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(Dedicated to Professor D. S. Hooda on His 80^{th} Birth Anniversary Celebrations)

CERTAIN RESULTS OF GENERALIZED BARNES TYPE DOUBLE SERIES RELATED TO THE HURWITZ-LERCH ZETA FUNCTIONS OF TWO VARIABLES

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Abstract

In these researches we introduce a generalized Barnes type double series and then, discuss its convergent conditions. We obtain some of its results related to the known and new Hurwitz -Lerch zeta function of two variables and also derive Eulerian and Mellin- Barnes type integral representations of these functions and analyze various properties these functions.

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1. Introduction, Definitions and Preliminaries

Rainville [30, p. 309] and MacRobert [24, p. 180] studied a second order elliptic function of Weierstrass, $\wp(z)$, in the double series as

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n=-\infty}^{\infty}{}' \left\{ \frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2} \right\}, \tag{1.1}$$

where, $\Omega_{m,n} = 2m\omega_1 + 2n\omega_2, \forall m, n \in \mathbb{Z}, \mathbb{Z}$ is a set of integers, $2\omega_1$ and $2\omega_2$ are fundamental periods , and $2\omega_1$ and $2\omega_2 \in \mathbb{C}, \mathbb{C}$ is set of complex numbers, the summation $\sum_{m,n=-\infty}^{\prime\infty}(.)$ indicates that the indices m, n of summation are not both to be zero at once. The function (1.1) is a doubly periodic in the parallelogram of a complex plane and even function, whereas, MacRobert [24, p. 91] shows that the function

$$\mathcal{F}(\epsilon) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty}{}' \left\{ \frac{1}{(\Omega_{m,n})^{\epsilon}} \right\}, \text{ converges for } \epsilon > 2. \tag{1.2}$$

Val [50] discussed various properties and applications of the elliptic functions (1.1) and (1.2). Erëmenko and Lyubich [11] studied their many dynamical properties. Pastras [29] has applied these functions in classical and quantum mechanics.

For $\Re(s) > 2$, Barnes [4] in 1901, (see also in the researches of Matsumoto [26]), introduced a double zeta function by the meromorphic continuation of the series given as

$$\zeta_2(s; \alpha, w) = \sum_{m,n=0}^{\infty} (\alpha + m + nw)^{-s}, \tag{1.3}$$

provided that $\alpha > 0, s \in \mathbb{C}$ and w is a non-zero complex number in $\mathbb{C}$, with $|\arg(w)| < \pi$. Each term in the right hand side of (1.3) understand as

$$\exp[-s \log(\alpha + m + nw)],$$

where, the logarithm takes the principal branch. The series (1.3) is convergent absolutely in the parallelogram of a complex plane and its continuation is holomorphic with respect to s except for the poles at $s = 1$ and $s = 2$. A famous single series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \tag{1.4}$$

is a Riemann zeta function, $\forall s \in \mathbb{C}, \Re(s) > 1$, (see in [25, p. 13], [49] and others).

A generalization of (1.4) in the form ([3], p. 249)

$$\zeta(s,\alpha)=\sum_{n=0}^{\infty}(n+\alpha)^{-s}, \tag{1.5}$$

is the Hurwitz zeta function where, $\alpha>0, s\in\mathbb{C}, \Re(s)>1$.

A further generalization of (1.4) and (1.5) in the Hurwitz - Lerch zeta function is presented as in [10, p. 27]

$$\Phi(z,s,\alpha)=\sum_{n=0}^{\infty}\frac{z^n}{(n+\alpha)^s}, \tag{1.6}$$

where, $\alpha\in\mathbb{C}\backslash\mathbb{Z}_0^-, \mathbb{Z}_0^-=\{0,-1,-2,-3,\ldots\}, s\in\mathbb{C}$. The series in (1.6) is convergent for all $s\in\mathbb{C}$ and $|z|<1$, and when $z=1$, $\Re(s)>1$. The meromorphic continuation is also studied in ([1], [2], [3], [27] and [51], etc.).

In 2017, Choi and Parmar [7] made extension of the functions (1.4) - (1.6) in another way to define and analyze a two variables Hurwitz-Lerch type zeta function for $\alpha\in\mathbb{C}\backslash\mathbb{Z}_0^-, s\in\mathbb{C}$, in the form

$$\Phi_{a,b,b',c}(x,y,s,\alpha)=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}m!n!}\frac{x^my^n}{(m+n+\alpha)^s}, \tag{1.7}$$

provided that $|x|<1, (x\neq 1); |y|<1, (y\neq 1)$ and $\Re(s)>0$. Again, for $x=1, y=1$, $\Re(c+s-a-b-b')>1$.

Here, for $a\neq 0$, the Pochhammer symbol [45, p. 22] is defined as factorial function by

$$\frac{\Gamma(a+n)}{\Gamma(a)}=(a)_n=\begin{cases}a(a+1)(a+2)\ldots(a+n-1); n\geq 1,\\ 1; n=0.\end{cases}. \tag{1.8}$$

By Gauss formula for gamma function (see Mathai and Haubold [25, p. 3], Srivastava and Manocha [45, p. 20]), and in view of the symbol in Eqn. (1.8), a relation with the Eulerian gamma function

$$\Gamma(z)=\int_0^{\infty}e^{-t}t^{z-1}dt, \Re(z)>0,$$

may be put in the form

$$\lim_{n\to\infty}(s)_n\Gamma(s)=\lim_{n\to\infty}\Gamma(n)n^s, s\neq 0,-1,-2,\ldots, \text{ and } s\in\mathbb{C}. \tag{1.9}$$

In our study, we suppose large values for the numbers M and N, and recall a Kampé de Fériet function [46], in the form

$$F^{p:q;k}_{l:u;v;(M,N)}\left[{(a_p):(b_q);(c_k);\atop (\alpha_l):(\beta_u);(\gamma_v);}x,y\right]=\sum_{m=M,n=N}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{m+n}\prod_{j=1}^{q}(b_j)_m\prod_{j=1}^{k}(c_j)_n}{\prod_{j=1}^{l}(\alpha_j)_{u+v}\prod_{j=1}^{u}(\beta_j)_m\prod_{j=1}^{v}(\gamma_j)_n}\frac{x^my^n}{m!n!}, \tag{1.10}$$

which converges for $|x|<\infty, |y|<\infty$; if $l+u+1-p-q>0$, $l+v+1-p-k>0$; or if $l+u+1-p-q=0$, $l+v+1-p-k=0$, then the function (1.10) converges for $|x|^{\frac{1}{(p-l)}}+|y|^{\frac{1}{(p-l)}}<1$; if $p>l$ and for $\max\{|x|,|y|\}<1$; if $p\leq l$.

Lemma 1.1. *If $\alpha\in\mathbb{C}\backslash\mathbb{Z}_0^-, s\in\mathbb{C}$, then for large values M and N, the Hurwitz-Lerch type zeta function in the double series (1.7) acts as Kampé de Fériet function (1.10). Thus for* $\max\{|x|,|y|\}<1$, *it is represented as*

$$\Phi_{a,b,b',c}(x,y,s,\alpha)=\sum_{m,n=0}^{<M,<N}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}m!n!}\exp[-slog(\alpha+m+n)]x^my^n$$
$$+\frac{\Gamma(\alpha)}{\Gamma(s+\alpha)}F^{2:1;1}_{2:0;0;(M,N)}\left[\begin{cases}a,\alpha:b;b';\\ c,s+\alpha:-;-;\end{cases}x,y\right]. \tag{1.11}$$

Proof. We plug the series in (1.7) in two parts in which second series has summation indices m and n large values M and N respectively and then $\forall\ s\in\mathbb{C}, \alpha>0$, on applying the formula (1.9) in it, we have

$$\Phi_{a,b,b',c}(x,y,s,\alpha)=\sum_{m,n=0}^{<M,<N}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}m!n!}\frac{x^my^n}{(m+n+\alpha)^s}+\sum_{m,n=M,N}^{\infty}\frac{(a)_{m+n}\Gamma(\alpha+m+n)(b)_m(b')_n}{(c)_{m+n}\Gamma(s+\alpha+m+n)}\frac{x^my^n}{m!n!}. \tag{1.12}$$

Here, in the relation (1.12) first series is finite and second series acts as Kampé de Fériet series (1.10) and hence, on making an appeal to the conditions of the function (1.10), the second series in (1.12) converges for $\max\{|x|,|y|\}<1$. Again on applying (1.3) in the first series of (1.12), each term understands as $\frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}m!n!}x^my^nexp[-s\log(\alpha+m+n)]$, and then, we find the formula (1.11) and thus Lemma 1.1 is followed.

Lemma 1.2. *For all $n \in \mathbb{Z}$, and $z \neq 0$, one of an Eulerian type integral representations of the Weierstrass elliptic function (1.1) is found by the formula*

$$\wp(z) = \int_0^\infty t_1 e^{-zt_1} dt_1 - 2\left\{\int_0^\infty t_2 e^{-\frac{z}{2}t_2} \left\{\sum_{m,n=-\infty}^{\infty}{}' \sin h(\frac{z}{2} - \Omega_{m,n})t_2\right\} dt_2\right\}. \tag{1.13}$$

Proof. Make the use of the Eulerian integral functions studied in ([12, [13]), in formula (1.1), we write

$$\wp(z) = \int_0^\infty t_1 e^{-zt_1} dt_1 + \sum_{m,n=-\infty}^{\infty}{}' \left\{\int_0^\infty t_2 e^{-(z-\Omega_{m,n})t_2} dt_2 - \int_0^\infty t_3 e^{-\Omega_{m,n}t_3} dt_3\right\}. \tag{1.14}$$

Then rearrange the integrands in last two integrals of Eqn. (1.14) and then define the sine hyperbolic function, we obtain (1.13) as

$$\wp(z) = \int_0^\infty t_1 e^{-zt_1} dt_1 - 2\left\{\int_0^\infty t_2 e^{-\frac{z}{2}t_2} \left\{\sum_{m,n=-\infty}^{\infty}{}' \sin h(\frac{z}{2} - \Omega_{m,n})t_2\right\} dt_2\right\}. \tag{1.15}$$

Hence, the Lemma 1.2 is followed.

Lemma 1.3. $\forall \alpha \in \mathbb{C}\backslash\mathbb{Z}_0^-$, *$s$, $w \in \mathbb{C}$, and $\Re(s) > 0$, an Eulerian type integral representation of the Barnes type zeta function (1.3) is obtained by*

$$\zeta_2(s;\alpha,w) = \frac{1}{\Gamma(s)}\int_0^\infty e^{-\alpha t}(1-e^{-t})^{-1}(1-e^{-wt})^{-1}t^{s-1}dt. \tag{1.16}$$

Proof. The right hand side of (1.16)

$$= \frac{1}{\Gamma(s)}\int_0^\infty e^{-\alpha t}(1-e^{-t})^{-1}(1-e^{-wt})^{-1}t^{s-1}dt$$

$$= \sum_{m,n=0}^{\infty}\frac{1}{\Gamma(s)}\int_0^\infty e^{-(m+\alpha+nw)t}t^{s-1}dt$$

$= \sum_{m,n=0}^{\infty}(\alpha+m+nw)^{-s}$ = Left hand side of (1.16), (on use of the formulae in ([12], [13]).

Lemma 1.1 shows that the Hurwitz-Lerch type zeta function in the double series (1.7) becomes a hypergeometric function in the form of the Kampé de Fériet function (1.10) when indices of the series are large, as given in (1.11), and hence last large terms of the series of (1.11) converges according to the ruling of convergence of the Kampé de Fériet series (1.10). By Lemmas 1.2 and 1.3, we show that Weierstrass elliptic function (1.1) and the Barnes type zeta function (1.3) are represented by Eulerian type integrals and under the given conditions therein and accordingly given conditions in Lemma 1.2 and 1.3, these functions are convergent.

To explore new ideas in the field of analytic number theory studied in the researches of ([1], [2], [3], [10], [27], ([34]-[38]), [51] and others) and by the above motivations (1.1)-(1.16), we introduce a generalized Barnes type double series and then, discuss its convergent conditions and obtain its related Hurwitz -Lerch zeta function of two variables. We also derive its Mellin- Barnes type integral representation to analyze various properties and relations with other known special functions, we refer to the researches given in the books ([1], [10], [24], [27], [51]), for introductory number theory in ([1], [2], [3], [10], [27], [51]), for convergence conditions of double series see in ([4], [5], [12], [15], [16], [17], [18], [28] and [44]) and for special functions see in ([13], [25], [30], [34], [40], [41], [44], [45] and others). Zeta functions, Hurwitz-Lerch zeta functions and their related functions are studied in the researches of ([2], [6], [7], [8], [9], [14], [16], [17], [22], [23], ([31]-[33]), ([37]-[39]), [43], [47], [48] and others).

2. Generalized Barnes type Double Series, its Convergence Conditions and its related Hurwitz-Lerch Zeta Functions and Special Functions

In this section to explore extensions in analytic number theory, we make a generalization of Barnes type double series (1.3) in following double series. We study its convergence conditions and its related Hurwitz-Lerch zeta functions.

A generalized Barnes type double series is introduced and defined as

$$\Phi_A^{\mu,\nu,\gamma,\delta}(x,y,(\sigma,\rho),(\alpha,\beta),w_1,w_2) = \sum_{m,n=0}^{\infty}\frac{\mathcal{A}(m,n)}{\Gamma(\mu m+\mu n+\gamma)\Gamma(\nu m+\delta)}\frac{x^m y^n}{(mw_1+\alpha)^\sigma(mw_1+nw_2+\beta)^\rho}. \tag{2.1}$$

Here, the double function $A(\tau,\kappa)$ satisfying certain restrictions, $\mu,\nu,\gamma,\delta,\sigma,\rho,w_1,x,y$ and w_2 are complex numbers such that $\Re(\rho) > 0, \beta \geq \alpha > 0, \alpha,\beta \in \mathbb{R}$, *i.e.* $\alpha \in \mathbb{R}\backslash\{0\}, |(\frac{kw_2+r}{mw_1+\alpha})| < 1$, for all $k,m \in \mathbb{N}$ and $r \in \mathbb{N}_0$.

Now to obtain convergence conditions of the function (2.1), we present following theorems:

Theorem 2.1. *In* (2.1) *considering* $w_1, w_2, \mu, \nu \in \mathbb{R}^+$; $\alpha, \beta, \gamma, \delta, \rho, \sigma \in \mathbb{C}$ *and supposing that*

$$A(m,n) = H^{A\,:\,B;B'}_{C\,:\,D;D'}(m,n) = \frac{\prod_{j=1}^{A}\Gamma(a_j+\theta_j m+\vartheta_j n)}{\prod_{j=1}^{C}\Gamma(c_j+\delta_j m+\kappa_j n)}\frac{\prod_{j=1}^{B}\Gamma(b_j+\psi_j m)}{\prod_{j=1}^{D}\Gamma(d_j+\varphi_j m)}\frac{\prod_{j=1}^{B'}\Gamma(b'_j+\psi'_j n)}{\prod_{j=1}^{D'}\Gamma(d'_j+\varphi'_j n)\Gamma(m+1)\Gamma(n+1)}, \quad (2.2)$$

where, the coefficients are taken as $\theta_j, \vartheta_j \in \mathbb{R}^+$, *with* $(j = 1, 2, \ldots, A)$; $\psi_j \in \mathbb{R}^+$, *with* $(j = 1, 2, \ldots, B)$; $\psi'_j \in \mathbb{R}^+$ *with* $(j = 1, 2, \ldots, B')$; $\delta_j, \kappa_j \in \mathbb{R}^+$ *with* $(j = 1, 2, \ldots, C)$; $\varphi_j \in \mathbb{R}^+$, *with* $(j = 1, 2, \ldots, D)$; $\varphi'_j \in \mathbb{R}^+$ *with* $(j = 1, 2, \ldots, D')$; $\mathbb{R}^+ = \{$ *a set of positive real numbers* $\}$; *then for large values of* M *and* N, *the double series* (2.1) *is plugged as*

$$\Phi_A^{\mu,\nu,\gamma,\delta}(x,y,(\sigma,\rho),(\alpha,\beta),w_1,w_2)$$
$$= \sum_{m,n=0}^{<M,N} \frac{H^{A\,:\,B;B'}_{C\,:\,D;D'}(m,n)}{\Gamma(\mu m+\mu n+\gamma)\Gamma(\nu m+\delta)}\frac{x^m y^n}{(mw_1+\alpha)^{\sigma}(mw_1+nw_2+\beta)^{\rho}} + S^{A+1\,:\,B+1;B'}_{C+2\,:\,D+2;D'}(M,N)$$
$$\times\left(\begin{matrix}[(a):\theta,\vartheta],[\beta:w_1,w_2]:[(b):\psi],[\alpha:w_1];\left[(b'):\psi'\right];\\ [(c):\delta,\kappa],[\gamma:\mu,\mu],[\beta+\rho:w_1,w_2]:[(d):\varphi],[\delta:\nu],[\alpha+\sigma:w_1];\left[(d'):\varphi'\right];\end{matrix}x,y\right). \quad (2.3)$$

Further by (2.2) *and* (2.3), *the series* (2.1) *is convergent for* $|x| < \infty, |y| < \infty$, *for Large values* M, N *of under the conditions*

$$\sum_{j=1}^{C}\delta_j + \sum_{j=1}^{D}\varphi_j + \mu + \nu - \sum_{j=1}^{A}\theta_j - \sum_{j=1}^{B}\psi_j + 1 > 0;$$

$$\sum_{j=1}^{C}\kappa_j + \sum_{j=1}^{D'}\varphi'_j + \mu - \sum_{j=1}^{A}\vartheta_j - \sum_{j=1}^{B'}\psi'_j + 1 > 0.$$

On the other hand when

$$\sum_{j=1}^{C}\delta_j + \sum_{j=1}^{D}\varphi_j + \mu + \nu - \sum_{j=1}^{A}\theta_j - \sum_{j=1}^{B}\psi_j + 1 = 0;$$

$$\sum_{j=1}^{C}\kappa_j + \sum_{j=1}^{D'}\varphi'_j + \mu - \sum_{j=1}^{A}\vartheta_j - \sum_{j=1}^{B'}\psi'_j + 1 = 0. \quad (2.4)$$

Also, the function $S^{A+1\,:\,B+1;B'}_{C+2\,:\,D+2;D'}(M,N)(.,.)$ *in* (2.3) *converges for* $\max\{|x|,|y|\} < 1$, *provided that* $A+1 \le C+2 \forall A \le C$ *and for Large* M, N.

Proof. Under the conditions $w_1, w_2, \mu, \nu \in \mathbb{R}^+$, and $\alpha, \beta, \gamma, \delta, \rho$ and $\sigma \in \mathbb{C}$, and for large M, N, and on using the Gauss formula (1.9), the function (2.1) with (2.2), is written by

$$\Phi_A^{\mu,\nu,\gamma,\delta}(x,y,(\sigma,\rho),(\alpha,\beta),w_1,w_2) = \sum_{m,n=0}^{<M,N}\frac{H^{A\,:\,B;B'}_{C\,:\,D;D'}(m,n)}{\Gamma(\mu m+\mu n+\gamma)\Gamma(\nu m+\delta)}\frac{x^m y^n}{(mw_1+\alpha)^{\sigma}(mw_1+nw_2+\beta)^{\rho}}$$
$$+\sum_{m,n=M,N}^{\infty}\frac{H^{A\,:\,B;B'}_{C\,:\,D;D'}(m,n)\Gamma(mw_1+\alpha)\Gamma(mw_1+nw_2+\beta)}{\Gamma(\mu m+\mu n+\gamma)\Gamma(mw_1+nw_2+\beta+\rho)}\frac{x^m y^n}{\Gamma(\nu m+\delta)\Gamma(mw_1+\alpha+\sigma)}. \quad (2.5)$$

Now in the second series of (2.5), make an appeal to the Srivastava and Daoust double series ([40]-[42]) given as (also see in the researches of ([5], [19]-[21], [28])

$$S^{A\,:\,B;B'}_{C\,:\,D;D'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];[(b'):\psi'];\\ [(c):\delta,\kappa]:[(d):\varphi];[(d'):\varphi'];\end{matrix}x,y\right) = \sum_{m,n=0}^{\infty}H^{A\,:\,B;B'}_{C\,:\,D;D'}(m,n)x^m y^n, \quad (2.6)$$

it converges for $|x| < \infty, |y| < \infty$, under the conditions given by

$$\sum_{j=1}^{C}\delta_j + \sum_{j=1}^{D}\varphi_j - \sum_{j=1}^{A}\theta_j - \sum_{j=1}^{B}\psi_j + 1 > 0;$$

$$\sum_{j=1}^{C}\kappa_j+\sum_{j=1}^{D'}\varphi'_j-\sum_{j=1}^{A}\vartheta_j-\sum_{j=1}^{B'}\psi'_j+1>0. \tag{2.7}$$

On the other hand when

$$\sum_{j=1}^{C}\delta_j+\sum_{j=1}^{D}\varphi_j+\mu+\nu-\sum_{j=1}^{A}\theta_j-\sum_{j=1}^{B}\psi_j+1=0;$$

$$\sum_{j=1}^{C}\kappa_j+\sum_{j=1}^{D'}\varphi'_j+\mu-\sum_{j=1}^{A}\vartheta_j-\sum_{j=1}^{B'}\psi'_j+1=0, \tag{2.8}$$

it converges for $\max\{|x|,|y|\}<1$, provided that $A\leq C$.

Hence making an appeal to (2.6)-(2.8) in second series of (2.5), we prove the *Theorem* 2.1.

Theorem 2.2. *If for parameters $w_2,\gamma,\delta,\rho,\mu,\nu$ and $\sigma\in\mathbb{C}$, such that $\Re(\mu)>0,\Re(\nu)>0,\Re(\gamma)>0$, $\Re(\delta)>0$ and $w_1,\alpha,\beta,\lambda\in\mathbb{R}^+,\forall\beta\geq\alpha>0$, and $|\frac{A(m,n)}{\Gamma(\mu m+\mu n+\gamma)\Gamma(\nu m+\delta)}|<1\forall m\geq 1,n\geq 0$, then for $x=y=1$, the double series given in right hand side of* (2.1) *has the form*

$$\Phi_A^{\mu,\nu,\gamma,\delta}(x,y,(\sigma,\rho),(\alpha,\beta),w_1,w_2)=\sum_{m,n=0}^{\infty}\frac{A(m,n)}{\Gamma(\mu m+\mu n+\gamma)\Gamma(\nu m+\delta)}\frac{1}{(mw_1+\alpha)^{\sigma}(mw_1+nw_2+\beta)^{\rho}}, \tag{2.9}$$

and it is convergent when $\lambda+\Re(\sigma)>0,\Re(\rho)>\lambda+2$; where, λ is finite positive.

Proof. For $w_2\in\mathbb{C};w_1,\alpha,\beta,\lambda\in\mathbb{R}^+$ and $\beta\geq\alpha>0$, then $\forall m\geq 1,n\geq 0$, there exists an inequality

$$|mw_1+nw_2+\beta|\geq C(w_2)(mw_1+\beta)\forall n\geq 0,m\geq 1;$$

where

$$C(w_2)=\begin{cases}1;\ \text{if } 0\leq\arg(w_2)\leq\frac{\pi}{2};\\ \sin(\pi-\arg(w_2)),\ \text{if } \frac{\pi}{2}\leq\arg(w_2)\leq\pi.\end{cases} \tag{2.10}$$

Then, by inequality (2.10), for $\beta\geq\alpha>0$, there exists a formula $\frac{|mw_1+nw_2+\beta|}{(mw_1+\alpha)}\geq C(w_2)$ which with an appeal to techniques of Matsumoto [26, Lemma 1, p.11], gives the series (2.9) in the form

$$|\sum_{m,n=0}^{\infty}\frac{A(m,n)}{\Gamma(\mu m+\mu n+\gamma)\Gamma(\nu m+\delta)}\frac{1}{(mw_1+\alpha)^{\sigma}(mw_1+nw_2+\beta)^{\rho}}|$$

$$\leq\sum_{m,n=0}^{\infty}|\frac{A(m,n)}{\Gamma(\mu m+\mu n+\gamma)\Gamma(\nu m+\delta)}||\frac{1}{(mw_1+\alpha)^{\sigma}(mw_1+nw_2+\beta)^{\rho}}|$$

$$\leq\sum_{m,n=0}^{\infty}|\frac{A(m,n)}{\Gamma(\mu m+\mu n+\gamma)\Gamma(\nu m+\delta)}|\frac{1}{|(mw_1+\alpha)^{\sigma}||(mw_1+nw_2+\beta)^{\rho}|}$$

$$\leq e^{\pi\Im(\rho)}\sum_{m=0}^{\infty}|\frac{1}{\Gamma(\nu m+\delta)(mw_1+\alpha)^{\Re(\sigma)}}|\sum_{n=0}^{\infty}\frac{|A(m,n)|}{|\Gamma(\mu m+\mu n+\gamma)|(mw_1+nw_2+\beta)^{\Re(\rho)}}. \tag{2.11}$$

Since, $\lambda>0$, and $\lambda+\Re(\sigma)>0$, therefore making an appeal to (2.10) and (2.11), we get

$$|\sum_{m,n=0}^{\infty}\frac{A(m,n)}{\Gamma(\mu m+\mu n+\gamma)\Gamma(\nu m+\delta)}\frac{1}{(mw_1+\alpha)^{\sigma}(mw_1+nw_2+\beta)^{\rho}}|$$

$$<e^{\pi\Im(\rho)}|\frac{1}{(\alpha)^{\Re(\sigma)}}|\sum_{n=0}^{\infty}\frac{|A(0,n)|}{|\Gamma(\mu n+\gamma)|\Gamma(\delta)(nw_2+\beta)^{\Re(\rho)}}$$

$$+e^{\pi\Im(\rho)}\sum_{m=1}^{\infty}|\frac{1}{\Gamma(\nu m+\delta)}|\sum_{n=0}^{\infty}\frac{(mw_1+\alpha)^{\lambda}(mw_1+nw_2+\beta)^{-\Re(\rho)+\lambda}|A(m,n)|}{(mw_1+nw_2+\beta)^{\lambda}|\Gamma(\mu m+\mu n+\gamma)|}$$

$$<e^{\pi\Im(\rho)}|\frac{1}{(\alpha)^{\Re(\sigma)}}|\sum_{n=0}^{\infty}\frac{|A(0,n)|(nw_2+\beta)^{-\Re(\rho)}}{\Gamma(\delta)|\Gamma(\mu n+\gamma)|}$$

$$+e^{\pi\Im(\rho)}\{C(w_2)\}^{-\lambda}\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(mw_1+nw_2+\beta)^{-\Re(\rho)+\lambda}|A(m,n)|}{|\Gamma(\mu m+\mu n+\gamma)||\Gamma(\nu m+\delta)|}. \tag{2.12}$$

Now in (2.12), making an appeal to conditions given in the *Theorem* 2.2 as $|\frac{A(m,n)}{\Gamma(\mu m+\mu n+\gamma)\Gamma(\nu m+\delta)}|<1\forall m\geq 1, n\geq 0$, and to the convergence conditions given in (1.3), the series (2.9) converges for

$$\Re(\rho)-\lambda>2\Rightarrow\Re(\rho)>\lambda+2,\ \text{and}\ \lambda+\Re(\sigma)>0.$$

Hence, the *Theorem* 2.2 is followed.

Further to derive the relations involving the function (2.1) with known and new zeta functions and with other special functions, we employ the series rearrangement techniques in the double series of (2.1) and write it in the form

$$\Phi_A^{\mu,\nu,\gamma,\delta}(x,y,(\sigma,\rho),(\alpha,\beta),w_1,w_2)=\sum_{n=0}^{\infty}\frac{y^n}{\Gamma(\mu n+\gamma)}\sum_{m=0}^{n}\frac{A(m,n-m)}{\Gamma(\nu m+\delta)}\frac{(xy^{-1})^m}{(mw_1+\alpha)^{\sigma+\rho}}(1+\frac{(n-m)w_2+(\beta-\alpha)}{mw_1+\alpha})^{-\rho}. \tag{2.13}$$

Now in right hand side of Eqn. (2.12), suppose that $\lambda(n,m)=\frac{(n-m)w_2+(\beta-\alpha)}{mw_1+\alpha}$, and $|\lambda(n,m)|<1,\ \forall m\geq 0, n\geq 0$, and $\Re(\rho)>0, \beta\geq\alpha>0, \rho>0$, such that $\alpha,\beta\in\mathbb{R}^+$, $|y|<R^+$, R^+ is positive finite, then we get a relation for (2.1) involving the special function ${}_1F_0(.)$ as

$$\Phi_A^{\mu,\nu,\gamma,\delta}(x,y,(\sigma,\rho),(\alpha,\beta),w_1,w_2)=\sum_{n=0}^{\infty}\frac{y^n}{\Gamma(\mu n+\gamma)}\sum_{m=0}^{n}\frac{A(m,n-m)}{\Gamma(\nu m+\delta)}\frac{(xy^{-1})^m}{(mw_1+\alpha)^{\sigma+\rho}}{}_1F_0(\rho;-;-\lambda(n,m)). \tag{2.14}$$

Remark 2.1. In relation (2.1) *set* $A(m,n)=C(m)B(m+n), x=y$, *and suppose that*

$$K_{B,C}^{\nu,\mu,\gamma,\delta,\alpha,\beta}(y;\sigma,\rho,w_1,w_1)=\sum_{m=0}^{\infty}\frac{C(m)B(m)}{\Gamma(\mu m+\gamma)\Gamma(\nu m+\delta)}\frac{y^m}{(mw_1+\alpha)^{\sigma}(mw_1+\beta)^{\rho}},$$

then we get

$$\begin{aligned}
&\Phi_{B,C}^{\mu,\nu,\gamma,\delta}(x,y,(\sigma,\rho),(\alpha,\beta),w_1,w_2)\\
&=\sum_{m,n=0}^{\infty}\frac{C(m)B(m+n)}{\Gamma(\mu m+\mu n+\gamma)\Gamma(\nu m+\delta)}\frac{y^{n+m}}{(mw_1+\alpha)^{\sigma}(mw_1+nw_2+\beta)^{\rho}}\\
&=\sum_{m=0}^{\infty}\frac{C(m)B(m)}{\Gamma(\mu m+\gamma)\Gamma(\nu m+\delta)}\frac{y^m}{(mw_1+\alpha)^{\sigma}(mw_1+\beta)^{\rho}}\\
&+\sum_{n=1}^{\infty}\frac{B(n)y^n}{(nw_2+\beta)^{\rho}\Gamma(\mu n+\gamma)}\sum_{m=0}^{n}\frac{C(m)}{(mw_1+\alpha)^{\sigma}\Gamma(\nu m+\delta)}(1+\frac{m(w_1-w_2)}{nw_2+\beta})^{-\rho}\\
&=\sum_{m=0}^{\infty}\frac{C(m)B(m)}{\Gamma(\mu m+\gamma)\Gamma(\nu m+\delta)}\frac{y^m}{(mw_1+\alpha)^{\sigma}(mw_1+\beta)^{\rho}}+\frac{C(0)}{(\alpha)^{\sigma}\Gamma(\delta)}\sum_{n=1}^{\infty}\frac{B(n)y^n}{(nw_2+\beta)^{\rho}\Gamma(\mu n+\gamma)}\\
&+\sum_{r=0}^{\infty}\frac{(\rho)_r(-1)^r(w_1-w_2)^r}{r!}\sum_{n=1}^{\infty}\frac{B(n)y^n}{(nw_2+\beta)^{\rho+r}\Gamma(\mu n+\gamma)}\sum_{m=1}^{n}\frac{C(m)(m)^r}{(mw_1+\alpha)^{\sigma}\Gamma(\nu m+\delta)}.\\
&=K_{B,C}^{\nu,\mu,\gamma,\delta,\alpha,\beta}(y;\sigma,\rho,w_1)+\frac{C(0)}{(\alpha)^{\sigma}\Gamma(\delta)}K_{B,1}^{0,\mu,\gamma,1,\alpha,\beta}(y;0,\rho,w_2)\\
&+\sum_{r=0}^{\infty}\frac{(\rho)_r(-1)^r(w_1-w_2)^r}{r!}\sum_{n=1}^{\infty}\frac{B(n)y^n}{(nw_2+\beta)^{\rho+r}\Gamma(\mu n+\gamma)}K_{1,C}^{\nu,0,1,\delta,\alpha,0}(1;\sigma,-r,w_1,1;n).
\end{aligned} \tag{2.15}$$

Here,

$$K_{1,C}^{\nu,0,1,\delta,\alpha,0}(1;\sigma,-r,w_1,1;n)=\sum_{m=1}^{n}\frac{C(m)(m)^r}{(mw_1+\alpha)^{\sigma}\Gamma(\nu m+\delta)}. \tag{2.16}$$

Now, we obtain a relation for (2.1) with the Srivastava's zeta type function which is defined in ([34]-[36]) as

$$E_{\nu,\delta}(\phi;z,s,\alpha)=\sum_{n=0}^{\infty}\frac{\phi(n)}{(n+\alpha)^s}\frac{z^n}{\Gamma(\nu n+\delta)},(\nu,\delta\in\mathbb{C},\Re(\nu)>0). \tag{2.17}$$

Here in (2.17), ϕ is a suitably-restricted function, the parameters ν,δ,s and α are appropriately constrained.

Then making an appeal to (2.14) and (2.17), we obtain a relation in the form

$$\Phi_A^{\mu,\nu,\gamma,\delta}(x,y,(\sigma,\rho),(\alpha,\beta),w_1,w_2)=\sum_{n=0}^{\infty}\frac{y^n}{\Gamma(\mu n+\gamma)}\sum_{r=0}^{\infty}\frac{(\rho)_r}{r!}\{E_{\nu,\delta}(A(m,n-m)(-z)^r;(xy^{-1}),w_1,\sigma+\rho+r,\alpha)\}, \quad (2.18)$$

where, the function $E_{\nu,\delta}(.)$ in relation (2.18) is given by

$$E_{\nu,\delta}(A(m,n-m)(-z)^r;(xy^{-1}),w_1,\sigma+\rho+r,\alpha)=\sum_{m=0}^{n}\frac{A(m,n-m)(-z)^r(xy^{-1})^m}{\Gamma(\nu m+\delta)(mw_1+\alpha)^{\sigma+\rho+r}},$$

and

$$z=(n-m)w_2+(\beta-\alpha),\beta\geq\alpha>0,0\leq m\leq n. \quad (2.19)$$

Again in the general formula (2.1), setting $A(m,n)=\frac{(a)_{m+n}(b)_m(b')_n}{(1)_n},\mu=\nu=\delta=1,\gamma=c$, such that $c\neq 0,-1,-2,\ldots$,and $w_1=1,w_2=w$, we get a general Barnes type double series analogous to the Hurwitz-Lerch function in the form

$$\Phi_{a,b,b'}^{c}(x,y,(\sigma,\rho),(\alpha,\beta),w)=\frac{1}{\Gamma(c)}\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}(1)_m(1)_n}\frac{x^my^n}{(m+\alpha)^{\sigma}(m+nw+\beta)^{\rho}}. \quad (2.20)$$

The series (2.20) is absolutely convergent for all w, σ and ρ complex numbers, $\beta\geq\alpha>0$, and with $|\arg(w)|<\pi$ and $|x|<1,|y|<1$. Again, when $x=y=1$; it converges absolutely for , $\beta\geq\alpha>0$, w is a complex number with $|\arg(w)|<\pi$, $\Re(\sigma)>-\delta$ and $\Re(\rho)>2+\delta$, δ is any positive real number.

Evidently, employing (2.20) with (1.7) for $c\neq 0,-1,-2,\ldots$, we derive

$$\lim_{\sigma\to 0,w\to 1}\Phi_{a,b,b'}^{c}(x,y,(\sigma,\rho),(\alpha,\beta),w)=\frac{1}{\Gamma(c)}\Phi_{a,b,b',c}(x,y,\rho,\beta). \quad (2.21)$$

provided that $|x|<1,(x\neq 1);|y|<1,(y\neq 1)$ and $\Re(\rho)>0$. Again, for $x=1,y=1,\Re(\rho)>1$.

Again in (2.20) setting $a=c,b=1,b'=1,x=1,y=1,\sigma\to 0$ and $c\neq 0,-1,-2,\ldots$, we find it's a relation with the Barnes' type function (1.3) as

$$\lim_{\sigma\to 0}\Phi_{a,b,b'}^{a}(1,1,(\sigma,\rho),(\alpha,\beta),w)=\frac{1}{\Gamma(c)}\zeta_2(\rho;\beta,w), \quad (2.22)$$

provided that $\beta>0$ and w is a non-zero complex number with $|\arg(w)|<\pi$.

For all $n,m\in\mathbb{N}_0$ and $|\lambda(n,m)|<1,\lambda(n,m)=\frac{(n-m)w_2+(\beta-\alpha)}{mw_1+\alpha},\beta\geq\alpha>0,\alpha,\beta\in\mathbb{R}$, $w_1,w_2,\rho\in\mathbb{C},\Re(\rho)>0$, $n\geq m\geq 0$, we suppose a sequence of functions as

$$\psi(n)=\sum_{m=0}^{n}\frac{A(m,n-m)}{\Gamma(\nu m+\delta)}\frac{(xy^{-1})^m}{(mw_1+\alpha)^{\sigma+\rho}}{}_1F_0(\rho;-;-\lambda(n,m)). \quad (2.23)$$

Then employing (2.14) and (2.23), we find a general Wright type function [52] given by

$$\Phi_A^{\mu,\nu,\gamma,\delta}(x,y,(\sigma,\rho),(\alpha,\beta),w_1,w_2)=\sum_{n=0}^{\infty}\psi(n)\frac{y^n}{\Gamma(\mu n+\gamma)}. \quad (2.24)$$

3. Integral Representations of the Double Series (2.1)

In this section, we obtain different integral representations by the double series (2.1).

Theorem 3.1. *For all $n,m\in\mathbb{N}_0=N\cup\{0\}$ and $\beta\geq\alpha>0,\alpha,\beta\in\mathbb{R}$, $w_1,w_2,\sigma,\rho\in\mathbb{C}$, and $|\lambda(n,m)|=|\frac{(n-m)w_2+(\beta-\alpha)}{mw_1+\alpha}|<\pi,0\leq m\leq n,\forall n=0,1,2,\ldots,$ if a sequence of functions exists as*

$$F_n(x,y,z;t)=\frac{t^{\sigma+\rho-1}}{\Gamma(\sigma+\rho)}\sum_{m=0}^{n}\frac{A(m,n-m)}{\Gamma(\nu m+\delta)}\{\lambda(n,m)\}^z(xe^{-w_1t}y^{-1})^m, \quad (3.1)$$

then, for $\Re(\sigma+\rho)>0$, the function (2.1) is represented in a series of Mellin-Barnes type integrals as

$$\Phi_A^{\mu,\nu,\gamma,\delta}(x,y,(\sigma,\rho),(\alpha,\beta),w_1,w_2)=\sum_{n=0}^{\infty}\frac{y^n}{\Gamma(\mu n+\gamma)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\rho+z)\Gamma(-z)}{\Gamma(\rho)}[L\{F_n(x,y,z;t);\alpha\}]dz. \quad (3.2)$$

Proof. In the Eqn. (2.13), supposing $\lambda(n,m)=\frac{(n-m)w_2+(\beta-\alpha)}{mw_1+\alpha}$, $0\leq m\leq n$, and $|\lambda(n,m)|<\pi$, and then making an appeal to the formula due to ([4], [16], [17], [51])

$$(1+\lambda)^{-\nu}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\nu+z)\Gamma(-z)}{\Gamma(\nu)}\lambda^z dz, \quad (3.3)$$

where, $-\delta < c < 0, \lambda \neq 0, \arg(\lambda) < \pi, \delta > 0$, we find that

$$\Phi_A^{\mu,\nu,\gamma,\delta}(x,y,(\sigma,\rho),(\alpha,\beta),w_1,w_2) = \sum_{n=0}^{\infty} \frac{y^n}{\Gamma(\mu n+\gamma)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\rho+z)\Gamma(-z)}{\Gamma(\rho)} \sum_{m=0}^{n} \frac{A(m,n-m)}{\Gamma(\nu m+\delta)} \{\lambda(n,m)\}^z \frac{(xy^{-1})^m}{(mw_1+\alpha)^{\sigma+\rho}} dz. \tag{3.4}$$

Now for $n \geq m \geq 0$ and $\Re(\sigma+\rho) > 0$, on defining the Laplace transformation of the sequence of functions (3.1), with respect to the parameter α, *where*, $\alpha \in \mathbb{R}\backslash\{0\}$, as

$$L\{F_n(x,y,z;t);\alpha\} := \int_0^{\infty} e^{-\alpha t} F_n(x,y,z;t)dt,$$

we get a formula

$$L\{F_n(x,y,z;t);\alpha\} = \sum_{m=0}^{n} \frac{A(m,n-m)}{\Gamma(\nu m+\delta)} \frac{(xy^{-1})^m}{(mw_1+\alpha)^{\sigma+\rho}} \{\lambda(n,m)\}^z, \tag{3.5}$$

where, $|\lambda(n,m)| = |\frac{(n-m)w_2+(\beta-\alpha)}{mw_1+\alpha}| < \pi, \forall n \geq m \geq 0, \beta \geq \alpha > 0, \alpha,\beta \in \mathbb{R}$.

Ultimately, making an appeal to (3.4) and (3.5), we get (3.2).

Hence, the Theorem 3.1 is followed.

Theorem 3.2. *For $\beta \geq \alpha > 0, \alpha,\beta \in \mathbb{R}, w_1,w_2,\sigma,\rho,\eta_1,\eta_2 \in \mathbb{C}$, $\Re(\eta_1) > 0, \Re(\eta_2) > 0$, the function (2.1) is represented as double Eulerian integral*

$$\Phi_A^{\mu,\nu,\gamma,\delta}(x,y,(\sigma+\eta_1,\rho+\eta_2),(\alpha,\beta),w_1,w_2)$$
$$= \int_0^{\infty}\int_0^{\infty} e^{-\alpha t_1-\beta t_2}(t_1)^{\eta_1-1}(t_2)^{\eta_2-1} \Phi_A^{\mu,\nu,\gamma,\delta}(xe^{-w_1t_1-w_1t_2}, ye^{-w_2t_2},(\sigma,\rho),(\alpha,\beta),w_1,w_2)dt_1dt_2. \tag{3.6}$$

Proof. By the definition (2.1), the right hand side of (3.6)

$$= \int_0^{\infty}\int_0^{\infty} e^{-\alpha t_1-\beta t_2}(t_1)^{\eta_1-1}(t_2)^{\eta_2-1} \Phi_A^{\mu,\nu,\gamma,\delta}(xe^{-w_1t_1-w_1t_2}, ye^{-w_2t_2},(\sigma,\rho),(\alpha,\beta),w_1,w_2)dt_1dt_2$$
$$= \sum_{m,n=0}^{\infty} \frac{A(m,n)}{\Gamma(\mu m+\mu n+\gamma)\Gamma(\nu m+\delta)} \frac{x^m y^n}{(mw_1+\alpha)^{\sigma}(mw_1+nw_2+\beta)^{\rho}}$$
$$\times \int_0^{\infty}\int_0^{\infty} e^{-(\alpha+mw_1)t_1} e^{-(\beta+mw_1+nw_2)t_2}(t_1)^{\eta_1-1}(t_2)^{\eta_2-1}dt_1dt_2. \tag{3.7}$$

Now using the techniques of Lemma 1.3 in (3.7), we find

$$\int_0^{\infty}\int_0^{\infty} e^{-\alpha t_1-\beta t_2}(t_1)^{\eta_1-1}(t_2)^{\eta_2-1} \Phi_A^{\mu,\nu,\gamma,\delta}(xe^{-w_1t_1-w_1t_2}, ye^{-w_2t_2},(\sigma,\rho),(\alpha,\beta),w_1,w_2)dt_1dt_2$$
$$= \sum_{m,n=0}^{\infty} \frac{A(m,n)}{\Gamma(\mu m+\mu n+\gamma)\Gamma(\nu m+\delta)} \frac{x^m y^n}{(mw_1+\alpha)^{\sigma+\eta_1}(mw_1+nw_2+\beta)^{\rho+\eta_2}}. \tag{3.8}$$

Finally, in the right hand side of (3.8) employing (2.1), we get left hand side of the result (3.6).

Hence, the Theorem 3.2 is followed.

Theorem 3.3. *Consider that*

$$\xi_{B,C}^{\mu,\gamma,\nu,\delta}(x;(\sigma,\rho),(\alpha,\beta)) = \sum_{m=0}^{\infty} \frac{C(m)B(m)x^m}{\Gamma(\mu m+\gamma)\Gamma(\nu m+\delta)(m+\alpha)^{\sigma}(mw_2+\beta)^{\rho}}, \tag{3.9}$$

for $\beta \geq \alpha > 0, \alpha,\beta \in \mathbb{R}$, and set $w_1 = w_2$, such that $w_2,\sigma,\rho \in \mathbb{C}$, $A(m,n) = C(m)B(m+n)$, then the function (2.1) becomes in the form

$$\Phi_{B,C}^{\mu,\nu,\gamma,\delta}(x,y,(\sigma,\rho),(\alpha,\beta),w_2,w_2) = \sum_{m,n=0}^{\infty} \frac{C(m)B(m+n)}{\Gamma(\mu m+\mu n+\gamma)\Gamma(\nu m+\delta)} \frac{x^m y^n}{(m+\alpha)^{\sigma}(mw_2+nw_2+\beta)^{\rho}}, \tag{3.10}$$

and then (3.10) is represented as series of Mellin-Barnes type integral as

$$\Phi_{B,C}^{\mu,\nu,\gamma,\delta}(x,y,(\sigma,\rho),(\alpha,\beta),w_2,w_2) = \xi_{B,C}^{\mu,\gamma,\nu,\delta}(x;(\sigma,\rho),(\alpha,\beta))$$
$$+\frac{1}{(\beta)^{\rho}}\frac{1}{2\pi i}\sum_{n=1}^{\infty} \frac{B(n)y^n}{\Gamma(\mu n+\gamma)} E_{\nu,\delta}^{(n)}(\phi;\frac{x}{y},s,\alpha) \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\rho+z)\Gamma(-z)}{\Gamma(\rho)}(\frac{nw_2}{\beta})^z dz,$$

where,

$$E_{\nu,\delta}^{(n)}(\phi;\frac{x}{y},s,\alpha) = \sum_{m=0}^{n} \frac{C(m)}{\Gamma(\nu m+\delta)} \frac{(\frac{x}{y})^m}{(m+\alpha)^{\sigma}}. \tag{3.11}$$

Proof. For $m \geq 0, n \geq 1, \max\{-\Re(\rho), 1-\Re(\sigma+\rho)\} < c < -1$, making an appeal to the function (3.9), we write the formula (3.10) in the form

$$\Phi_{B,C}^{\mu,\nu,\gamma,\delta}(x,y,(\sigma,\rho),(\alpha,\beta),w_2,w_2) = \sum_{m,n=0}^{\infty} \frac{C(m)B(m+n)}{\Gamma(\mu m+\mu n+\gamma)\Gamma(\nu m+\delta)} \frac{x^m y^n}{(m+\alpha)^{\sigma}(mw_2+nw_2+\beta)^{\rho}}. \tag{3.12}$$

Now employing a series rearrangement techniques in the relation (3.12), we write

$$= \xi_{B,C}^{\mu,\gamma,\nu,\delta}(x;(\sigma,\rho),(\alpha,\beta)) + \sum_{n=1}^{\infty} \frac{B(n)}{\Gamma(\mu n+\gamma)} \frac{y^n}{(nw_2+\beta)^{\rho}} \sum_{m=0}^{n} \frac{C(m)}{\Gamma(\nu m+\delta)} \frac{(\frac{x}{y})^m}{(m+\alpha)^{\sigma}}.$$

$$= \xi_{B,C}^{\mu,\gamma,\nu,\delta}(x;(\sigma,\rho),(\alpha,\beta)) + \frac{1}{(\beta)^{\rho}} \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{B(n)y^n}{\Gamma(\mu n+\gamma)} E_{\nu,\delta}^{(n)}(\phi;\frac{x}{y},s,\alpha) \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\rho+z)\Gamma(-z)}{\Gamma(\rho)} (\frac{nw_2}{\beta})^z dz,$$

(on applying (3.3)), where,

$$E_{\nu,\delta}^{(n)}(\phi;\frac{x}{y},s,\alpha) = \sum_{m=0}^{n} \frac{C(m)}{\Gamma(\nu m+\delta)} \frac{(\frac{x}{y})^m}{(m+\alpha)^{\sigma}}.$$

4. Concluding Remarks

These researches are very useful in analytic continuation and number theory studied earlier in ([1], [2], [3], [51], and others). Since the function (2.1), introduced here in this work, is related with various analytic functions for example ([4], [16], [17] and others), and with the Srivastava-Daoust two variables hypergeometric function (2.5), a generalization of Kampe' de Fe'riet function [46] given in (1.10) and various Hurwitz-Lerch type functions and Barnes functions of two variables.

References

[1] M. Abramowitz and I. A. Stegun (Editors), *Handbook of Mathematical Functions with Formulas; Graphs; and Mathematical Tables*, Applied Mathematics Series 55, National Bureau of Standards, Washington, D.C., 1964; Reprinted by Dover Publications, New York, 1965.

[2] V. S. Adamchik and H. M. Srivastava, Some series of the zeta and related functions, *Analysis,* **18** (1998), 131-144.

[3] T. M. Apostol, *Introduction to Analytic Number Theory* , Springer-Verlag, New York, Heidelberg and Berlin, 1976.

[4] E. W. Barnes, The theory of the double gamma function, *Philos. Trans. Roy. Soc.* (A), **196** (1901), 265-387.

[5] R. C. S. Chandel and H. Kumar, Contour integral representations of two variables generalized hypergeometric function of Srivastava and Daoust with their applications to initial value problems of arbitrary order, *Jnānā bha,* **50** (1) (2020), 232-242.

[6] J. Choi, D. S. Jang and H. M. Srivastava, A generalization of the Hurwitz-Lerch zeta function, *Integral Transforms Spec. Funct.*, **19** (2008), 65-79.

[7] J. Choi, and R. K. Parmar, An extension of the generalized Hurwitz-Lerch zeta function of two variables, *Filomat,* **31** (2017), 91-96.

[8] J. Choi, R. Sahin, O. Yağci and D. Kim, Note on the Hurwitz-Lerch zeta function of two variables, *Symmetry,* **12** (2020), Article ID 1431, 1-11.

[9] O. A. Daman and M. A. Pathan, A further generalization of the Hurwitz Zeta function, *Math. Sci. Res. J.,* **16** (2012), 251-259.

[10] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill Book Company, New York, Toronto and London, 1953.

[11] A. E. Erëmenko and M. Yu. Lyubich, Dynamical properties of some classes of entire functions, *Ann. Inst. Fourier (Grenoble)*, **42**(4) (1992), 989-1020.

[12] H. Exton, *Multiple Hypergeometric Functions and Applications*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1976.

[13] H. Exton, *Handbook of Hypergeometric Integrals: Theory; Applications; Tables and Computer Programs*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1978.

[14] P. L. Gupta, R. C. Gupta, S.-H. Ong and H. M. Srivastava, A class of Hurwitz-Lerch zeta distributions and their applications in reliability, *Appl. Math. Comput.,* **196** (2008), 521-531.

[15] N. T. H*a*i, O. I. Marichev and H. M. Srivastava, A note on the convergence of certain families of multiple hypergeometric series, *J. Math. Anal. Appl.,* **164** (1992), 104-115.

[16] M. Katsurada, An application of Mellin-Barnes' type integrals to the mean square of Lerch zeta-functions, *Collect. Math.,* **48** (1997), 137-153.

[17] M. Katsurada, An application of Mellin-Barnes type of integrals to the mean square of L-functions, *Liet. Mat. Rink.*, **38** (1998), 98-112. = *Lithuanian Math. J.*, **38** (1998), 77-88.

[18] H. Kumar, On convergence properties and applications of two variable generalized Mittag-Leffler function, *Jnānā bha,* **47**(1) (2017), 125-138.

[19] H. Kumar, Application in initial value problems via operational techniques on a contour integral for Srivastava-Daoust function of two variables, *Jñānābha* **50**(2) (2020), 84-92.

[20] H. Kumar and R. C. S. Chandel, Relations and identities via contour integral representations involving Hurwitz-Lerch zeta type functions for two-variable Srivastava-Daoust functions, *Jnānābha,* **51** (1) (2021), 125-132.

[21] H. Kumar, R. C. S. Chandel and R. D. Agrawal, Phase shift involving multiple hypergeometric function of Srivasatava-Daoust, *Jñānābha* **29** (1999), 117-122.

[22] S.-D. Lin and H. M. Srivastava, Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations, *Appl. Math. Comput.,* **154** (2004), 725-733.

[23] S.-D. Lin, H. M. Srivastava and P.-Y. Wang, Some expansion formulas for a class of generalized Hurwitz-Lerch Zeta functions, *Integral Transforms Spec. Funct.,* **17** (2006), 817-822.

[24] T. M. MacRobert, *Functions OF A Complex Variable*, Glasgow, 1954.

[25] A. M. Mathai and H. J. Haubold, *Special Functions for Applied Scientists*, Springer, New York, 2008.

[26] K. Matsumoto, Asymptotic expansions of double zeta-functions of Barnes, of Shintani, and Eisenstein series, *Nagoya Math. J.*, **172** (2003), 59-102.

[27] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark (Editors), NIST *Handbook of Mathematical Functions*

[28] , U. S. Department of Commerce, National Institute of Standards and Technology, Washington, D. C., 2010; Cambridge University Press, Cambridge, London and New York, 2010 .

[29] M. A. Pathan and H. Kumar, Generalized multivariable Cauchy residue theorem and non-zero zeros of multivariable and multi-parameters generalized Mittag-Leffler functions, *South Asian Bull. Math.,* **43** (2019), 733-749.

[30] G. Patras, *The Weiertrass Elliptic Function and Application in Classical and Quantum Mechanics*, Springer Link, 2020.

[31] E. D. Rainville, *Special Functions*, MacMillan, New York, 1960.

[32] H. M. Srivastava, A new family of the λ-generalized Hurwitz-Lerch zeta functions with applications, *Appl. Math. Inform. Sci.*, **8** (2014), 1485-1500.

[33] H. M. Srivastava, The zeta and related functions: Recent developments, *J. Adv. Engrg. Comput.*, **3** (2019), 329-354.

[34] H. M. Srivastava, Some general families of the Hurwitz-Lerch zeta functions and their applications: Recent developments and directions for further researches, *Proc. Inst. Math. Mech. Nat. Acad. Sci.* Azerbaijan, **45** (2019), 234-269.

[35] H. M. Srivastava, Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations, *J. Nonlinear Convex Anal.*, **22** (2021), 1501-1520.

[36] H. M. Srivastava, An introductory overview of fractional-calculus operators based upon the Fox- Wright and related higher transcendental functions, *J. Adv. Engrg. Comput.*, **5** (2021), 135-166.

[37] H. M. Srivastava, A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics, *Symmetry,* **13** (2021), Article ID 2294, 1-22.

[38] H. M. Srivastava and A. A. Attiya, An integral operator associated with the Hurwitz-Lerch zeta function and differential subordination, *Integral Transforms Spec. Funct.*, **18** (2007), 207-216.

[39] H. M. Srivastava, R. C. Singh Chandel and H. Kumar, Some general Hurwitz-Lerch type zeta functions associated with the Srivastava-Daoust multiple hypergeometric functions, *J. Nonlinear Var. Anal.*, **6** (4) (2022), 299-315, Available online at http://jnva.biemdas.com, https://doi.org/10.23952/jnva.6.2022.4.01 .

[40] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

[41] H. M. Srivastava and M. C. Daoust, On Eulerian integrals associated with Kampé de Fériet's function, *Publ. Inst. Math.* (Beograd) (Nouvelle Ser.), **9**(23) (1969), 199-202.

[42] H. M. Srivastava and M. C. Daoust, Certain generalized Neumann expansions associated with the Kampé de Fériet function, *Nederl. Akad. Wetensch. Proc*. Ser. A, 72 = *Indag. Math.*, **31** (1969), 449-457.

[43] H. M. Srivastava and M. C. Daoust, A note on the convergence of Kampe' de Fe'riet's double hyper-geometric series, *Math. Nachr*., **53** (1972), 151-159.

[44] H. M. Srivastava, S. Gaboury and B.-J. Fugere, Further results involving a class of generalized Hurwitz-Lerch zeta functions, *Russian J. Math. Phys*., **21** (2014), 521-537.

[45] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

[46] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[47] H. M. Srivastava and R. Panda, An integral representation for the product of two Jacobi polynomials, *J. London Math. Soc.* (Ser.2), **12** (1976), 419-425.

[48] H. M. Srivastava, R. Sahin and O. Yağci, A family of incomplete Hurwitz-Lerch zeta functions of two variables, *Miskolc Math. Notes,* **21** (2020), 401-415.

[49] H. M. Srivastava, R. K. Saxena, T. K. Pog_any and R. Saxena, Integral and computational representations of the extended Hurwitz-Lerch zeta function, *Integral Transforms Spec. Funct*., **22** (2011), 487-506.

[50] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford University (Clarendon) Press, Oxford and London, 1951; Second Edition (Revised by D. R. Heath-Brown), 1986.

[51] Patrick Du Val, *Elliptic Functions and Elliptic curves* , Cambridge University Press, London, 1973.

[52] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; with an Account of the Principal Transcendental Functions,* Fourth Edition, Cambridge University Press, Cambridge, London and New York, 1927.

[53] E. M. Wright, The asymptotic expansion of integral functions defined by Taylor series. I, *Philos. Trans. Roy. Soc. London Ser. A Math. Phys. Sci.,* **238** (1940), 423-451.