ([5]). *Let* (X, q) *be a weak partial metric space. For all* $L, M, N \in CB^q(X)$,*we have*
(wh1) $H_q^+(L, L) \leq H_q^+(L, M)$,
(wh2) $H_q^+(L, M) = H_q^+(M, L)$,
(wh3) $H_q^+(L, M) \leq H_q^+(L, N) + H_q^+(N, M)$.

Definition 1.6 ([5]). *Let* (X, q) *be a weak partial metric space. For* $L, M \in CB^q(X)$, *define*

$$H_q^+(L, M) = \frac{1}{2}\{\delta_q(L, M) + \delta_q(M, L)\}.$$

The mapping $H_q^+ : CB^q(X) \times CB^q(X) \to [0, +\infty)$ *is called* H_q^+- *type Hausdorff metric induced by q.*

Definition 1.7 ([5]). *Let* (X, q) *be a weak partial metric space. A multi-valued map* $U : X \to CB^q(X)$ *is called* H_q^+-*contraction if*
(1) $\exists\ \alpha \in (0, 1)$ *such that*

$$H_q^+(U(x)\backslash\{x\}, U(y)\backslash\{y\}) \leq \alpha q(x, y) \quad for\ every\ x, y \in X$$

(2) *For every x* in *X, y* in $U(x)$ *and* $\epsilon > 0$, *there exists z in* $U(y)$ *such that*

$$q(y, z) \leq H_q^+(U(y), U(x)) + \epsilon.$$

Remark 1.2. *Since,* $max\{a, b\} \geq \frac{1}{2}(a+b)\ \ \forall\ a, b \geq 0$, *which follows that* H_q *contracion always implies* H_q^+- *contraction but the converse need not be true.*

A variant of Nadler's fixed point theorem is given by Beg and Pathak [5], which is stated as:

Theorem 1.2 ([5]). *Every* H_q^+- *type multi-valued contraction map* $U : X \to CB^q(X)$ *on a complete weak partial metric space has a fixed point.*

We define H_q^+-type hybrid contraction mapping as follows:

Definition 1.8. *Let* (X, q) *be a weak partial metric space. A mapping* $f : X \to X$ *be a single valued mapping and* $U : X \to CB^q(X)$ *be a multi-valued mapping. U is said to be a* H_q^+ *- hybrid contraction if*
(1) $\exists\ \alpha \in (0, 1)$ *such that*

$$H_q^+(U(x)\backslash\{x\}, U(y)\backslash\{y\}) \leq \alpha q(fx, fy) \quad for\ every\ x, y \in X$$

(2) *For every x in X, y in* $U(x)$ *and* $\epsilon > 0$, *there exists z in* $U(y)$ *such that*

$$q(y, z) \leq H_q^+(U(y), U(x)) + \epsilon.$$

2. Main Result

Theorem 2.1. *Let* (X, q) *be a weak partial metric space,* $f : X \longrightarrow X$ *be a single-valued mapping and* $U : X \longrightarrow$

$CB^q(X)$ be a H_q^+- type hybrid contraction mapping. Suppose fX is a complete subspace of X and $Ux \subset fX$. Then f and U have a coincidence point. Furthermore, if f is $T-$weakly commuting at coincidence points of f and U, then f and U have a common fixed point. Proof. Let x_0 be an arbitrary point of X and $y_0 = fx_0$ also let $\epsilon > 0$. We construct sequences $\{x_k\}, \{y_k\}$ in X respectively. Since $Ux \subset fX$, there exists $x_1 \in X$ such that $y_1 = fx_1 \in Ux_0$. If $q(fx_1, fx_0) = 0$, then x_0 is a coincidence point. Hence, assume $q(fx_1, fx_0) > 0$. Now, there exists $y_2 = fx_2 \in Ux_1$ such that $q(y_1, y_2) \leq H_q^+(Ux_0, Ux_1) + \epsilon$. Similarly, assume $q(y_1, y_2) > 0$. Again by (2) and the fact $Ux \subset fX$, there exists $y_3 = fx_3 \in Ux_2$ such that $q(y_2, y_3) \leq H_q^+(Ux_1, Ux_2) + \epsilon$, assume $q(y_2, y_3) > 0$.
Proceeding in this way, we can construct a sequence $y_{n+1} = fx_{n+1} \in Ux_n$, assume $q(y_n, y_{n+1}) > 0$ satisfying

$$q(y_n, y_{n+1}) \leq H_q^+(Ux_{n-1}, Ux_n) + \epsilon, \tag{2.1}$$

Now, by (2.1) and choosing $\epsilon = (\frac{1}{\sqrt{\alpha}} - 1)H_q^+(Ux_{n-1}, Ux_n)$, we have

$$\begin{aligned}
q(y_n, y_{n+1}) &\leq H_q^+(Ux_{n-1}, Ux_n) + (\frac{1}{\sqrt{\alpha}} - 1)H_q^+(Ux_{n-1}, Ux_n) \\
&\leq \frac{1}{\sqrt{\alpha}} H_q^+(Ux_{n-1}, Ux_n) \\
&= \frac{1}{\sqrt{\alpha}} H_q^+(Ux_{n-1}\backslash\{x_{n-1}\}, Ux_n\backslash\{x_n\}) \\
&\leq \frac{1}{\sqrt{\alpha}} .\alpha q(f(x_{n-1}), f(x_n)) \\
&= \sqrt{\alpha}.q(f(x_{n-1}), f(x_n)) \\
&= \sqrt{\alpha}.q(y_{n-1}, y_n).
\end{aligned}$$

Adopting similar process, we obtain

$$q(y_n, y_{n+1}) \leq (\sqrt{\alpha})^n q(y_0, y_1).$$

Using property (WP4) of a weak partial metric, for any $m \in \mathbb{N}$, we have

$$\begin{aligned}
q^s(y_n, y_{n+m}) &\leq q(y_n, y_{n+m}) \\
&\leq q(y_n, y_{n+1}) + q(y_{n+1}, y_{n+2}) + q(y_{n+2}, y_{n+3}) + ... + q(y_{n+m-1}, y_{n+m}) \\
&\leq (\sqrt{\alpha})^n q(y_0, y_1) + (\sqrt{\alpha})^{n+1} q(y_0, y_1) + (\sqrt{\alpha})^{n+2} q(y_0, y_1) + ... + (\sqrt{\alpha})^{n+m-1} q(y_0, y_1) \\
&= ((\sqrt{\alpha})^n + \sqrt{\alpha})^{n+1} + \sqrt{\alpha})^{n+2} + ... + \sqrt{\alpha})^{n+m-1}) q(y_0, y_1) \\
&\leq \frac{\sqrt{\alpha})^n}{1 - \sqrt{\alpha}} .q(y_0, y_1) \longrightarrow 0 \quad as \quad n \to \infty.
\end{aligned}$$

This implies that $\{y_k\} = \{fx_k\}$ where $k = 1, 2, 3, ...$; is a Cauchy sequence in (X, q^s). Since fX is complete $\exists\ w \in X$ such that the sequence $y_n = fx_n$ converges to fw as $n \longrightarrow \infty$ w.r.t. the metric q^s, that is, $\lim_{n\to\infty} q^s(fx_n, fw) = 0$. Moreover, we have

$$q(fw, fw) = \lim_{n\to\infty} q(y_n, fw) = \lim_{n\to\infty} q(y_n, y_n) = 0.$$

We now show that $fw \in Uw$.
By triangle inequality,

$$\begin{aligned}
q(fw, Uw) &\leq q(fw, fx_k) + q(fx_k, Uw) \\
&\leq q(fw, fx_k) + H_q^+(Ux_{k-1}, Uw) \\
&= q(fw, fx_k) + H_q^+(Ux_{k-1}\backslash\{x_{k-1}\}, Uw\backslash\{w\}) \\
&\leq q(fw, fx_k) + \alpha q(fx_{k-1}, fw),
\end{aligned}$$

$\forall\ k = 1, 2, 3, ..$ now we follow from $fx_k \to fw$ as $k \to \infty$ that $q(fw, fx_k)$ and $q(fx_{k-1}, fw) \to 0$ as $k \to \infty$. Therefore all terms in right hand side tend to 0 as $k \to \infty$ which implies that $q(fw, Uw) = 0$. Since Uw is closed, $fw \in Uw$. Therefore, f and U have a coincidence point $w \in X$. Let $t = fw \in Uw$. It follows from the definition of H_q^+- type Hausdroff metric that

$$q(t, ft) \leq q(t, Ut) = q(fw, Ut)$$

$$\leq H_q^+(Uw, Ut)$$
$$= H_q^+(Uw\backslash\{w\}, Ut\backslash\{t\})$$
$$\leq \alpha q(fw, ft)$$
$$= \alpha q(t, ft)$$
$$\implies q(t, ft) = 0.$$

It follows from $q(ft, Ut) = q(fw, Ut) \leq H_q^+(Uw, Ut) = 0$. Since Ut is closed, $t = ft \in Ut$. Thus f and U have a common fixed point. Now, we give an example to support our result.

Example 2.1. Let (X, q) be a weak partial metric space w.r.t. weak partial metric $q : X \times X \to [0, \infty)$ where $X = \left\{0, \frac{1}{2}, 1\right\}$ and q is defined by

$$q(x, y) = |x - y| + max\{x, y\} \quad \forall \quad x, y \in X.$$

Define the maps $U : X \to CB^q(X)$ and such that

$$U(x) = \begin{cases} \{0\}, & if \ x = \{0, 1\} \\ \left\{0, \frac{1}{2}\right\}, & if \ x = \left\{\frac{1}{2}\right\} \end{cases}$$

and $f : X \to X$ such that

$$f(\frac{1}{2}) = 0, \quad f(0) = 1, \quad f(1) = \frac{1}{2}.$$

Since $q(1, 1) = 1 \neq 0$, $q(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2} \neq 0$. Hence q is not a metric on X. Here $Ux \subset fX$. Also,

$$x \in \overline{\{0\}} \Leftrightarrow q(x, \{0\}) = q(x, x)$$
$$\Leftrightarrow 2x = x \Leftrightarrow x = 0$$
$$\Leftrightarrow x \in \{0\}.$$

Thus, $\{0\}$ is closed with respect to q.

$$x \in \overline{\left\{0, \frac{1}{2}\right\}} \Leftrightarrow q\left(x, \left\{0, \frac{1}{2}\right\}\right) = q(x, x)$$
$$\Leftrightarrow min\left\{2x, |x - \frac{1}{2}| + max\{x, \frac{1}{2}\}\right\} = x$$
$$\Leftrightarrow x \in \left\{0, \frac{1}{2}\right\}.$$

Hence, $\left\{0, \frac{1}{2}\right\}$ is closed with respect to q. Now, for all $x, y \in X$, we shall show that the contractive condition (1) is satisfied. For this, consider the following cases:

(i) $x = 0, y = 0$. We have

$$H_q^+(U(0)\backslash\{0\}, U(0)\backslash\{0\}) = H_q^+(\phi, \phi) = 0$$

and (1) is satisfied.

(ii) $x = 0, y = \frac{1}{2}$. We have

$$H_q^+(U(0)\backslash\{0\}, U(\frac{1}{2})\backslash\{\frac{1}{2}\}) = H_q^+(\phi, \{0\}) = 0,$$

and (1) is satisfied.

(iii) $x = \frac{1}{2}, y = 0$. We have

$$H_q^+(U(\frac{1}{2})\backslash\{\frac{1}{2}\}, U(0)\backslash\{0\}) = H_q^+(\{0\}, \phi) = 0$$

and (1) is satisfied.

(iv) $x = 0, y = 1$. We have

$$H_q^+(U(0)\backslash\{0\}, U(1)\backslash\{1\}) = H_q^+(\phi, \{0\}) = 0$$

and (1) is satisfied.

(v) $x = 1, y = 0$. We have

$$H_q^+(U(1)\backslash\{1\}, U(0)\backslash\{0\}) = H_q^+(\{0\}, \phi) = 0$$

and (1) is satisfied.

(vi) $x = \frac{1}{2}, y = \frac{1}{2}$. We have

$$H_q^+(U(\frac{1}{2})\backslash\{\frac{1}{2}\}, U(\frac{1}{2})\backslash\{\frac{1}{2}\}) = H_q^+(\{0\}, \{0\}) = 0$$

and (1) is satisfied.

(vii) $x = \frac{1}{2}, y = 1$. We have

$$H_q^+(U(\frac{1}{2})\backslash\{\frac{1}{2}\}, U(1)\backslash\{1\}) = H_q^+(\{0\}, \{0\}) = 0$$

and (1) is satisfied.

(viii) $x = 1, y = \frac{1}{2}$. We have

$$H_q^+(U(1)\backslash\{1\}, U(\frac{1}{2})\backslash\{\frac{1}{2}\}) = H_q^+(\{0\}, \{0\}) = 0$$

and (1) is satisfied.

(ix) $x = 1, y = 1$. We have

$$H_q^+(U(1)\backslash\{1\}, U(1)\backslash\{1\}) = H_q^+(\{0\}, \{0\}) = 0$$

and (1) is satisfied.

Further, we shall show that for every x in X, y in $U(x)$ and $\epsilon > 0$, $\exists$ z in $U(y)$ such that $q(y, z) \le H_q^+(U(y), U(x)) + \epsilon$. Indeed,

(1) if $x = 0, y \in U(0) = \{0\}, \epsilon > 0$, $\exists$ $z \in U(y) = \{0\}$ such that

$$0 = q(y, z) \le H_q^+(U(y), U(x)) + \epsilon$$

(2a) if $x = \frac{1}{2}$, $y \in U\left(\frac{1}{2}\right) = \left\{0, \frac{1}{2}\right\}$, say $y = 0, \epsilon > 0$, $\exists$ $z \in U(y) = \{0\}$, such that

$$0 = q(y, z) < 1 + \epsilon = H_q^+(U(y), U(x)) + \epsilon$$

(2b) if $x = \frac{1}{2}$, $y \in U\left(\frac{1}{2}\right) = \left\{0, \frac{1}{2}\right\}$, say $y = \frac{1}{2}, \epsilon > 0$, $\exists$ $z \in U(y) = \left\{0, \frac{1}{2}\right\}$, such that

$$\frac{1}{2} = q(y, z) < \frac{1}{2} + \epsilon = H_q^+(U(y), U(x)) + \epsilon$$

(3) If $x = 1, y \in U(1) = \{0\}, \epsilon > 0$ $\exists$ $z \in U(0) = \{0\}$ such that

$$0 = q(y, z) \le H_q^+(U(y), U(x)) + \epsilon$$

Hence, all the conditions of theorem are satisfied. Here $x = \frac{1}{2}$ is a coincidence point of f and U. In this example f is not T- weakly commuting at coincidence point.

Example 2.2. Let (X, q) be a weak partial metric space w.r.t. weak partial metric $q : X \times X \to [0, \infty)$ where $X = \left\{0, \frac{1}{6}, 1\right\}$ and q is defined by

$$q(x, y) = |x - y| + \frac{1}{3} max\{x, y\} \quad \forall \quad x, y \in X.$$

Define the maps $U : X \to CB^q(X)$ such that

$$U(x) = \begin{cases} \{0\}, & if \ x = \left\{0, \frac{1}{6}\right\} \\ \left\{1, \frac{1}{6}\right\}, & if \ x = \{1\} \end{cases}$$

and $f: X \to X$ such that

$$f(x) = x \ \ \forall \ \ x, y \in X$$

Since $q(1,1) = \frac{1}{3} \neq 0$, $q\left(\frac{1}{6}, \frac{1}{6}\right) = \frac{1}{18} \neq 0$. Hence q is not a metric on X. Here $Ux \subset fX$. Also,

$$\begin{aligned} x \in \overline{\{0\}} &\Leftrightarrow q(x, \{0\}) = q(x, x) \\ &\Leftrightarrow \frac{4}{3}x = \frac{x}{3} \Leftrightarrow x = 0 \\ &\Leftrightarrow x \in \{0\}. \end{aligned}$$

Thus, $\{0\}$ is closed with respect to q.

$$\begin{aligned} x \in \overline{\left\{1, \frac{1}{6}\right\}} &\Leftrightarrow q\left(x, \left\{1, \frac{1}{6}\right\}\right) = q(x, x) \\ &\Leftrightarrow min\left\{|x-1| + \frac{1}{3}max\{x, 1\}, |x - \frac{1}{6}| + max\{x, \frac{1}{6}\}\right\} = \frac{x}{3} \\ &\Leftrightarrow x \in \left\{1, \frac{1}{6}\right\}. \end{aligned}$$

Hence, $\left\{1, \frac{1}{6}\right\}$ is closed with respect to q. Now, for all $x, y \in X$, we shall show that the contractive condition (1) is satisfied. For this, consider the following cases:

(i) $x = 0, y = 0$. We have

$$H_q^+(U(0)\backslash\{0\}, U(0)\backslash\{0\}) = H_q^+(\phi, \phi) = 0$$

and (1) is satisfied.

(ii) $x = 0, y = \frac{1}{6}$. We have

$$H_q^+(U(0)\backslash\{0\}, U(\frac{1}{6})\backslash\{\frac{1}{6}\}) = H_q^+(\phi, \{0\}) = 0,$$

and (1) is satisfied.

(iii) $x = \frac{1}{6}, y = 0$. We have

$$H_q^+(U(\frac{1}{6})\backslash\{\frac{1}{6}\}, U(0)\backslash\{0\}) = H_q^+(\{0\}, \phi) = 0$$

and (1) is satisfied.

(iv) $x = 0, y = 1$. We have

$$H_q^+(U(0)\backslash\{0\}, U(1)\backslash\{1\}) = H_q^+(\phi, \{\frac{1}{6}\}) = 0$$

and (1) is satisfied.

(v) $x = 1, y = 0$. We have

$$H_q^+(U(1)\backslash\{1\}, U(0)\backslash\{0\}) = H_q^+(\{\frac{1}{6}\}, \phi) = 0$$

and (1) is satisfied.

(vi) $x = \frac{1}{6}, y = \frac{1}{6}$. We have

$$H_q^+(U(\frac{1}{6})\backslash\{\frac{1}{6}\}, U(\frac{1}{6})\backslash\{\frac{1}{6}\}) = H_q^+(\{0\}, \{0\}) = 0$$

and (1) is satisfied.

(vii) $x = \frac{1}{6}, y = 1$. We have

$$H_q^+(U(\frac{1}{6})\backslash\{\frac{1}{6}\}, U(1)\backslash\{1\}) = H_q^+(\{0\}, \{\frac{1}{6}\}) = \frac{2}{9} \leq \alpha.\frac{7}{6}$$

and (1) is satisfied.

(viii) $x = 1, y = \frac{1}{6}$. We have

$$H_q^+(U(1)\backslash\{1\}, U(\frac{1}{6})\backslash\{\frac{1}{6}\}) = H_q^+(\{\frac{1}{6}\}, \{0\}) = \frac{2}{9} \leq \alpha.\frac{7}{6}$$

and (1) is satisfied.

(ix) $x = 1, y = 1$. We have

$$H_q^+(U(1)\backslash\{1\}, U(1)\backslash\{1\}) = H_q^+(\{\frac{1}{6}\}, \{\frac{1}{6}\}) = \frac{1}{9} \leq \alpha.\frac{1}{3}$$

and (1) is satisfied.

Further, we shall show that for every x in X, y in $U(x)$ and $\epsilon > 0$, $\exists$ z in $U(y)$ such that $q(y,z) \leq H_q^+(U(y), U(x)) + \epsilon$. Indeed,

(1) if $x = 0, y \in U(0) = \{0\}, \epsilon > 0,\ \exists\, z \in U(y) = \{0\}$ such that

$$0 = q(y,z) \leq H_q^+(U(y), U(x)) + \epsilon$$

(2a) if $x = 1,\ y \in U(1) = \left\{1, \frac{1}{6}\right\}$, say $y = 1, \epsilon > 0,\ \exists\, z \in U(y) = \left\{1, \frac{1}{6}\right\}$ $z = 1$, such that

$$\frac{1}{3} = q(y,z) < \frac{1}{3} + \epsilon = H_q^+(U(y), U(x)) + \epsilon$$

(2b) if $x = 1,\ y \in U(1) = \left\{1, \frac{1}{6}\right\}$, say $y = \frac{1}{6}, \epsilon > 0,\ \exists\, z \in U(y) = \{0\}$, such that

$$\frac{2}{9} = q(y,z) < \frac{7}{9} + \epsilon = H_q^+(U(y), U(x)) + \epsilon$$

(3) If $x = \frac{1}{6}, y \in U\left(\frac{1}{6}\right) = \{0\}, \epsilon > 0\ \exists\, z \in U(y) = U(0) = \{0\}$ such that

$$0 = q(y,z) \leq H_q^+(U(y), U(x)) + \epsilon$$

Here $x = 0, 1$ are the coincidence points of f and U. Now we shall show that f is T-weakly commuting at coincidence points.

(i) For $x = 0,\ ff(0) = 0$ and $Uf(0) = \{0\}$
Thus $ff(0) \in Uf(0)$.

(ii) For $x = 1,\ ff(1) = 1$ and $Uf(1) = \left\{1, \frac{1}{6}\right\}$
Thus $ff(1) \in Uf(1)$.

(iii) For $x = \frac{1}{6},\ ff\left(\frac{1}{6}\right) = \frac{1}{6}$ and $Uf\left(\frac{1}{6}\right) = \{0\}$
Thus $ff\left(\frac{1}{6}\right) \notin Uf\left(\frac{1}{6}\right)$

Hence, all the conditions of theorem are satisfied. Here **x = 0, 1** are the common fixed points of f and U.

3. Conclusion

In this article, we established a coincidence and common fixed point theorem for hybrid contraction in weak partial metric space. We give a counter example to show that it is necessary to f satisfies T-weakly commuting condition on coincidence point for obtaining the common fixed point. We also give an example in support of our result.

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