ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 52(2) (2022), 202-208**
(Dedicated to Professor D. S. Hooda on His 80^{th} Birth Anniversary Celebrations)

A NOTE ON POSITIVE INTEGER TRIPLES (a, b, c) GENERATED BY THE EQUATION $a^{-1} + b^{-1} = c^{-1}$

J. A. De Ia Cruz[1], J. N. Singh[2] and M. Shakil[3]
[1,2]Department of Mathematics and Computer Science, Barry University, Miami Shores, Florida 33161, USA
[3]Department of Mathematics, Miami Dade College, Hialeah, FL, USA.
Email: jdelacruz@barry.edu, jsingh@barry.edu, mshakil@mdc.edu
(Received : October 03*,* 2022*; In format ; October* 09*,* 2022*; Revised: November* 11*,* 2022*; Accepted: November* 13*,* 2022*)*

DOI: https://doi.org/10.58250/jnanabha.2022.52223

Abstract

This paper develops some general formulas to generate positive integers triples $(a, b, c) \in Z_+^3$ satisfying equation $a^{-1} + b^{-1} = c^{-1}$, which is based on a positive integer parameter $\lambda = b - c$ and $a \geq b > c$. The paper also investigates the area formed by the arcs arising from the equation.
2020 Mathematical Sciences Classification: 11A07, 18C15, 97G30.
Keywords and Phrases: Primitive, Triples, Area.

1. Introduction

Positive integer triples $(a, b, c) \in Z_+^3$ (where Z_+ is the set of positive integers) have been a subject of great interest in mathematics, particularly those connected with the equation $a^n + b^n = c^n$, where n is a positive integer. The famous Pythagorean theorem corresponds to the case of $n = 2$. Euclid's formula is a tool for generating all possible Pythagorean triples (a, b, c) such that $a^2 + b^2 = c^2$ exactly once for a set of positive integers k and m > n (m relatively prime to n, exactly one of the m and n is odd)

$$a = k(m^2 - n^2), b = k(2mn), c = k(m^2 + n^2).$$

In recent years the trees of primitive Pythagorean triples are being investigated as a rooted ternary trees. Some useful and interesting information about these integers' triples can be found in ([2],[3],[4],[7] and [8]).

It is well known that the equation $a^n + b^n = c^n$ does not have positive integer solutions when n ≥ 3(Fermat Last Theorem). Motivated by the great properties of such positive integer triples (a, b, c), we have tried to investigate the equation $a^n + b^n = c^n$ when $n = -1$.

In this note, we present some properties of positive integers triples $(a, b, c) \in Z_+^3$ satisfying the equation

$$a^{-1} + b^{-1} = c^{-1}, \tag{1.1}$$

based on a parameter $\lambda \in Z_+$.

We consider the case of positive integer triples (a, b, c), with$a \geq b > c$, and $b = c + \lambda$.

a.) First, we consider the case when $\lambda \neq 4k$, and $k \in Z_+$, (i.e., $\lambda = 1, 2, 3, 5, 6, 7, 9, \ldots$), we can start probing the smallest values for a and b, analyzing $a^{-1} + b^{-1} >=< c^{-1}$ until we find those values satisfying the equation $a^{-1} + b^{-1} = c^{-1}$.

b.) Second, we analyze the case for $\lambda = 4k$ and $k \in Z_+$, (i.e., λ = 4, 8, 12, 16, 20, ...). For each "λ" we started probing since the smallest n values for a and b, analyzing $a^{-1} + b^{-1} >=< c^{-1}$ until we find those values that satisfy the equation $a^{-1} + b^{-1} = c^{-1}$

It is important to note that the two subsets $Z_+^1 = \{\lambda : \lambda \neq 4k\}$ and $Z_+^2 = \{\lambda : \lambda = 4k\}$ defines a partition of Z_+ of the set of positive integers.

c.) The paper also investigates the area of the region formed by the arcs arising from these equations.

The organization of the paper is as follows. The section 1 is mainly the introduction of the proposed work and its relevance. In section 2 we propose a theorem and some corollaries to generate positive integer triples (a, b, c) satisfying equation $a^{-1} + b^{-1} >= c^{-1}$ and provide the proof of the theorem by mathematical induction. In section 3 we investigate the region formed by certain arcs arising from these equations. In section 4, we present some remarks associated with the equation $a^{-1} + b^{-1} >= c^{-1}$, and its connections with Fermat last theorem, and the Pythagorean triples.

Finally, in section 5 we present some concluding remarks.

2. Interesting Theorem

In this section, we propose a theorem and some corollaries to generate positive integer triples $(a, b, c) \in Z_{+}^{3}$ satisfying equation $a^{-1} + b^{-1} = c^{-1}$, which is based on a parameter $\lambda \in Z_{+}$ and prove them using mathematical induction.

Theorem 2.1. *If $(a, b, c) \in Z_{+}^{3}$, such that $a^{-1} + b^{-1} = c^{-1}$, where $a \geq b > c$, $b = c + \lambda$, if $\lambda \in Z_{+}^{1}, a = \lambda n(n+1)$, $b = \lambda(n+1)$, and $c = \lambda n$; and if $\lambda \in Z_{+}^{2}$, $a = k(n+1)(n+3)$, $b = 2k(n+3)$, $c = 2k(n+1)$.*

Corollary 2.1. *The integers that satisfy the Theorem* 2.1 *must have the following combinations for a, b, and c.*

***Table* 2.1**

a	b	c
even	even	odd
even	odd	even
even	even	even
odd	even	even

Corollary 2.2. *The integers that satisfy the Theorem 2.1 never can have the following combinations for a, b, and c.*

***Table* 2.2**

a	b	c
odd	odd	odd
odd	even	odd
odd	odd	even
even	odd	odd

Proof of Theorem 2.1.

Case I: when $\lambda \in Z_{+}^{1}$.

For $n = 1$ $\frac{1}{2} + \frac{1}{2} = \frac{1}{1}$

Hypothesis:

$$\frac{1}{k(k+1)} + \frac{1}{k+1} = \frac{1}{k} \qquad \text{for } k \in Z. \tag{1a}$$

Thesis:

$$\frac{1}{(k+1)(k+2)} + \frac{1}{k+2} = \frac{1}{k+1}. \tag{1b}$$

From (1a)

$$\frac{1}{k+1} = \frac{1}{k} - \frac{1}{k(k+1)}. \tag{1c}$$

Substituting (1c) in (1b)

$$\frac{1}{(k+1)(k+2)} + \frac{1}{k+2} = \frac{1}{k} - \frac{1}{k(k+1)}$$

i.e.

$$\frac{1 + (k+1)}{(k+1)(k+2)} = \frac{(k+1) - 1}{k(k+1)}$$

Thus

$$\frac{1}{k+1} = \frac{1}{k+1}.$$

Similarly, for $\lambda = 2, 3, 5, 6, 7, 9, \ldots$ we can obtain $a = \lambda n(n+1)$, $b = \lambda(n+1)$ and $c = \lambda n$.

Empirical results were obtained using theorem 1 when $\lambda \in Z_{+}^{1}$ can be depicted in the table below.

Table **2.3**

λ	a	c	$b=c+\lambda$	a^{-1}	b^{-1}	c^{-1}
1	2	1	2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{1}$
1	6	2	3	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$
1	12	3	4	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{3}$
...	...	...	...	...	...	...
λ	a	c	$b=c+\lambda$	a^{-1}	b^{-1}	c^{-1}
2	4	2	4	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
2	12	4	6	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$
2	24	6	8	$\frac{1}{24}$	$\frac{1}{8}$	$\frac{1}{6}$
...	...	...	...	...	...	...
λ	a	c	$b=c+\lambda$	a^{-1}	b^{-1}	c^{-1}
3	6	3	6	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
3	18	6	9	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$
3	36	9	12	$\frac{1}{36}$	$\frac{1}{12}$	$\frac{1}{9}$
...	...	...	...	...	...	...

Case II: When $\lambda \in Z_+^2$

For $n = 1$ result $\frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

Hypothesis:

$$\frac{1}{(k+1)(k+3)} + \frac{1}{2(k+3)} = \frac{1}{2(k+1)} \qquad \text{for } k \in Z_+. \tag{2a}$$

Thesis:

$$\frac{1}{(k+2)(k+4)} + \frac{1}{2(k+4)} = \frac{1}{2(k+2)} \tag{2b}$$

Thus

$$\frac{1}{2(k+2)} = \frac{1}{2(k+1)+2} = \frac{1}{2(k+1)} - \frac{2}{2(k+1)2(k+2)}$$

$$= \frac{1}{2(k+1)} - \frac{1}{2(k+1)(k+2)} \tag{2c}$$

Substituting (2c) in (2b)

$$\frac{1}{(k+2)(k+4)} + \frac{1}{2(k+4)} = \frac{1}{2(k+1)} - \frac{1}{2(k+1)(k+2)}. \tag{2d}$$

Substituting (2a) in (2d)

$$\frac{1}{(k+2)(k+4)} + \frac{1}{2(k+4)} = \frac{1}{(k+1)(k+3)} + \frac{1}{2(k+3)} - \frac{1}{2(k+1)(k+2)}$$

$$k^3 + 8k^2 + 19k + 12 = k^3 + 8k^2 + 19k + 12.$$

The cases for $\lambda = 4, 8, 12, \ldots$ can be obtained from $a = k(n+1)(n+3)$, $b = 2k(n+3)$, $c = 2k(n+1)$. Empirical results were obtained using theorem 1 when $\lambda \in Z_+^2$ can be depicted in the table below.

Table **2.4**

λ	a	c	$b=c+\lambda$	a^{-1}	b^{-1}	c^{-1}
4	8	4	8	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$
4	15	6	10	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{1}{6}$
4	24	8	12	$\frac{1}{24}$	$\frac{1}{12}$	$\frac{1}{8}$
...	...	...	...	...	...	...
λ	a	c	$b=c+\lambda$	a^{-1}	b^{-1}	c^{-1}
8	16	8	16	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$
8	30	12	20	$\frac{1}{130}$	$\frac{1}{30}$	$\frac{1}{12}$
8	48	16	24	$\frac{1}{48}$	$\frac{1}{24}$	$\frac{1}{16}$
...	...	...	...	...	...	...

3. Area of a region enclosed by arcs

In this section, we obtain the area of the region enclosed by arcs (i.e., the area formed by the arcs arising from the equation). Further details of such regions can be found in [2], [5], and [6].

For $\lambda = 1$ let us find the area of the region formed by three arcs (arc-triangle), with sides $\frac{1}{x}$ from 0 to ∞, $\frac{1}{x(x+1)}$ from 0 to the intersection with $\frac{1}{x+1}$, that is in $x = 1$, and $\frac{1}{x+1}$ from 1 to ∞ . The details of such an arc triangle can be found in [1, 2, and 5].

$$A = \lim_{b\to 0}\left\{\int_b^1 \frac{1}{x}dx - \int_b^1 \frac{1}{x(x+1)}dx\right\} + \lim_{b\to\infty}\left\{\int_1^b \frac{1}{x}dx - \int_1^b \frac{1}{x+1}dx\right\}$$

$$= \lim_{b\to 0}\int_b^1 \frac{1}{x+1}dx + \lim_{b\to\infty}\{lnb - ln1 - ln(b+1) + ln2\}$$

$$= \lim_{b\to 0}\{ln2 - ln(b+1)\} + \lim_{b\to\infty} ln(\frac{2b}{b+1})$$
$$= 2ln2.$$

For $\lambda = 2$ let us find the area of the region formed by three arcs (arc-triangle), with sides $\frac{1}{2x}$ from 0 to ∞, $\frac{1}{2x(x+1)}$ from 0 to the intersection with $\frac{1}{2(x+1)}$, that is in $x = 1$, and $\frac{1}{2(x+1)}$ from 1 to ∞.

$$A = \lim_{b\to 0}\left\{\int_b^1 \frac{1}{2x}dx - \int_b^1 \frac{1}{2x(x+1)}dx\right\} + \lim_{b\to\infty}\left\{\int_1^b \frac{1}{2x}dx - \int_1^b \frac{1}{2(x+1)}dx\right\}$$
$$= \frac{1}{2}\lim_{b\to 0}\left\{\int_b^1 \frac{1}{x}dx - \int_b^1 \frac{1}{x(x+1)}dx\right\} + \frac{1}{2}\lim_{b\to\infty}\left\{\int_1^b \frac{1}{x}dx - \int_1^b \frac{1}{x+1}dx\right\}$$
$$= ln2.$$

For $\lambda = 3$ let us find the area of the region formed by the three arcs in figure 3.1 with sides $\frac{1}{3x}$ from 0 to ∞, $\frac{1}{3x(x+1)}$ from 0 to the intersection with $\frac{1}{3(x+1)}$, that is in $x = 1$, and $\frac{1}{3(x+1)}$ from 1 to ∞ .

$$A = \lim_{b\to 0}\left\{\int_b^1 \frac{1}{3x}dx - \int_b^1 \frac{1}{3x(x+1)}dx\right\} + \lim_{b\to\infty}\left\{\int_1^b \frac{1}{3x}dx - \int_1^b \frac{1}{3(x+1)}dx\right\}$$
$$= \frac{1}{3}\lim_{b\to 0}\left\{\int_b^1 \frac{1}{x}dx - \int_b^1 \frac{1}{x(x+1)}dx\right\} + \frac{1}{3}\lim_{b\to\infty}\left\{\int_1^b \frac{1}{x}dx - \int_1^b \frac{1}{x+1}dx\right\}$$
$$= \frac{1}{3}(2ln2) = \frac{2}{3}ln2.$$

From the previous results we can arrive at the following theorem:

Theorem 3.1. *The areas of curved triangles with sides given by the curves* $\frac{1}{\lambda x}$ *from* $x = 0$ *to* ∞, $\frac{1}{\lambda x(x+1)}$ *from* $x = 0$ *to the intersection with* $\frac{1}{\lambda(x+1)}$ *, that is in* $x = 1$*, and* $\frac{1}{\lambda(x+1)}$ *from* $x = 1$ *to* ∞ *are given by:*

$$A = \frac{2}{\lambda}ln2, \lambda \in Z_+^1, \tag{3.1}$$

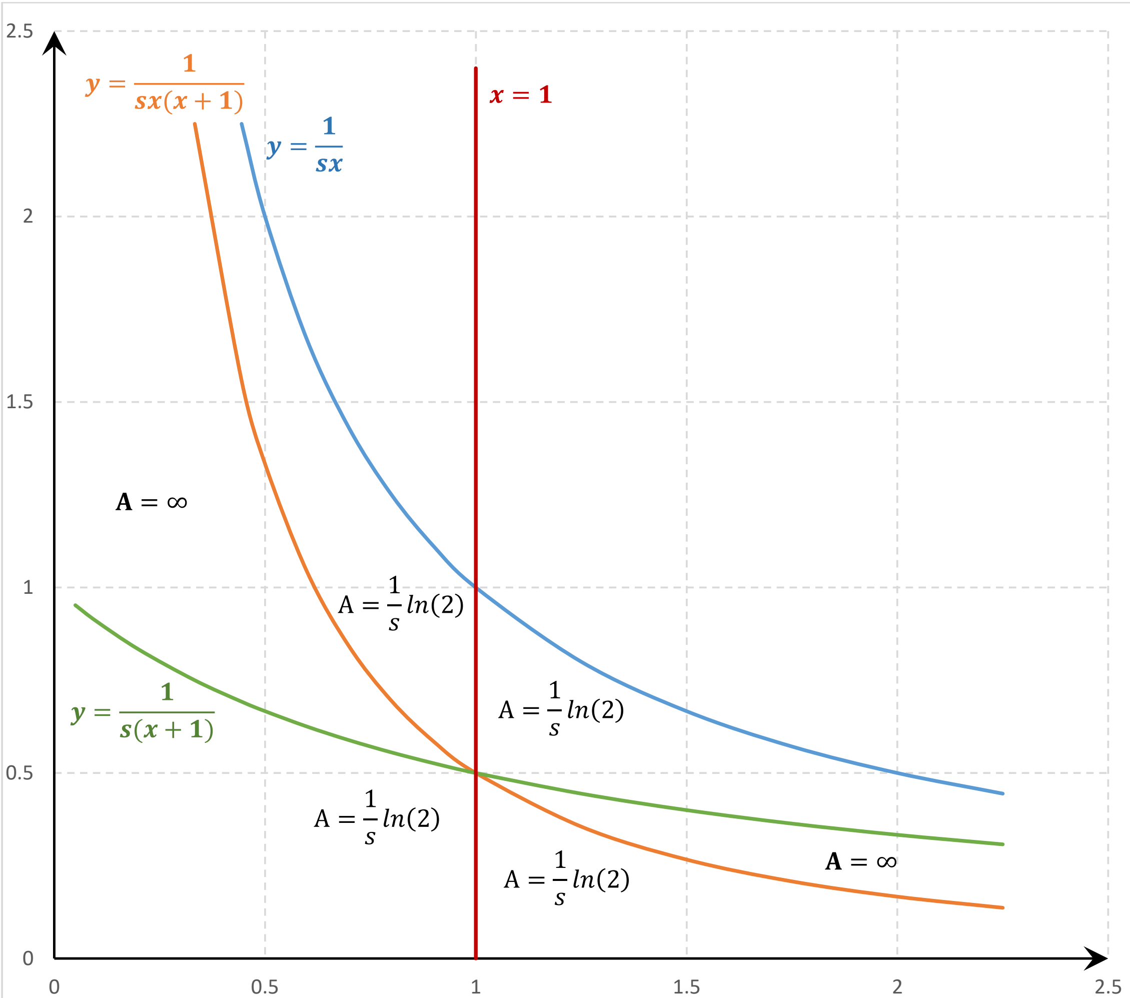

***Figure* 3.1:** Show areas and curves for the general case when $\lambda \in Z_+^1$

For $\lambda = 4$ *let us find the area of the curved region, with curved-sides* $\frac{1}{2(x+1)}$ *from* 0 *to* ∞*,* $\frac{1}{(x+1)(x+3)}$ *from* 0 *to the intersection with* $\frac{1}{2(x+3)}$*, that is,* $x = 1$*, and* $\frac{1}{2(x+3)}$ *from 1 to* ∞*.*

$$A = \lim_{b\to 0}\left\{\int_b^1 \frac{1}{2(x+1)}dx - \int_b^1 \frac{1}{(x+1)(x+3)}dx\right\} + \lim_{b\to\infty}\left\{\int_1^b \frac{1}{2(x+1)}dx - \int_1^b \frac{1}{2(x+3)}dx\right\}$$
$$= \lim_{b\to 0}\int_b^1 \frac{1}{2(x+3)}dx + \frac{1}{2}\lim_{b\to\infty}[ln(x+1) - ln(x+3)]_1^b$$
$$= \lim_{b\to 0}[\tfrac{1}{2}ln(x+3)]_b^1 + \frac{1}{2}\lim_{b\to\infty}\left\{ln(\tfrac{b+1}{b+3}) - ln2 + ln4\right\}$$
$$= \lim_{b\to 0}\frac{1}{2}ln(\tfrac{4}{b+3}) + \frac{1}{2}ln(\tfrac{4}{2})$$
$$= \frac{1}{2}ln(\tfrac{4}{3}) + \frac{1}{2}ln2.$$

For $\lambda = 8$ *let us find the area of the curved triangle in figure above, with sides* $\frac{1}{4(x+1)}$ *from* 0 *to* ∞*,* $\frac{1}{2(x+1)(x+3)}$ *from* 0 *to the intersection with* $\frac{1}{4(x+3)}$*, that is in* $x = 1$*, and* $\frac{1}{4(x+3)}$ *from* 1 *to* ∞ *.*

$$A = \lim_{b\to 0}\left\{\int_b^1 \frac{1}{4(x+1)}dx - \int_b^1 \frac{1}{2(x+1)(x+3)}dx\right\} + \lim_{b\to\infty}\left\{\int_1^b \frac{1}{4(x+1)}dx - \int_1^b \frac{1}{4(x+3)}dx\right\}$$
$$= \lim_{b\to 0}\int_b^1 \frac{1}{4(x+1)}dx + \frac{1}{4}\lim_{b\to\infty}\{ln(x+1) - ln(x+3)\}_1^b$$
$$= \lim_{b\to 0}\left[\tfrac{1}{4}ln(x+3)\right]_b^1 + \frac{1}{4}\lim_{b\to\infty}\left\{ln(\tfrac{b+1}{b+3}) - ln2 + ln4\right\}$$
$$= \lim_{b\to 0}\frac{1}{4}ln(\tfrac{4}{b+3}) + \frac{1}{4}ln\frac{4}{2}$$
$$= \frac{1}{4}ln(\tfrac{4}{3}) + \frac{1}{4}ln2.$$

From the previous results we can arrive at the following theorem:

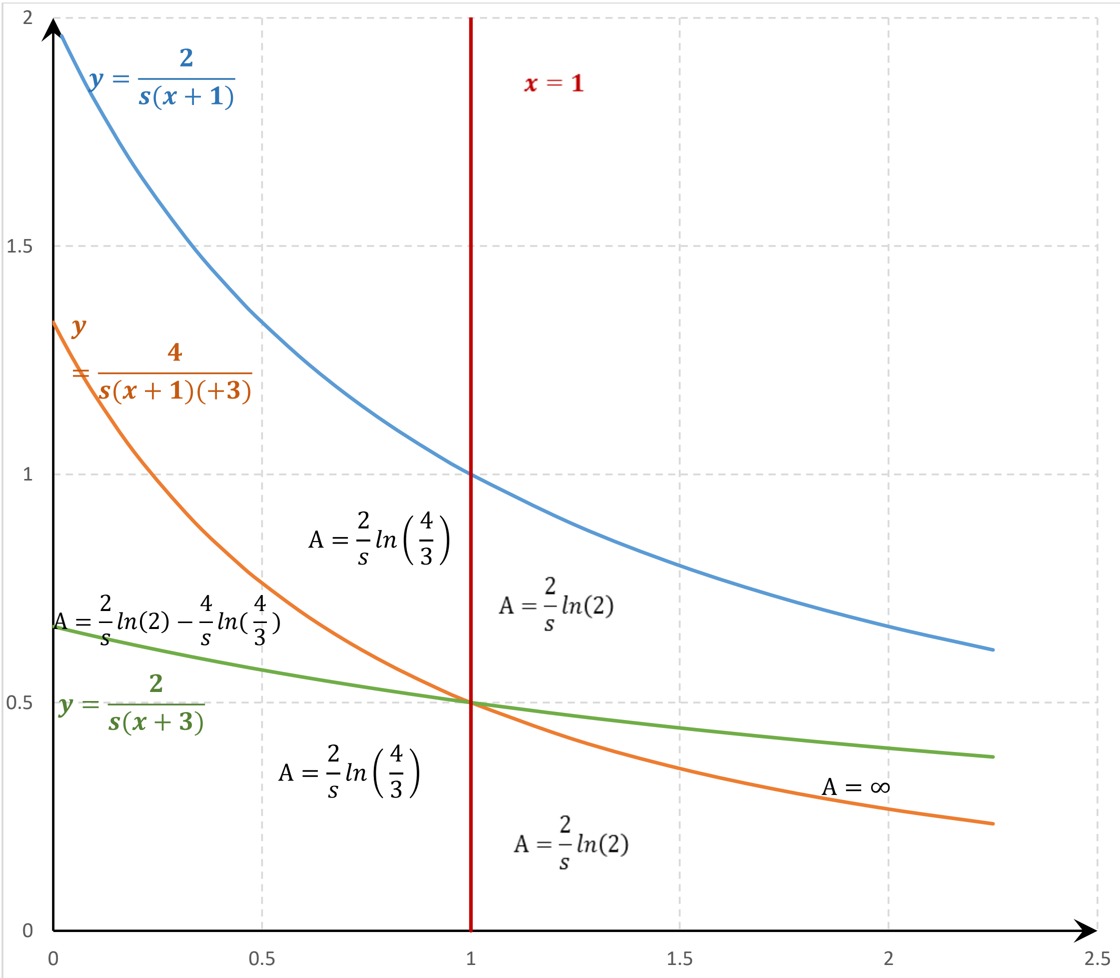

***Figure* 3.2:** Show areas and curves for the general case when $\lambda \in Z_+^2$

.

Theorem 3.2. *The areas of curved regions with sides given by the curves* $\frac{2}{\lambda(x+1)}$ *from* $x = 0$ *to* ∞ , $\frac{4}{\lambda(x+1)(x+3)}$ *from* $x = 0$ *to the intersection with* $\frac{2}{\lambda(x+3)}$, *that is in* $x = 1$, *and* $\frac{2}{\lambda(x+3)}$ *from* $x = 1$ *to* ∞ *are given by:*

$$A = \frac{2}{\lambda} ln(\frac{4}{3}) + \frac{2}{\lambda} ln2, \lambda \in Z_+^2 \tag{3.2}$$

4. Further Remarks

Remark 4.1. For $\lambda = 1$, see Table 1, we have $\frac{1}{a} = \frac{1}{n(n+1)}$.

Let us consider $\frac{1}{n(n+1)} = \frac{d}{n} + \frac{e}{n+1}$ i. e, $d(n+1) + en = 1$.
For $n = 0$ we obtain $d = 1$.
For $n = -1$ we get $e = -1$, then $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, i.e., $\frac{1}{n(n+1)} + \frac{1}{n+1} = \frac{1}{n}$
It means that $a = n(n+1)$, $b = n+1$, and $c = n$, then $a = bc$ and $(a, b, c) \equiv (n(n+1), n+1, n)$
These results for $\lambda = 1$ can be well described in the formula (1)
$\frac{1}{n(n+1)} + \frac{1}{n+1} = \frac{1}{n}$, where $n \in Z$.

Remark 4.2. We get the conclusion that the first term a^{-1}, see Table 2, can be well described in the formula $\frac{1}{(n+1)(n+3)}$ then the other two terms can be obtained for simple fractions as

$\frac{1}{(n+1)(n+3)} = \frac{r}{n+1} + \frac{s}{n+3}$
$r(n+3) + s(n+1) = 1$

For $n = -1$ we obtain $r = \frac{1}{2}$ and for $n = -3$ we get $s = -\frac{1}{2}$ then

$$\frac{1}{(n+1)(n+3)} + \frac{1}{2(n+3))} = \frac{1}{2(n+1)}.$$

It means that $a = (n+1)(n+3)$, $b = (n+3)$, and $c = (n+1)$, then $a = bc/\lambda$ and

$$(a, b, c) \equiv ((n+1)(n+3), (n+3), (n+1))$$

The results in table 2 for $\lambda = 4$ can be well described in the formula (5)

$$\frac{1}{(n+1)(n+3)} + \frac{1}{2(n+3)} = \frac{1}{2(n+1)}, \text{ where } n \in Z_+ \tag{4.1}$$

Remark 4.3. Fermat's Last Theorem states that no three positive integers a, b, and c satisfy the equation

$$a^n + b^n = c^n, \text{ when } n > 2. \tag{4.2}$$

If we divide both sides of equation (4.2) by $a^n b^n c^n$ we get $\frac{1}{b^n c^n} + \frac{1}{a^n c^n} = \frac{1}{a^n b^n}$ i.e., $\frac{1}{(ac)^n} + \frac{1}{(bc)^n} = \frac{1}{(ab)^n}$. Now if we take $A = (bc)^n$, $B = (ac)^n$, and $C = (ab)^n$ then equation (4.2) can be expressed in the form

$$A^{-1} + B^{-1} = C^{-1}, \tag{4.3}$$

as Z_+ is closed under multiplication. Thus, equation (4.2) can be expressed as equation (4.3), but getting a solution to equation (4.2) does not imply a solution to equation (4.2).

Remark 4.4. All Pythagorean triples are related to inverse triples.

Theorem 4.1. *For each Pythagorean Triple (a, b, c) being $a < b < c$ arise from a circular permutation $a \to b \to c \to a$ that $A > B > C$, being $A = (bc)^2$, $= (ac)^2$, $C = (ab)^2$ results $A^{-1} + B^{-1} = C^{-1}$.*

Let's consider Euclid's formula for generating all possible Pythagorean triples (a, b, c) such that $a^2 + b^2 = c^2$ exactly once for a set of positive integers k and m > n (m relatively prime to n, exactly one of m and n odd)

$$a = k(m^2 - n^2), b = k(2mn), c = k(m^2 + n^2).$$

Then $(a, b, c) \equiv (k(m^2 - n^2), k(2mn), k(m^2 + n^2))$ then $A = (bc)^2 = A = (k(2mn)k(m^2 + n^2))^2$, $B = (ac)^2 =(k(m^2 - n^2)k(m^2 + n^2))^2$, $C = (ab)^2 = (k(m^2 - n^2)k(2mn))^2$.

Now, $\frac{1}{A} + \frac{1}{B} = \frac{1}{C}$ implies $\frac{1}{(k^2(2mn)(m^2+n^2))^2} + \frac{1}{(k^2(m^2-n^2)(m^2+n^2))^2} = \frac{1}{(k^2(m^2-n^2)(2mn))^2}$.

This results as an equation $(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$, which is an identity.

5. Concluding Remarks

This paper develops some formulas to generate positive integer triples (a, b, c) satisfying equation $a^{-1} + b^{-1} = c^{-1}$, and shows that the equation $a^n + b^n = c^n$ can be expressed as $A^{-1} + B^{-1} = C^{-1}$. It is expected that this connection will be helpful in investigating the data structures arising from the rooted ternary trees for $n = 2$.

Acknowledgement. Authors are very much thankful to the Editors and Reviewers for their suggestions to bring the paper in its present form.

References

[1] J. E. Brock, The Inertia Tensor for a Spherical Triangle, *J. Appl. Mech.,* **42**(1)(1975), 239-239 (one page).

[2] T. Cătinas, Iterates of Bernstein Type Operators on a Triangle with All Curved Sides, *Abstract and Applied Analysis* , (published 6 March 2014), Article ID 820130, (7 Pages) available at http://dx.doi.org/10.1155/2014/820130.

[3] J. A. De la Cruz and J. Goehl Jr., Two Interesting Integer Parameters of Integer-sided Triangles, Forum *Geometricorum* **17** (2017), 411-517.

[4] A. Hall, Genealogy of Pythagorean Triads, *Math. Gazette,* **54**(390) (1970), 377-379.

[5] T. J. S. Jothi and K. Srinivasan, Acoustic Characteristics of Jets Emanating from Curved Triangular Topologies, *International Journal of Aeroacoustics*, **9**(6) (2010), 821-848.

[6] D. M. Makow, A spherical triangle computer for marine and air navigation, *IEEE Transactions on Aerospace and Navigational Electronics*, **ANE-10**(4) (December 1963), 324-328 Available at: https://ieeexplore.ieee.org/document/4502137

[7] H. L. Price, *The Pythagorean Tree: A new Species* (published September 2008). Available at: https://arxiv.org/pdf/0809.4324.pdf

[8] Kevin Ryde, "Trees of Primitive Pythagorean Triples" (published May 2020). Available at: https://download.tuxfamily.org/user42/triples/triples.pdf