ON CLOSED FILTERS IN CI-ALGEBRAS

Pulak Sabhapandit¹, Kuljit Pathak² and Manoj Kumar Sarma³

¹Department of Mathematics, Biswanath College, Biswanath Chariali, Assam, 784176, India
²Department of Mathematics, B.H. College, Howly, Assam, 781316, India
³Department of Mathematics, Guwahati College, Guwahati - 21, Assam, India.

Abstract

The filter theory of CI-algebras was first introduced by Meng(2009). This concept plays a significant role in the study of CI-algebras. In this paper, we continue to study the notion of closed filter in CI-algebra with some characteristic properties. We also study closed filters in the Cartesian product of CI-algebra and the function algebra of CI-algebra.

Keywords and Phrases: CI-algebra, BE-algebra, filter, closed filter.

1. Introduction

In 1966 the concepts of BCK and BCI-algebras were first introduced by Imai and Iseki[7] and Iseki[8]. Since then mathematicians have made further studies in the field and developed concepts of similar type such as BCH [6], BH [9], d [14], etc. In 2006, the idea of BE -algebras [11] was introduced by Kim and Kim as a generalization of dual BCK-algebras [10]. In 2010, Meng[12] propounded the notion of a new algebraic structure called CI-algebras as a generalization of BE-algebras. In his paper Meng[12] defined the filter theory of CI-algebras which is considered to be very significant in the study of CI -algebras. Closed filters in CI-algebras was also studied in details by Meng[13]. The present author(s)[15] developed the concept of Cartesian product of CI-algebras which led to further research in the direction and findings were had in 2018 in respect of some special types of CI-algebras [17] obtained from a given CI-algebra. In this paper we continue to study closed filters in CI-algebras and investigate some new properties of it in Cartesian product of CI-algebras and the function algebra of CI-algebras.

2. Preliminaries

Definition 2.1. An algebraic system $(S; \oplus, 1)$ is called a BE-algebra if it satisfies the following axioms:

(A1) $p \oplus p = 1$,
(A2) $p \oplus 1 = 1$,
(A3) $1 \oplus p = p$,
(A4) $p \oplus (t \oplus r) = t \oplus (p \oplus r)$ for all $p, t, r \in S$.

Definition 2.2 ([12,16,18]). An algebraic system $(S; \oplus, 1)$ is called a CI-algebra if it satisfies the following axioms:

(A1) $p \oplus p = 1$,
(A3) $1 \oplus p = p$,
(A4) $p \oplus (t \oplus r) = t \oplus (p \oplus r)$ for all $p, t, r \in S$.

Example 2.1. Let $A$ be a non-empty set and let $F(A)$ be the set of all function $f : A \to (0, \infty)$. For $h, k \in F(A)$, we define a binary operation $*$ as

$$(h * k)(t) = \frac{k(t)}{h(t)}, t \in A.$$

If we put $1(t) = 1$ for all $t \in A$, then $1 \in F(A)$ and simple computation proves that $(F(A); *, 1)$ is a CI-algebra. In $S$, we can define a binary relation $\leq$ by $s \leq t$ iff $s \oplus t = 1$.

Lemma 2.1 ([12]). In a CI-algebra $(S; \oplus, 1)$ following results are true:

1. $s \oplus ((s \oplus t) \oplus t) = 1$,
2. $(s \oplus t) \oplus 1 = (s \oplus 1) \oplus (t \oplus 1)$ for all $s, t \in S$. 

224
Notation 2.1 ([12]). If \((S; \otimes, 1)\) is a CI-algebra and let \(B(S) = \{x \in S : x \otimes 1 = 1\}\). \(B(S)\) is called the BE-part of \(S\). Clearly \(B(S)\) is non-empty, since \(1 \in B(S)\).

Definition 2.3 ([12]). If for any non-empty subset \(A\) of \(S\) we have \(x \in A\) and \(y \in A\) imply \(x \oplus y \in A\), then \(A\) is called a sub-algebra.

Theorem 2.1 ([15]). Let \((S; \otimes, 1)\) be a system consisting of a non-empty set \(S\), a binary operation \(\oplus\) and a distinct element \(1\). Let \(Q = S \times S = \{(p_1, p_2) : p_1, p_2 \in S\}\). For \(p, t \in Q\) with \(p = (p_1, p_2)\) and \(t = (t_1, t_2)\), we define an operation \(\Phi\) in \(Q\) as

\[ p \Phi t = (p_1 \oplus t_1, p_2 \oplus t_2). \]

Then \((Q; \Phi, (1, 1))\) is a CI-algebra iff \((S; \otimes, 1)\) is a CI-algebra.

Corollary 2.1 ([15]). If \((S; \otimes, 1)\) and \((Q; o, e)\) are two CI-algebras, then \(R = S \times Q\) is also a CI-algebra under the following binary operation:

\[ \text{For } p = (p_1, t_1) \text{ and } t = (p_2, t_2) \text{ in } R, \]
\[ p \Phi t = (p_1 \oplus p_2, t_1 \circ t_2). \]

Here \((1, e)\) is the distinct element of \(R\).

Theorem 2.2 ([17]). Let \((S; \otimes, 1)\) be a CI-algebra and let \(F(S)\) be the class of all functions \(f : S \to S\). Let a binary operation \((\cdot)\) be defined in \(F(S)\) as follows:

\[ \text{For } f, f_1 \in F(S) \text{ and } p \in S, \]
\[ (f \cdot f_1)(p) = f(p) \oplus f_1(p). \]

Then \((F(S); \otimes, 1^{-})\) is a CI-algebra where \(1^{-}\) is defined as \(1^{-}(p) = 1\) for all \(p \in S\). Here two functions \(f, f_1 \in F(S)\) are equal iff \(f(p) = f_1(p)\) for all \(p \in S\).

Notation 2.2 ([17]). (a) For any set \(S_1 \to S\), let \(F(S_1)\) denote the set of all functions \(f \in F(S)\) such that \(f(p) \in S_1\) for all \(p \in S\).

(b) For any \(t \in S\), we consider \(f_t \in F(S)\) defined as \(f_t(p) = t\) for all \(p \in S\).

Definition 2.4 ([12]). A nonempty subset \(F\) of a CI-algebra \(S\) is called a filter of \(S\) if it satisfies

1. \(I \in F\).
2. \(x \oplus y \in F\) and \(x \in F\) then \(y \in F\).

Note 2.1. In example (2.3), \(F = \{1, a, b, c\}\) is a filter in \(S\). However, \(E = \{1, a, b\}\) is not a filter in \(S\).

Lemma 2.2. Let \(F\) be a filter of a CI-algebra \((S; \otimes, 1)\).

If \(x \oplus y = 1\) (i.e., \(x \leq y\)) and \(x \in F\) then \(y \in F\).

3. Closed Filters of CI-Algebras

Definition 3.1 ([13]). A filter \(F\) of a CI-algebra \(S\) is said to be closed if \(x \in F\) if \(x \oplus l \in F\).

Example 3.1 ([13]). Let \(S = R^+ = \{x \in R : x > 0\}\).

For \(x, y \in S\), we define \(x \oplus y = xy\).

Then \((S; \otimes, 1)\) is a CI-algebra. Let us consider \(G = \{2^n : n \in Z\}\).

Then \(F\) is a closed filter in \(S\). However, the set \(F_1 = \{2^n : n \in N\}\) is a filter of \(S\) but it is not closed.

Theorem 3.1. (a) Every filter of a BE-algebra is closed.

(b) Every filter of a finite CI-algebra is closed.

(c) The BE-part \(B(S)\) of a CI-algebra is a closed filter of \(S\).

Theorem 3.2. A filter \(F\) of a CI-algebra \(S\) is closed if it is a subalgebra of \(S\).

Proof. It follows directly from the definition.

Theorem 3.3. \(F\) is a closed filter in \(S \in F(F)\) is a closed filter in \(F(S)\).

Proof. Let \(F\) be a closed filter in \(S\). Then

(i) \(1 \in F\),

(ii) for any \(x, y \in S, x \in F\) and \(x \oplus y \in F\) if \(y \in F\) and
(iii) $x \in \mathcal{F} \in x \oplus 1 \in \mathcal{F}$.

Now $1^- (x) = 1 \in F$ for any $x \in S$.
This means that $1^- \in F(\mathcal{F})$.
Let $f, g \in F(S), f \in F(\mathcal{F})$ and $f \circ g \in F(\mathcal{F})$.
Then $f(x) \in \mathcal{F}$ and $(f \circ g)(x) \in \mathcal{F}$.
Then $f(x) \oplus g(x) \in \mathcal{F}$ for all $x \in S$.

Above conditions imply that $g(x) \in \mathcal{F}$. So $g \in F(\mathcal{F})$.

Let $f \in F(\mathcal{F})$. Then $f(x) \in \mathcal{F}$ for any $x \in S$.
This means that $f(x) \oplus 1 \in \mathcal{F}$ for any $x \in S$.
So $f(x) \oplus 1^- (x) = (f \circ 1^-)(x) \in F$ for any $x \in S$.

This proves that $f \circ 1^- \in F(\mathcal{F})$. Hence $F(\mathcal{F})$ is a closed filter of $F(S)$.

Conversely, suppose that $F(\mathcal{F})$ is a closed filter of $F(S)$. Then $1^- \in F(\mathcal{F})$.
This means that for any $x \in S$, $1^- (x) = 1 \in \mathcal{F}$.

Also $f, g \in F(S), f \in F(\mathcal{F})$ and $f \in g \in F(\mathcal{F}) \in g \in F(\mathcal{F})$.

Let $x, y \in S, x \in F$ and $x \oplus y \in F$. We consider functions $f_x, f_y \in F(\mathcal{F})$ defined as in Notation (2.2)(b). Then $f_x \in F(\mathcal{F})$ and $(f_x \circ f_y)(t) = f_x(t) \oplus f_y(t) = x \oplus y \in \mathcal{F}$ $\Rightarrow$ $f_x \circ f_y \in F(\mathcal{F})$. So from above we have $f_t \in F(\mathcal{F})$.
This gives $f_x(t) = y \in \mathcal{F}$ for all $t \in S$.
Also $f \in F(\mathcal{F})$ $\Rightarrow$ $f \circ 1^- \in F(\mathcal{F})$.
Let $x \in F$. Then $f_x \in F(\mathcal{F})$ $\Rightarrow$ $f_x \circ 1^- \in F(\mathcal{F})$.

This proves that $F$ is a closed filter of $S$.

**Theorem 3.4.** Let $F_1$ and $F_2$ be subsets of a CI-algebra $(S; \oplus, 1)$ and let $F = F_1 \times F_2$. Then $F$ is a closed filter of $Q = S \times S$ iff $F_1$ and $F_2$ are closed filters of $S$.

**Proof.** Let $F$ be a closed filter of $Q$. Then

(i) $(1, 1) \in F$,
(ii) $u \Phi v \in F$ and $u \in F$ $\Rightarrow$ $v \in F$ where $u = (x_1, y_1), v = (x_2, y_2)$ and $x_1, x_2 F_1, y_1, y_2 F_2$ and
(iii) $u = (x, y)F \rightarrow u \Phi (1, 1) = (x \oplus 1, y \oplus 1)F$.

Now

(a) $(1, 1) \in F \Rightarrow 1 \in F_1, 1 \in F_2$.
(b) Let $x_1 \oplus x_2 F_1$ and $x_1 F_1$. We consider $(x_1, 1), (x_2, 1) \in Q$.

Then $(x_1, 1) \Phi (x_2, 1) = (x_1 \oplus x_2, 1)F$ and $(x_1, 1) \in F$.
So from condition (ii) we get $(x_2, 1)F$. This implies that $x_2 F_1$ and hence $F_1$ is a filter of $S$.

Again let $y_1 \oplus y_2 F_2, y_1 F_2$. We consider $(1, y_1), (1, y_2) \in Q$.
Then $(1, y_1) \Phi (1, y_2) = (1, y_1 \oplus y_2)F$ and $(1, y_1) \in F$.

So from condition (ii) we get $(1, y_2)F$. This implies that $y_2 \in F_2$. So $F_2$ is a filter of $S$.

(c) Let $xF_1$. Then $(x, 1)F$. So condition (iii) implies that $(x, 1) \Phi (1, 1)F$, i.e., $(x \oplus 1, 1)F$ which implies that $x \oplus 1 F_1$.

Similarly $y F_2$ implies $y \oplus 1 F_2$.

Hence $F_1$ and $F_2$ are closed filters of $S$.

Conversely, we assume that $F_1$ and $F_2$ are closed filters of $S$. Then $1 \in F_1, 1 \in F_2$ and so $(1, 1)F$.

Let $u = (x, y), u \Phi v \in F$ and $u \in F$.
Then $(x, y) \oplus (x_1, y_1) \in G$ and $(x, y)(x_1, y_1)F$. This implies that $x_1 \oplus x_2 F_1, x_1 F_1, y_1 \oplus y_2 F_2$ and $y_1 \oplus F_2$.

Since $F_1$ and $F_2$ are filters so we get $x_2 F_1, y_2 F_2$ which means that $(x_2, y_2)F$, i.e., $(x, y) \in F$. So $F$ is a filter of $Q$.
Finally, we assume that $u = (x, y) \in F$. Then $x \in F_1$ and $y \in F_2$. Since $F_1$ and $F_2$ are closed filters of $S, x \oplus 1 \in F_1$ and $y \oplus 1 \in F_2$. This proves that $(x \oplus 1, y \oplus 1)F$ i.e., $(x, y) \Phi (1, 1)F$. Hence $F$ is closed.
Using the same technique it can be proved that

**Corollary 3.1.** Let $F_1$ and $F_2$ be subsets of a CI-algebra $(S; \oplus, 1)$ and $(Q; \circ, e)$ respectively. Then $F_1 \times F_2$ is a closed filter of $R = S \times Q$ if $F_1$ and $F_2$ are closed filters of $S$ and $Q$ respectively.
4. Conclusion
Here we want to mention the summary of the results included in the paper. In the preliminary section we include some definitions and basic results. Theorem 3.1 and 3.2 contain some basic results of closed filters in CI-algebra. In Theorem 3.3 we obtain a necessary and sufficient for a function algebra of a closed filter in a CI-algebra to be a closed filter. In Theorem 3.4 we have studied behaviour of cartesian product of two closed filters in a CI-algebra.

Acknowledgement. We would like to record our appreciation and gratefulness to the persons and institutions to whom, we came in contact during the course of our research work for the paper. We express our deep sense of gratitude to all the faculty members in the Departments of Mathematics, B. H. College, Biswanath College and Guwahati College. We take this opportunity to thank the authorities of the aforesaid three colleges for all the excellent facilities that they have provided.

References