HYBRID CONTRACTION IN WEAK PARTIAL METRIC SPACES

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Abstract

In this paper, a fixed point theorem is established for hybrid contraction in weak partial metric space. Our result is supported by examples.


Keywords and Phrases: Hybrid contraction mapping, Weak Partial metric space, Partial Hausdroff metric, Coincidence Point.

1. Introduction and Preliminaries

The theory of non-linear analysis has emerged as a fascinating field. Many authors have generalized and extended Banach contraction principle. In 1969, Nadler [7] initiated the study of fixed points for multi-valued contraction mappings using Hausdorff metric.

Let \( (X, d) \) be a non-empty metric space and \( CB(X) \), the class of all nonempty closed and bounded subsets of \( X \). The Hausdorff metric [3] induced by \( d \) on \( CB(X) \) is

\[
H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\},
\]

for every \( A, B \in CB(X) \), where \( d(a, B) = \inf \{d(a, b) : b \in B \} \) is the distance from \( a \) to \( B \subseteq X \).

Let \( f : X \to X \) be a single-valued mapping and \( U : X \to CB(X) \) be a multi-valued mapping.

(i) A point \( w \in X \) is a fixed point of \( f \) (resp. \( U \)) if \( fw = w \) (resp. \( w \in U(w) \)).

The set of all fixed points of \( f \) (resp. \( U \)) is denoted by \( \text{Fix}(f) \) (resp. \( \text{Fix}(U) \)).

(ii) A point \( w \in X \) is a coincidence point of \( f \) and \( U \) if \( fw \in Uw \).

The set of all coincidence points of \( f \) and \( U \) is denoted by \( C(f, U) \).

(iii) A point \( w \in X \) is a common fixed point of \( f \) and \( U \) if \( w = fw \in Uw \).

The set of all common fixed points of \( f \) and \( U \) is denoted by \( \text{Fix}(f, U) \).

Nadler [7] proved the following

**Theorem 1.1** ([7]). Let \( (X, d) \) be a complete metric space and \( U : X \to CB(X) \) be a multi-valued mapping satisfying

\[
H(Ux, Uy) \leq kd(x, y), \quad \forall x, y \in X
\]

where \( k \in [0, 1) \) then \( \exists x \in X \) such that \( x \in Ux \).

Afterward, a rapid progress has been observed using weak and generalized contraction mappings. Multi-valued contraction mapping has many applications in differential equations, control theory and economics.

Singh and Mishra [9] introduced the concept of \((IT)\)-commutativity for a hybrid pair of single-valued and multivalued mappings. Further, in 2004, Kamran [12] introduced the notion of \( (T) \)-weak commutativity for a hybrid pair of single-valued and multivalued maps which is weaker than \((IT)\)-commutativity. The definitions of \((IT)\)-commutativity and \(T\)-weak commutativity are as follows ([9]). A mapping \( f : X \to X \) and \( U : X \to CB(X) \) are said to be \((IT)\)-commuting at \( w \in X \) if \( fUw \subseteq Ufw \).

**Definition 1.1** ([12]). Let \( f : X \to X \) and \( U : X \to CB(X) \), the map \( f \) is said to be \( T \)-weakly commuting at \( w \in X \) if \( fUw \subseteq Ufw \).

On the other hand, the distance notion in the metric fixed point theory has been introduced and generalized in several different ways by many authors. In 1992, Mathews [8] introduced the notion of partial metric space as a part of the study of denotational semantics of data flow networks. He presented a modified version of Banach contraction principle. Several authors have done work in this direction ([4],[2],[6]).
Definition 1.2. Let $X$ be a non empty set. Then a mapping $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on $X$ if for all $x, y, z \in X$,

(P1) $x = y \iff p(x, x) = p(x, y) = p(y, y)$;
(P2) $p(x, x) \leq p(x, y)$;
(P3) $p(x, y) = p(y, x)$;
(P4) $p(x, y) \leq p(x, z) + p(z, x) - p(z, z)$.

The pair $(X, p)$ is called a partial metric space.

Recently, a weaker form of partial metric space is introduced by Ismat Beg and H. K. Pathak [5] known as Weak Partial Metric Space and defined as:

Definition 1.4 ([5]). Let $X$ be a non empty set. A function $q : X \times X \to \mathbb{R}^+$ is called a weak partial metric on $X$ if for all $x, y, z \in X$, the following conditions hold:

(WP1) $q(x, x) = q(y, y) \iff x = y$;
(WP2) $q(x, x) \leq q(x, y)$;
(WP3) $q(x, y) = q(y, x)$;
(WP4) $q(x, y) \leq q(x, z) + q(z, x)$.

The pair $(X, q)$ is a weak partial metric space. Further, many authors have worked on weak partial metric space ([1], [10], [11]).

Example 1.1.

(i) $(\mathbb{R}^+, q)$, where $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ defines as

$$q(x, y) = e^{iy - y} \forall x, y \in \mathbb{R}^+.$$

(ii) $(\mathbb{R}^+, q)$, where $q : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ defines as

$$q(x, y) = |x - y| + \frac{1}{3} \max \{x, y\} \forall x, y \in \mathbb{R}^+.$$

Observe that

○ If $q(x, y) = 0$, then (WP1) and (WP2) $\Rightarrow x = y$. But the converse need not be true.
○ (P1) $\Rightarrow$ (WP1), but the converse need not be true.
○ (P4) $\Rightarrow$ (WP4), but the converse need not be true.

Each weak partial metric $q$ on $X$ generates a $T_0$ topology $\tau_q$ on $X$. Topology $\tau_q$ has as a base the family of open $q$-balls $\{B_q(x, e) : x \in X, e > 0\}$, where $B_q(x, e) = \{y \in X : q(x, y) < q(x, x) + e\}$ for all $x \in X$ and $e > 0$.

If $q$ is weak partial metric on $X$, then the function $q^* : X \times X \to \mathbb{R}^+$ given by

$$q^*(x, y) = q(x, y) - \frac{1}{2}[q(x, x) + q(y, y)]$$

defines a metric on $X$.

Definition 1.5 ([5]). Let $(X, q)$ be a weak partial metric space. Then

(i) $P$ is said to be a bounded subset in $(X, q)$ if $\exists x \in X$ and $L \geq 0$ such that $\forall p \in P$, we have $p \in B_q(x_0, L)$ that is

$$q(x_0, p) < q(p, p) + L.$$

(ii) A sequence $\{x_n\}$ in $(X, q)$ converges to a point $x \in X$ w.r.t. $\tau_q$ if $q(x, x) = \lim_{n \to \infty} q(x, x_n)$. Moreover, a sequence $\{x_n\}$ converges in $(X, q^*)$ to a point $x \in X$ if

$$\lim_{n \to \infty} q(x_n, x_m) = \lim_{n \to \infty} q(x_n, x) = q(x, x).$$

(iii) A sequence $\{x_n\}$ in $X$ is said to be a Cauchy sequence if $\lim_{n,m \to \infty} q(x_n, x_m)$ exists and is finite.

(iv) $(X, q)$ is called complete if every Cauchy sequence $\{x_n\}$ in $X$ converges to $x \in X$ with respect to topology $\tau_q$.

Lemma 1.1 ([5]). Let $(X, q)$ be a weak partial metric space. Then

(a) A sequence $\{x_n\}$ in $X$ is Cauchy sequence in $(X, q)$ if and only if it is a Cauchy sequence in the metric space $(X, q^*)$.
(b) $(X, q)$ is called complete iff the metric space $(X, q^*)$ is complete.
For \( L, M \in \text{CB}^0(X) \) and \( x \in X \) define \( q(x, L) = \inf \{ q(x, l) : l \in L \} \), \( \delta_q(L, M) = \sup \{ q(l, M) : l \in L \} \) and \( \delta_q(M, L) = \sup \{ q(m, L) : m \in M \} \).

Clearly \( q(x, L) = 0 \Rightarrow q^*(x, L) = 0 \) where \( q^*(x, L) = \inf \{ q^*(x, l) : l \in L \} \).

Remark 1.1 ([4]). Let \( (X, q) \) be a weak partial metric space and \( L \) be any non-empty set in \( (X, q) \), then
\[
 l \in \bar{L} \Leftrightarrow q(l, L) = q(l, l)
\]
where \( \bar{L} \) denotes the closure of \( L \) with respect to weak partial metric \( q \). Observe that \( L \) is closed in \( (X, q) \) iff \( L = \bar{L} \).

Now, we study the following properties of the mapping \( \delta_q : \text{CB}^0(X) \times \text{CB}^0(X) \to [0, \infty) \).

**Proposition 1.1 ([5]).** Let \( (X, q) \) be a weak partial metric space. For all \( L, M, N \in \text{CB}^0(X) \), we have the following:

(a) \( \delta_q(L, L) = \sup \{ q(l, l) : l \in L \} \),
(b) \( \delta_q(L, L) \leq \delta_q(L, M) \),
(c) \( \delta_q(M, L) = 0 \Rightarrow L \subseteq M \),
(d) \( \delta_q(M, L) \leq \delta_q(L, N) + \delta_q(N, M) \).

**Proposition 1.2 ([5]).** Let \( (X, q) \) be a weak partial metric space. For all \( L, M, N \in \text{CB}^0(X) \), we have

(wh1) \( H_q^+(L, N) \leq H_q^+(L, M) \),
(wh2) \( H_q^+(L, M) = H_q^+(M, L) \),
(wh3) \( H_q^+(L, M) \leq H_q^+(L, N) + H_q^+(N, M) \).

**Definition 1.6 ([5]).** Let \( (X, q) \) be a weak partial metric space. For \( L, M \in \text{CB}^0(X) \), define
\[
 H_q^+(L, M) = \frac{1}{2} (\delta_q(L, M) + \delta_q(M, L)).
\]

The mapping \( H_q^+ : \text{CB}^0(X) \times \text{CB}^0(X) \to [0, +\infty) \) is called \( H_q^+ \)-type Hausdorff metric induced by \( q \).

**Definition 1.7 ([5]).** Let \( (X, q) \) be a weak partial metric space. A multi-valued map \( U : X \to \text{CB}^0(X) \) is called \( H_q^+ \)-contraction if

1. \( \exists \alpha \in (0, 1) \) such that \( H_q^+(U(x), (x), U(y), (y)) \leq \alpha q(x, y) \) for every \( x, y \in X \)
2. For every \( x \) in \( X \), \( y \) in \( U(x) \) and \( \epsilon > 0 \), there exists \( z \) in \( U(y) \) such that \( q(y, z) \leq H_q^+(U(y), U(x)) + \epsilon \).

Remark 1.2. Since, \( \max \{a, b\} \geq \frac{1}{2} (a + b) \) \( \forall \ a, b \geq 0 \), which follows that \( H_q \) contraction always implies \( H_q^+ \)-contraction but the converse need not be true.

A variant of Nadler’s fixed point theorem is given by Beg and Pathak [5], which is stated as:

**Theorem 1.2 ([5]).** Every \( H_q^+ \)-type multi-valued contraction map \( U : X \to \text{CB}^0(X) \) on a complete weak partial metric space has a fixed point.

We define \( H_q^+ \)-type hybrid contraction mapping as follows:

**Definition 1.8.** Let \( (X, q) \) be a weak partial metric space. A mapping \( f : X \to X \) be a single valued mapping and \( U : X \to \text{CB}^0(X) \) be a multi-valued mapping. \( U \) is said to be a \( H_q^+ \)-hybrid contraction if

1. \( \exists \alpha \in (0, 1) \) such that \( H_q^+(U(x), (x), U(y), (y)) \leq \alpha q(fx, fy) \) for every \( x, y \in X \)
2. For every \( x \) in \( X \), \( y \) in \( U(x) \) and \( \epsilon > 0 \), there exists \( z \) in \( U(y) \) such that \( q(y, z) \leq H_q^+(U(y), U(x)) + \epsilon \).

2. **Main Result**

**Theorem 2.1.** Let \( (X, q) \) be a weak partial metric space, \( f : X \to X \) be a single-valued mapping and \( U : X \mapsto \text{CB}^0(X) \) be a multi-valued mapping. If \( \exists \alpha \in (0, 1) \) such that \( H_q^+(U(x), (x), U(y), (y)) \leq \alpha q(fx, fy) \) for every \( x, y \in X \), then \( x \) in \( X \) such that \( q(y, z) \leq H_q^+(U(y), U(x)) + \epsilon \).
\( CB(X) \) be a \( H_q^* \)-type hybrid contraction mapping. Suppose \( fX \) is a complete subspace of \( X \) and \( Ux \subset fX \). Then \( f \) and \( U \) have a coincidence point. Furthermore, if \( f \) is \( I \)-weakly commuting at coincidence points of \( f \) and \( U \), then \( f \) and \( U \) have a common fixed point. Proof. Let \( x_0 \) be an arbitrary point of \( X \) and \( y_0 = f x_0 \) also let \( \epsilon > 0 \). We construct sequences \( \{x_n\}, \{y_n\} \) in \( X \) respectively. Since \( Ux \subset fX \), there exists \( x_1 \in X \) such that \( y_1 = f x_1 = U x_0 \). If \( q(f x_1, f x_0) = 0 \), then \( x_0 \) is a coincidence point. Hence, assume \( q(f x_1, f x_0) > 0 \). Now, there exists \( y_2 = f x_2 \in X x_1 \) such that \( q(y_1, y_2) \leq H_q^*(U x_0, U x_1) + \epsilon \). Similarly, assume \( q(y_1, y_2) > 0 \). Again by (2) and the fact \( Ux \subset fX \), there exists \( y_3 = f x_3 \in U x_2 \) such that \( q(y_2, y_3) \leq H_q^*(U x_1, U x_2) + \epsilon \), assume \( q(y_2, y_3) > 0 \). Proceeding in this way, we can construct a sequence \( y_{n+1} = f x_{n+1} \in U x_n \), assume \( q(y_n, y_{n+1}) > 0 \) satisfying
\[
q(y_n, y_{n+1}) \leq H_q^*(U x_{n-1}, U x_n) + \epsilon,
\]
(2.1)

Now, by (2.1) and choosing \( \epsilon = (\frac{1}{\sqrt{\alpha}} - 1)^m \), we have
\[
q(y_n, y_{n+1}) \leq H_q^*(U x_{n-1}, U x_n) + (\frac{1}{\sqrt{\alpha}} - 1)^m H_q^*(U x_{n-1}, U x_n)
\]
\[
\leq \frac{1}{\sqrt{\alpha}} H_q^*(U x_{n-1}, U x_n)
\]
\[
= \frac{1}{\sqrt{\alpha}} H_q^*(U x_{n-1} \setminus x_{n-1}, U x_n \setminus x_n)
\]
\[
\leq \frac{1}{\sqrt{\alpha}} \alpha f(x_{n-1}), f(x_n))
\]
\[
= \sqrt{\alpha} q(f(x_{n-1}), f(x_n))
\]
\[
= \sqrt{\alpha} q(y_{n-1}, y_n).
\]

Adopting similar process, we obtain
\[
q(y_n, y_{n+1}) \leq (\sqrt{\alpha})^n q(y_0, y_1).
\]

Using property (WP4) of a weak partial metric, for any \( m \in \mathbb{N} \), we have
\[
q^m(y_n, y_{n+m}) \leq q(y_n, y_{n+1}) + q(y_{n+1}, y_{n+2}) + q(y_{n+2}, y_{n+3}) + \ldots + q(y_{n+m-1}, y_{n+m})
\]
\[
\leq (\sqrt{\alpha})^n q(y_0, y_1) + (\sqrt{\alpha})^{n+1} q(y_0, y_1) + (\sqrt{\alpha})^{n+2} q(y_0, y_1) + \ldots + (\sqrt{\alpha})^{n+m-1} q(y_0, y_1)
\]
\[
= ((\sqrt{\alpha})^n + (\sqrt{\alpha})^{n+1} + (\sqrt{\alpha})^{n+2} + \ldots + (\sqrt{\alpha})^{n+m-1}) q(y_0, y_1)
\]
\[
\leq \frac{1}{1 - \sqrt{\alpha}} q(y_0, y_1) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

This implies that \( \{y_k\} = \{f x_k\} \) where \( k = 1, 2, 3, \ldots \) is a Cauchy sequence in \( (X, q^*) \). Since \( fX \) is complete \( \exists \) \( w \in X \) such that the sequence \( y_n = f x_n \) converges to \( f w \) as \( n \rightarrow \infty \) w.r.t. the metric \( q^* \). that is, \( \lim_{n \rightarrow \infty} q^*(f x_n, f w) = 0 \). Moreover, we have
\[
q(f w, f w) = \lim_{n \rightarrow \infty} q(y_n, f w) = \lim_{n \rightarrow \infty} q(y_n, y_n) = 0.
\]

We now show that \( f w \in U w \).

By triangle inequality,
\[
q(f w, U w) \leq q(f w, f x_k) + q(f x_k, U w)
\]
\[
\leq q(f w, f x_k) + H_q^*(U x_{k-1}, U w)
\]
\[
= q(f w, f x_k) + H_q^*(U x_{k-1} \setminus x_{k-1}, U w \setminus w)
\]
\[
\leq q(f w, f x_k) + \alpha q(f x_{k-1}, f w),
\]
\[
\forall \ k = 1, 2, 3, \ldots \text{ now we follow from } f x_k \rightarrow f w \text{ as } k \rightarrow \infty \text{ that } q(f w, f x_k) \text{ and } q(f x_k, f w) \rightarrow 0 \text{ as } k \rightarrow \infty. \] Therefore all terms in right hand side tend to 0 as \( k \rightarrow \infty \) which implies that \( q(f w, U w) = 0 \). Since \( U w \) is closed, \( f w \in U w \). Therefore, \( f \) and \( U \) have a coincidence point \( w \in X \). Let \( t = f w \in U w \). It follows from the definition of \( H_q^* \)-type Hausdorff metric that
\[
q(t, ft) \leq q(t, Ut) = q(f w, U t)
\]
\[ H^*_q(Uw, Ut) \leq H^*_q(Uw\setminus\{w\}, U\setminus\{t\}) \]
\[ = H^*_q(Uw, Ut) \leq \alpha q(fw, ft) = \alpha q(t, ft) \]

\[ \implies q(t, ft) = 0. \]

It follows from \( q(ft, Ut) = q(fw, Ut) \leq H^*_q(Uw, Ut) = 0. \) Since \( Ut \) is closed, \( t = ft \in Ut. \) Thus \( f \) and \( U \) have a common fixed point. Now, we give an example to support our result.

**Example 2.1.** Let \((X, q)\) be a weak partial metric space w.r.t. weak partial metric \( q : X \times X \to [0, \infty) \) where \( X = \{0, \frac{1}{2}, 1\} \) and \( q \) is defined by
\[ q(x, y) = |x - y| + \max\{x, y\} \quad \forall \ x, y \in X. \]

Define the maps \( U : X \to CB^q(X) \) and such that
\[ U(x) = \begin{cases} \{0\}, & \text{if } x = \{0, 1\} \\ \{0, \frac{1}{2}\}, & \text{if } x = \{\frac{1}{2}\} \end{cases} \]
and \( f : X \to X \) such that
\[ f\left(\frac{1}{2}\right) = 0, \quad f(0) = 1, \quad f(1) = \frac{1}{2}. \]

Since \( q(1, 1) = 1 \neq 0, \ q\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} \neq 0. \) Hence \( q \) is not a metric on \( X. \) Here \( Ux \subset fX. \) Also,
\[ x \in \{0\} \iff q(x, \{0\}) = q(x, x) \]
\[ \iff 2x = x \iff x = 0 \]
\[ \iff x \in \{0\}. \]

Thus, \( \{0\} \) is closed with respect to \( q. \)
\[ x \in \left\{0, \frac{1}{2}\right\} \iff q\left(x, \left\{0, \frac{1}{2}\right\}\right) = q(x, x) \]
\[ \iff \min\left\{2x, \left|x - \frac{1}{2}\right| + \max\{x, \frac{1}{2}\}\right\} = x \]
\[ \iff x \in \left\{0, \frac{1}{2}\right\}. \]

Hence, \( \left\{0, \frac{1}{2}\right\} \) is closed with respect to \( q. \) Now, for all \( x, y \in X, \) we shall show that the contractive condition (1) is satisfied. For this, consider the following cases:

(i) \( x = 0, y = 0. \) We have
\[ H^*_q(U(0)\setminus\{0\}, U(0)\setminus\{0\}) = H^*_q(\phi, \phi) = 0 \]
and (1) is satisfied.

(ii) \( x = 0, y = \frac{1}{2}. \) We have
\[ H^*_q(U(0)\setminus\{0\}, U\left(\frac{1}{2}\right)\setminus\{\frac{1}{2}\}) = H^*_q(\phi, \{0\}) = 0. \]
and (1) is satisfied.

(iii) \( x = \frac{1}{2}, y = 0. \) We have
\[ H^*_q(U\left(\frac{1}{2}\right)\setminus\{\frac{1}{2}\}, U(0)\setminus\{0\}) = H^*_q(\{0\}, \phi) = 0 \]
and (1) is satisfied.
Indeed, further, we shall show that for every $x$
\[(vii) x = 1, y = \frac{1}{2}. \text{ We have}\]
\[H_q^*(U(1), U(1)|1)) = H_q^*(\{0\}, \{0\}) = 0\]
and (1) is satisfied.
\[(viii) x = 1, y = \frac{1}{2}. \text{ We have}\]
\[H_q^*(U(1), U(1)|1)) = H_q^*(\{0\}, \{0\}) = 0\]
and (1) is satisfied.
\[(ix) x = 1, y = 1. \text{ We have}\]
\[H_q^*(U(1), U(1)|1)) = H_q^*(\{0\}, \{0\}) = 0\]
and (1) is satisfied.

Further, we shall show that for every $x$ in $X$, $y$ in $U(x)$ and $\epsilon > 0$, $\exists \ z$ in $U(y)$ such that $q(y, z) \leq H_q^*(U(y), U(x)) + \epsilon$.

Indeed, (1) if $x = 0$, $y \in U(0) = \{0\}$, $\epsilon > 0$, $\exists \ z \in U(y) = \{0\}$ such that
\[0 = q(y, z) \leq H_q^*(U(y), U(x)) + \epsilon\]
(2a) if $x = \frac{1}{2}$, $y \in U\left(\frac{1}{2}\right) = \left\{0, \frac{1}{2}\right\}$, say $y = 0$, $\epsilon > 0$, $\exists \ z \in U(y) = \{0\}$, such that
\[0 = q(y, z) < 1 + \epsilon = H_q^*(U(y), U(x)) + \epsilon\]
(2b) if $x = \frac{1}{2}$, $y \in U\left(\frac{1}{2}\right) = \left\{0, \frac{1}{2}\right\}$, say $y = \frac{1}{2}$, $\epsilon > 0$, $\exists \ z \in U(y) = \left\{0, \frac{1}{2}\right\}$, such that
\[\frac{1}{2} = q(y, z) < 1 + \epsilon = H_q^*(U(y), U(x)) + \epsilon\]
(3) if $x = 1$, $y \in U(1) = \{0\}$, $\epsilon > 0$ $\exists \ z \in U(0) = \{0\}$ such that
\[0 = q(y, z) \leq H_q^*(U(y), U(x)) + \epsilon\]

Hence, all the conditions of theorem are satisfied. Here $x = \frac{1}{2}$ is a coincidence point of $f$ and $U$. In this example $f$ is not $T$- weakly commuting at coincidence point.

**Example 2.2.** Let $(X, q)$ be a weak partial metric space w.r.t. weak partial metric $q : X \times X \to [0, \infty)$ where $X = \left\{0, \frac{1}{6}, 1\right\}$ and $q$ is defined by
\[q(x, y) = |x - y| + \frac{1}{3} \ max \{x, y\} \ \forall \ x, y \in X.\]

Define the maps $U : X \to CB^0(X)$ such that
\[U(x) = \left\{\begin{array}{ll}
\{0\}, & \text{if } x = \left\{0, \frac{1}{6}\right\} \\
\left\{0, \frac{1}{6}\right\}, & \text{if } x = \{1\}
\end{array}\right.\]
and \( f : X \to X \) such that 

\[
f(x) = x \quad \forall \ x, y \in X
\]

Since \( q(1,1) = \frac{1}{3} \neq 0, \ q\left(\frac{1}{6}, \frac{1}{6}\right) = \frac{1}{18} \neq 0 \). Hence \( q \) is not a metric on \( X \). Here \( Ux \subset fX \). Also, 

\[
x \in \{0\} \iff q(x,\{0\}) = q(x, x)
\]

\[
\iff \frac{4}{3}x = \frac{x}{3} \iff x = 0
\]

\[
\iff x \in \{0\}.
\]

Thus, \( \{0\} \) is closed with respect to \( q \).

\[
x \in \left\{\frac{1}{6}\right\} \iff q\left(x, \left\{\frac{1}{6}\right\}\right) = q(x, x)
\]

\[
\iff \min \left\{ |x - 1| + \frac{1}{3} \max \{|x, 1|, |x - \frac{1}{6}| + \max \{|x, \frac{1}{6}|\} \right\} = \frac{x}{3}
\]

\[
\iff x \in \left\{\frac{1}{6}\right\}.
\]

Hence, \( \left\{\frac{1}{6}\right\} \) is closed with respect to \( q \). Now, for all \( x, y \in X \), we shall show that the contractive condition (1) is satisfied. For this, consider the following cases:

(i) \( x = 0, y = 0 \). We have 

\[
H_q^*(U(0)\setminus\{0\}, U(0)\setminus\{0\}) = H_q^*(\phi, \phi) = 0
\]

and (1) is satisfied.

(ii) \( x = 0, y = \frac{1}{6} \). We have 

\[
H_q^*(U(0)\setminus\{0\}, U(\frac{1}{6})\setminus\{\frac{1}{6}\}) = H_q^*(\phi, \{0\}) = 0,
\]

and (1) is satisfied.

(iii) \( x = \frac{1}{6}, y = 0 \). We have 

\[
H_q^*(U(\frac{1}{6})\setminus\{\frac{1}{6}\}, U(0)\setminus\{0\}) = H_q^*(\{0\}, \phi) = 0
\]

and (1) is satisfied.

(iv) \( x = 0, y = 1 \). We have 

\[
H_q^*(U(0)\setminus\{0\}, U(1)\setminus\{1\}) = H_q^*(\phi, \{\frac{1}{6}\}) = 0
\]

and (1) is satisfied.

(v) \( x = 1, y = 0 \). We have 

\[
H_q^*(U(1)\setminus\{1\}, U(0)\setminus\{0\}) = H_q^*(\frac{1}{6}, \phi) = 0
\]

and (1) is satisfied.

(vi) \( x = \frac{1}{6}, y = \frac{1}{6} \). We have 

\[
H_q^*(U(\frac{1}{6})\setminus\{\frac{1}{6}\}, U(\frac{1}{6})\setminus\{\frac{1}{6}\}) = H_q^*(\{0\}, \{0\}) = 0
\]

and (1) is satisfied.

(vii) \( x = \frac{1}{6}, y = 1 \). We have 

\[
H_q^*(U(\frac{1}{6})\setminus\{\frac{1}{6}\}, U(1)\setminus\{1\}) = H_q^*(\{0\}, \{\frac{1}{6}\}) = \frac{2}{9} \leq \alpha \cdot \frac{7}{6}
\]

and (1) is satisfied.
(viii) $x = 1, y = \frac{1}{6}$. We have
\[ H_q^*(U(1) \setminus \{1\}, U(\frac{1}{6}) \setminus \{\frac{1}{6}\}) = H_q^*(\frac{1}{6}, \{0\}) = \frac{2}{9} \leq \alpha \frac{7}{6} \]
and (1) is satisfied.
(ix) $x = 1, y = 1$. We have
\[ H_q^*(U(1) \setminus \{1\}, U(1) \setminus \{1\}) = H_q^*(\frac{1}{6}, \{\frac{1}{6}\}) = \frac{1}{9} \leq \alpha \frac{1}{3} \]
and (1) is satisfied.

Further, we shall show that for every $x$ in $X, y$ in $U(x)$ and $\epsilon > 0$, $\exists z$ in $U(y)$ such that $q(y, z) \leq H_q^*(U(y), U(x)) + \epsilon$. Indeed,
(1) if $x = 0, y \in U(0) = \{0\}, \epsilon > 0$, $\exists z \in U(y) = \{0\}$ such that
\[ 0 = q(y, z) \leq H_q^*(U(y), U(x)) + \epsilon \]
(2a) if $x = 1, y \in U(1) = \{1, \frac{1}{6}\}$, say $y = 1, \epsilon > 0$, $\exists z \in U(y) = \{1, \frac{1}{6}\}$ $z = 1$, such that
\[ \frac{1}{3} = q(y, z) < \frac{1}{3} + \epsilon = H_q^*(U(y), U(x)) + \epsilon \]
(2b) if $x = 1, y \in U(1) = \{1, \frac{1}{6}\}$, say $y = \frac{1}{6}, \epsilon > 0$, $\exists z \in U(y) = \{0\}$, such that
\[ \frac{2}{9} = q(y, z) < \frac{7}{9} + \epsilon = H_q^*(U(y), U(x)) + \epsilon \]
(3) If $x = \frac{1}{6}, y \in U\left(\frac{1}{6}\right) = \{0\}, \epsilon > 0 \exists z \in U(y) = U(0) = \{0\}$ such that
\[ 0 = q(y, z) \leq H_q^*(U(y), U(x)) + \epsilon \]
Here $x = 0$, 1 are the coincidence points of $f$ and $U$. Now we shall show that $f$ is $T$-weakly commuting at coincidence points.
(i) For $x = 0$, $ff(0) = 0$ and $Uf(0) = \{0\}$
Thus $ff(0) \in Uf(0)$.
(ii) For $x = 1$, $ff(1) = 1$ and $Uf(1) = \left\{1, \frac{1}{6}\right\}$
Thus $ff(1) \in Uf(1)$.
(iii) For $x = \frac{1}{6}$, $ff\left(\frac{1}{6}\right) = \frac{1}{6}$ and $Uf\left(\frac{1}{6}\right) = \{0\}$
Thus $ff\left(\frac{1}{6}\right) \notin Uf\left(\frac{1}{6}\right)$

Hence, all the conditions of theorem are satisfied. Here $x = 0, 1$ are the common fixed points of $f$ and $U$.

3. Conclusion

In this article, we established a coincidence and common fixed point theorem for hybrid contraction in weak partial metric space. We give a counter example to show that it is necessary to $f$ satisfies $T$-weakly commuting condition on coincidence point for obtaining the common fixed point. We also give an example in support of our result.

References


