ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 52(2) (2022), 224-227**
(Dedicated to Professor D. S. Hooda on His 80th Birth Anniversary Celebrations)

ON CLOSED FILTERS IN *CI*-ALGEBRAS

Pulak Sabhapandit[1], Kuljit Pathak[2] and Manoj Kumar Sarma[3]
[1]Department of Mathematics, Biswanath College, Biswanath Chariali, Assam, 784176, India
[2]Department of Mathematics, B.H. College, Howly, Assam, 781316, India
[3]Department of Mathematics, Guwahati College, Guwahati - 21, Assam, India.
Email: pulaksabhapandit@gmail.com, kulajitpathak79@gmail.com, mksghy3009@gmail.com
(Received: March 23, 2022; In format: April 10, 2022; Revised: September 23, 2022; Accepted:)

DOI: https://doi.org/10.58250/jnanabha.2022.52226

Abstract

The filter theory of *CI*-algebras was first introduced by Meng(2009). This concept plays a significant role in the study of *CI*-algebras. In this paper, we continue to study the notion of closed filter in *CI*-algebra with some characteristic properties. We also study closed filters in the Cartesian product of *CI*-algebra and the function algebra of *CI* -algebra.
2020 Mathematical Sciences Classification: 06F35, 03G25, 03B52.
Keywords and Phrases: *CI*-algebra, *BE*-algebra, filter, closed filter.

1. Introduction

In 1966 the concepts of *BCK* and *BCI*-algebras were first introduced by Imai and Iseki[7] and Iseki[8]. Since then mathematicians have made further studies in the field and developed concepts of similar type such as *BCH* [6], BH [9], d [14], etc. In 2006, the idea of *BE* -algebras [11] was introduced by Kim and Kim as a generalization of dual *BCK*-algebras [10]. In 2010, Meng[12] propounded the notion of a new algebraic structure called *CI*-algebras as a generalization of *BE*-algebras. In his paper Meng[12] defined the filter theory of *CI*-algebras which is considered to be very significant in the study of *CI* -algebras. Closed filters in *CI*-algebras was also studied in details by Meng[13]. The present author(s)[15] developed the concept of Cartesian product of *CI*-algebras which led to further research in the direction and findings were had in 2018 in respect of some special types of *CI*-algebras [17] obtained from a given *CI*-algebra. In this paper we continue to study closed filters in *CI*-algebras and investigate some new properties of it in Cartesian product of *CI*-algebras and the function algebra of *CI*-algebras

2. Preliminaries

Definition 2.1. *An algebraic system (S; ⊕, 1) is called a BE-algebra if it satisfies the following axioms:*
(A1) $p \oplus p = 1$,
(A2) $p \oplus 1 = 1$,
(A3) $1 \oplus p = p$,
(A4) $p \oplus (t \oplus r) = t \oplus (p \oplus r)$ *for all* $p, t, r \in S$.

Definition 2.2 ([12,16,18])**.** *An algebraic system* $(S; \oplus, 1)$ *is called a CI-algebra if it satisfies the following axioms:*
(A1) $p \oplus p = 1$,
(A3) $1 \oplus p = p$,
(A4) $p \oplus (t \oplus r) = t \oplus (p \oplus r)$ *for all* $p, t, r \in S$.

Example 2.1. Let A be a non-empty set and let $F(A)$ be the set of all function $f : A \to (0, \infty)$. For $h, k \in F(A)$, we define a binary operation * as

$$(h * k)(t) = \frac{k(t)}{h(t)}, t \in A.$$

If we put $1(t) = 1$ for all $t \in A$, then $1 \in F(A)$ and simple computation proves that $(F(A); *, 1)$ is a *CI*-algebra.
In S, we can define a binary relation $\leq$ by $s \leq t$ iff $s \oplus t = 1$.

Lemma 2.1 ([12])**.** *In a CI-algebra* $(S; \oplus, 1)$ *following results are true:*
1. $s \oplus ((s \oplus t) \oplus t) = 1$,
2. $(s \oplus t) \oplus 1 = (s \oplus 1) \oplus (t \oplus 1)$ *for all* $s, t \in S$,

Notation 2.1 ([12])**.** *If* $(S;\oplus,1)$ *is a CI - algebra and let* $B(S) = \{x \in S : x \oplus 1 - 1\}$. $B(S)$ *is called the BE - part of* S. *Clearly* $B(S)$ *is non - empty, since* $1 \in B(S)$.

Definition 2.3 ([12])**.** *If for any non-empty subset A of S we have* $x \in A$ *and* $y \in A$ *imply* $x \oplus y \in A$, *then A is called a sub-algebra.*

Theorem 2.1 ([15])**.** *Let* $(S;\oplus,1)$ *be a system consisting of a non-empty set* S, *a binary operation* $\oplus$ *and a distinct element* 1. *Let* $Q = S \times S = \{(p_1, p_2) \colon p_1, p_2 \in S\}$. *For* $p, t \in Q$ *with* $p = (p_1, p_2), t = (t_1, t_2)$, *we define an operation* Φ *in* Q *as*

$$p\Phi t = (p_1 \oplus t_1, p_2 \oplus t_2).$$

Then $(Q;\Phi,(1,1))$ *is a CI-algebra iff (S;* $\oplus$*, 1) is a CI-algebra.*

Corollary 2.1 ([15])**.** *If* $(S;\oplus,1)$ *and* $(Q;o,e)$ *are two CI-algebras, then* $R = S \times Q$ *is also a CI-algebra under the following binary operation:*
For $p = (p_1, t_1)$ *and* $t = (p_2, t_2)$ *in R,*
$p\Phi t = (p_1 \oplus p_2, t_1 o t_2)$.
Here $(1, e)$ *is the distinct element of* R.

Theorem 2.2 ([17])**.** *Let* $(S;\oplus,1)$ *be a CI-algebra and let* $F(S)$ *be the class of all functions* $f : S \to S$. *Let a binary operation* $(\cdot)$ *be defined in* $F(S)$ *as follows:*
For $f, f_1 \in F(S)$ *and* $p \in S$,
$(f \in f_1)(p) = f(p) \oplus f_1(p)$.
Then $(F(S);\odot,1^{\sim})$ *is a CI-algebra where* $1^{\sim}$ *is defined as* $1^{\sim}(p) = 1$ *for all* $p \in S$.
Here two functions $f, f_1 \in F(S)$ *are equal iff* $f(p) = f_1(p)$ *for all* $p \in S$.

Notation 2.2 ([17])**.** *(a) For any set* $S_1 \to S$, *let* $F(S_1)$ *denote the set of all functions* $f \in F(S)$ *such that* $f(p) \in S_1$ *for all* $p \in S$.
(b) For any $t \in S$, *we consider* $f_t \in F(S)$ *defined as* $f_t(p) = t$ *for all* $p \in S$.

Definition 2.4 (12)**.** *A nonempty subset F of a CI-algebra S is called a filter of S if it satisfies*
1. *1* $\in F$.
2. $x \oplus y \in F$ *and* $x \in F \in y \in F$.

Note 2.1. In example (2.3), $F = \{1, a, b, c\}$ is a filter in$S$. However, $E = \{1, a, b\}$ is a not a filter in S.

Lemma 2.2. *Let F be a filter of a CI-algebra* $(S;\oplus,1)$.
If $x \oplus y = 1$ *(i.e.,* $x \leq y$*) and* $x \in F$ *then* $y \in F$.

3. Closed Filters of CI-Algebras

Definition 3.1 ([13])**.** *A filter F of a CI-algebra S is said to be closed if* $x \in F \in x \oplus 1 \in F$.

Example 3.1 ([13])**.** Let $S = R^+ = \{u \in R : x > 0\}$.
For $x, y \in S$, we define $x \oplus y = yx$.
Then $(S;\oplus,1)$ is a CI-algebra. Let us consider $G = \{2^n : n \in Z\}$.
Then F is a closed filter in S. However, the set $F_1 = \{2^n : n \in N\}$ is a filter of S but it is not closed.

Theorem 3.1. *(a) Every filter of a BE-algebra is closed,*
(b) Every filter of a finite CI-algebra is closed,
(c) The BE-part $B(S)$ *of a CI-algebra is a closed filter of* S.

Theorem 3.2. *A filter F of a CI-algebra S is closed iff it is a subalgebra of* S.

Proof. It follows directly from the definition.

Theorem 3.3. F *is a closed filter in* $S \in F(F)$ *is a closed filter in* $F(S)$.

Proof. Let F be a closed filter in S. Then

(i) $1 \in \mathcal{F}$,
(ii) for any $x, y \in S$, $x \in \mathcal{F}$ and $x \oplus y \in \mathcal{F} \in y \in \mathcal{F}$ and

(iii) $x \in \mathcal{F} \in x \oplus 1 \in \mathcal{F}$.

Now $1^{\sim}(x) = 1 \in F$ for any $x \in S$.
This means that $1^{\sim} \in F(\mathcal{F})$.
Let $f, g \in F(S), f \in F(\mathcal{F})$ and $f \odot g \in F(\mathcal{F})$.
Then $f(x) \in \mathcal{F}$ and $(f \odot g)(x) \in \mathcal{F}$.
Then $f(x) \oplus g(x) \in \mathcal{F}$ for all $x \in S$.
Above conditions imply that $g(x) \in \mathcal{F}$. So $g \in F(\mathcal{F})$.
Let $f \in F(\mathcal{F})$. Then $f(x) \in \mathcal{F}$ for any $x \in S$.
This means that $f(x) \oplus 1 \in \mathcal{F}$ for any $x \in S$.
So $f(x) \oplus 1^{\sim}(x) = (f \odot 1^{\sim})(x) \in F$ for any $x \in S$.
This proves that $f \odot 1^{\sim} \in F(\mathcal{F})$. Hence $F(\mathcal{F})$ is a closed filter of $F(S)$.
Conversely, suppose that $F(\mathcal{F})$ is a closed filter of $F(S)$. Then $1^{\sim} \in F(\mathcal{F})$.
This means that for any $x \in S, 1^{\sim}(x) = 1 \in \mathcal{F}$.
Also $f, g \in F(S), f \in F(\mathcal{F})$ and $f \in g \in F(\mathcal{F}) \in g \in F(\mathcal{F})$.
Let $x, y \in S, x \in F$ and $x \oplus y \in F$. We consider functions $f_x, f_y \in F(\mathcal{F})$ defined as in Notation (2.2)(b). Then $f_x \in F(\mathcal{F})$ and $(f_x \odot f_y)(t) = f_x(t) \oplus f_y(t) = x \oplus y \in \mathcal{F} \Rightarrow f_x \odot f_y \in F(\mathcal{F})$. So from above we have $f_y \in F(\mathcal{F})$. This gives $f_y(t) = y \in \mathcal{F}$ for all $t \in S$.
Also $f \in F(\mathcal{F}) \Rightarrow f \odot 1^{\sim} \in F(\mathcal{F})$.
Let $x \in F$. Then $f_x \in F(\mathcal{F}) \Rightarrow f_x \odot 1^{\sim} \in F(\mathcal{F})$.

$$\Rightarrow (f_x \odot 1^{\sim})(t) = f_x(t) \oplus 1^{\sim}(t) = x \oplus 1 \in F, t \in S.$$

This proves that F is a closed filter of S.

Theorem 3.4. *Let F_1 and F_2 be subsets of a CI-algebra $(S; \oplus, 1)$ and let $F = F_1 \times F_2$. Then F is a closed filter of $Q = S \times S$ iff F_1 and F_2 are closed filters of S.*

Proof. Let F be a closed filter of Q. Then
(i) $(1, 1) \in F$,
(ii) $u\Phi v \in F$ and $u \in F \Rightarrow v \in F$ where $u = (x_1, y_1), v = (x_2, y_2)$ and $x_1, x_2 F_1, y_1, y_2 F_2$ and
(iii) $u = (x, y)F \rightarrow u\Phi(1, 1) = (x \oplus 1, y \oplus 1)F$.
Now

(a) $(1, 1) \in F \Rightarrow 1 \in F_1, 1 \in F_2$.
(b) Let $x_1 \oplus x_2 F_1$ and $x_1 F_1$. We consider $(x_1,1),(x_2,1) \in Q$.

Then $(x_1,1)\Phi(x_2,1) = (x_1 \oplus x_{2,}1)F$ and $(x_{1,}1) \in F$.

So from condition (ii) we get $(x_{2,}1)F$. This implies that $x_2 F_1$ and hence F_1 is a filter of S.

Again let $y_1 \oplus y_2 F_2, y_1 F_2$. We consider $(1, y_1),(1, y_2) \in Q$.

Then $(1, y_1)\Phi(1, y_2)=(1, y_1 \oplus y_2)F$ and $(1, y_1)F$.

So from condition (ii) we get $(1, y_2)F$. This implies that $y_2 \in F_2$. So F_2 is a filter of S.

(c) Let xF_1. Then $(x,1)F$. So condition (iii) implies that $(x,1)\Phi(1,1)F$, i.e., $(x \oplus 1, 1)F$ which implies that $x \oplus 1F_1$. Similarly yF_2 implies $y \oplus 1F_2$.

Hence F_1 and F_2 are closed filters of S.

Conversely, we assume that F_1 and F_2 are closed filters of S. Then $1 \in F_1, 1 \in F_2$ and so $(1, 1)F$.
Let $u = (x_1, y_1), u\Phi v \in F$ and $u \in F$.
Then $(x_1 \oplus x_2, y_1 \oplus y_2)G$ and $(x_1, y_1)F$. This implies that $x_1 \oplus x_2 F_1, x_1 F_1, y_1 \oplus y_2 F_2$ and $y_1 \oplus F_2$. Since F_1 and F_2 are filters so we get $x_2 F_1, y_2 F_2$ which means that $(x_2, y_2)F$, i.e., $v \in F$. So F is a filter of Q.
Finally, we assume that $u = (x, y) \in F$. Then $x \in F_1$ and $y \in F_2$. Since F_1 and F_2 are closed filters of $S, x \oplus 1 \in F_1$ and $y \oplus 1F_2$. This proves that $(x \oplus 1, y \oplus 1)F$ i.e., $(x, y)\Phi(1, 1) \in F$. Hence F is closed.
Using the same technique it can be proved that

Corollary 3.1. *Let F_1 and F_2 be subsets of a CI-algebra $(S; \oplus, 1)$ and $(Q; o, e)$ respectively. Then $F_1 \times F_2$ is a closed filter of $R = SxQiff F_1$ and F_2 are closed filters of S and Q respectively.*

4. Conclusion

Here we want to mention the summary of the results included in the paper. In the preliminary section we include some definitions and basic results. Theorem 3.1 and 3.2 contain some basic results of closed filters in *CI*-algebra. In Theorem 3.3 we obtain a necessary and sufficient for a function algebra of a closed filter in a CI-algebra to be a closed filter. In Theorem 3.4 we have studied behaviour of cartesian product of two closed filters in a *CI* -algebra.

Acknowledgement. We would like to record our appreciation and gratefulness to the persons and institutions to whom, we came in contact during the course of our research work for the paper. We express our deep sense of gratitude to all the faculty members in the Departments of Mathematics, B. H. College, Biswanath College and Guwahati College. We take this opportunity to thank the authorities of the aforesaid three colleges for all the excellent facilities that they have provided.

References

[1] D. Das, L. N. Mishra, V. N. Mishra, H. G. Rosales, A. Dhaka, F. E. Lopez Monteagudo E. G. Fernandez and T. A. Ramirez-delReal, C^*-Algebra Valued Modular G-Metric Spaces with Applications in Fixed Point Theory, *Symmetry,* **13**(11) (2021), Article ID: 2003. DOI: https://doi.org/10.3390/sym13112003

[2] D. Das, N. Goswami, and V. N. Mishra, Some Results on Fixed Point Theorems in Algebras, *Int. J. Anal. Appl*., **13**(1) (2017), 32 - 40

[3] V. N. Mishra, S. Delen and I. N. Cangul, Algebraic structure of graph operations in terms of degree sequences, *Int. J. Anal. Appl*., **16** (6) (2018), 809 - 821, DOI: 10.28924/2291-8639-16-2018-809

[4] D. Das, N. Goswami and V. N. Mishra, Some results on the projective cone normed tensor product spaces over Banach algebras, *Boletim da Sociedade Paranaense de Matematica*, **38**(1) (2020), 197 - 220, DOI: 10.5269/bspm.v38il.36450

[5] D. Das, V. N. Mishra and L. N. Mishra, C^*-Algebra Valued Modular SSS-Metric Spaces with Applications in Fixed Point Theory, *Tbilisi Mathematical Journal*, **14**(3) (2021) 111 - 126

[6] Q. P. Hu and X. Li, On BCH-algebras, *Math. Seminer Notes*, **11**(2) (1983), 313 - 320

[7] Y. Imai and K. Iseki On axiom systems of propositional calculi, XIV,*Proc. Japan* Acad., **42**(1) (1966), 19 - 22

[8] K. Iseki An algebra related with a propositional calculus, *Proc. Japan Acad.,* **42**(1) (1966), 1- 29.

[9] Y. B. Jun, E. H. Roh and H. S. Kim, On BH-algebras, *Scientiae Mathematicae,* **1**(3) (1998), 347 - 354

[10] K. H. Kim and Y. H. Yon, Dual BCK-algebra and MV-algebra, *Sci. Math. Japon.,* **66**(2) (2007), 247 - 253

[11] H. S. Kim and Y. H. Kim, On BE-algebras, *Sci. Math. Japon.,* **66**(1) (2007), 113 - 116.

[12] B. L. Meng, *CI*-algebras, *Sci. Math. Japonicae online,* (e-2009), 695 - 701.

[13] B. L. Meng, Closed filters in *CI*-algebras, *Sci. Math. Japonicae,* **71**(3) (2010), 1- 372 (e-2010) 265 - 270.

[14] J. Negger and H. S. Kim, On d-algebras, *Math. Slovaca ,* **49**(1) (1999), 19 - 26.

[15] K. Pathak and P. Sabhapandit, Cartesian Product of BE/CI-algebras, *J. Assam Acad.Math.,* **6**(2013), 33 - 40.

[16] A. Rezaei, A. B. Saeid and K. Y. S. Saber, On pseudo-CI algebras, *Soft Computing,* **23**(2) (2019), 4643 - 4654.

[17] P. Sabhapandit and K. Pathak, Some special type of CI-algebras, *International Journal of Mathematics Trends and Technology*, **64**(1) (2018), 6 - 11.

[18] F. Smarandache, A. Rezaei and H. S. Kim, A New Trend to Extensions of CI-algebras, *International Journal of Neutrosophic Science,* **5** (1) (2020), 8 - 15.

[19] A. Sarma, N. Goswami and V. N. Mishra, Some properties of bilinear mappings on the tensor product of C^*-algebras, *Korean J. Math.,* **27**(4) (2019), 977 - 1003.

[20] A. Sarma, N. Goswami and V. N. Mishra, Some results for a class of Extended centralizers on C^*-algebras, Discrete Mathematics, *Algorithms and Applications*, **12**(6) (2020), DOI: https://doi.org/10.1142/S1793830920500871