ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 52(2) (2022), 168-181**
*(Dedicated to Professor D. S. Hooda on His $80^{th}$ Birth Anniversary Celebrations)*

# A SCHAUDER TYPE HYBRID FIXED POINT THEOREM IN A PARTIALLY ORDERED METRIC SPACE WITH APPLICATIONS TO NONLINEAR FUNCTIONAL INTEGRAL EQUATIONS

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*(Received: June 01, 2022; In format : July 18, 2022; Revised : August 09, 2022; Accepted : November: 10, 2022)*

DOI: https://doi.org/10.58250/jnanabha.2022.52220

**Abstract**

In this paper we prove some hybrid fixed point theorems for the monotonic nondecreasing mappings in a partially ordered metric space which includes the Schauder type fixed point theorems in an ordered Banach space proved by Dhage (2013,2014) and Dhage *et al.* (2022) in a partially ordered Banach space as the special cases. As an application, we discuss a nonlinear functional integral equation of Fredholm type and a nonlinear functional boundary value problem for proving the existence and approximation of solution by constructing the algorithms via Dhage monotone iteration method under some natural condtions.

**2020 Mathematical Sciences Classification:** 47H10, 34A08, 34A12, 34A34
**Keywords and Phrases:** Ordered metric space; Hybrid fixed point principle; Functional integral equation; Dhage iteration method; Existence and approximation theorem.

## 1. Introduction

It is well-known that the fixed point theory is very much useful in the theory of nonlinear equations for proving the different qualitative aspects of the solutions (see Granas and Dugundji [14] and the references therein). Similarly, the hybrid fixed point theory initiated by the present author by mixing different arguments from different branches of mathematics finds numerous applications in solving qualitatively the nonlinear equations that arise in several natural and physical processes of the universe( see [1, 2, 3, 4, 5, 7] and the references therein). Recently, the present author developed a hybrid fixed point theory in an ordered abstract spaces and discussed several nonlinear differential and integral equations via construction of the algorithms. The novelty of this approach lies in the fact that we obtain the algorithms to calculate the approximate solution rather than only existence of the solution. Before proving the main hybrid fixed point theorem we give some preliminary definitions.

Throughout this paper, unless otherwise mentioned, let $(E, \preceq, d)$ denote a partially ordered metric space. Two elements $x$ and $y$ in $E$ are said to be **comparable** if either the relation $x \preceq y$ or $y \preceq x$ holds. A non-empty subset $C$ of $E$ is called a **chain** or **totally ordered** if all the elements of $C$ are comparable. It is known that $E$ is **regular** if $\{x_n\}$ is a nondecreasing (resp. nonincreasing) sequence in $E$ such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. The conditions guaranteeing the regularity of $E$ may be found in Guo and Lakshmikatham [13] and the references therein.

We need the following definitions (see Dhage [1, 2, 3, 4, 5] and the references therein) in what follows.

A mapping $\mathcal{T} : E \to E$ is called **isotone** or **monotone nondecreasing** if it preserves the order relation $\preceq$, that is, if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$. Similarly, $\mathcal{T}$ is called **monotone nonincreasing** if $x \preceq y$ implies $\mathcal{T}x \succeq \mathcal{T}y$ for all $x, y \in E$. Finally, $\mathcal{T}$ is called **monotonic** or simply **monotone** if it is either monotone nondecreasing or monotone nonincreasing on $E$.

A mapping $\mathcal{T} : E \to E$ is called **partially continuous** at a point $a \in E$ if for given $\epsilon > 0$ there exists a $\delta > 0$ such that $d(\mathcal{T}x, \mathcal{T}a) < \epsilon$ whenever $x$ is comparable to $a$ and $d(x, a) < \delta$. $\mathcal{T}$ is called partially continuous on $E$ if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on $E$, then it is continuous on every chain $C$ contained in $E$ and vice-versa.

A non-empty subset $S$ of the partially ordered metric space $E$ is called **partially bounded** if every chain $C$ in $S$ is bounded. $S$ is called **uniformly partially bounded** if all chains $C$ in $S$ are bounded by a unique constant. A non-empty subset $S$ of the partially ordered metric space $E$ is called **partially compact** if every chain $C$ in $S$ is a compact subset of $E$. $S$ is called **uniformly partially compact** if $S$ is a uniformly partially bounded and partially compact operator on $E$.

Finally we state a very fundamental, natural and most celebrated notion of a **Janhavi set** which forms the basis of hybrid fixed point theory in the partially ordered metric and Banach spaces and applications.

**Definition 1.1** (Dhage [1, 2, 3, 4, 5])**.** *The order relation $\preceq$ and the metric $d$ on a non-empty set $E$ are said to be $\mathcal{D}$-**compatible** if $\{x_n\}$ is a monotone sequence, that is, monotone nondecreasing or monotone nonincreasing sequence in $E$ and if a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to $x^*$ implies that the original sequence $\{x_n\}$ converges to $x^*$. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be $\mathcal{D}$-compatible if $\preceq$ and the metric $d$ defined through the norm $\|\cdot\|$ are $\mathcal{D}$-compatible. A subset $S$ of $E$ is called **Janhavi set** if the order relation $\preceq$ and the metric $d$ or the norm $\|\cdot\|$ are $\mathcal{D}$-compatible in $S$. In particular, if $S = E$, then $E$ is called a **Janhavi metric space** or **Janhavi Banach space**.*

Using above definition of Janhavi set, several hybrid fixed point theorems are proved in a partially ordered metric space having interesting applications to nonlinear equations( see Dhage [1, 2, 3, 4, 5, 6, 7] and refernces therein). However, to the best of our knowledge, no fixed point theorem is obtained in a non-empty subset of the ordered metric space which may have some interesting applications to nonlinear differential integral equations. This prompted us to discuss some results in this direction. In this paper, we formulate a Schauder type hybrid fixed point theorem in the partially ordered metric and Banach spaces and derive an interesting applicable hybrid fixed point theorem in an ordered Banach space giving applications to nonlinear nonlinear functional integral equations and nonlinear two point boundary value problems of functional differential equations.

## 2. The Hybrid Fixed Point Theory

In this section we shall prove a couple of hybrid fixed point theorems useful for applications to nonlinear equations for proving the existence and approximation of solutions. Our main hybrid fixed point theorem in a partially ordered metric space is as follows.

**Theorem 2.1.** *Let $S$ be a non-empty partially compact subset of a regular partially ordered metric space $(E, \preceq, d)$ with every chain $C$ in $S$ is Janhavi set and let $\mathcal{T} : S \to S$ be a monotone nondecreasing, partially continuous mapping. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x = x$ has a solution $\xi^*$ in $S$ and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to $\xi^*$.*

*Proof.* Assume first that we have an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ and define a sequence $\{x_n\}_{n=0}^\infty$ of points in $S$ by

$$x_{n+1} = \mathcal{T}x_n, \ n = 0, 1, 2, \dots. \tag{2.1}$$

From the monotonic nndecreasing nature of $\mathcal{T}$, it follows that $\{x_n\}_{n=0}^\infty$ is a nondecreasing sequence of point in $S$, i.e., we have

$$x_0 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots. \tag{2.2}$$

Consequently, $\{x_n\}_{n=0}^\infty$ is a chain in $S$. As $S$ is partially compact the chain $\{x_n\}_{n=0}^\infty$ is compact chain in $S$. As a result, $\{x_n\}_{n=0}^\infty$ has a convergent subsequence $\{x_{n_k}\}_{k=0}^\infty$ converging to a point, say, $\xi^*$. Again, by hypothesis, $C = \{x_n\}_{n=0}^\infty$ is a Janhavi set in $S$, so the original sequence $\{x_n\}_{n=0}^\infty$ converges monotone nondecreasingly to $\xi^*$. Since $(E, \preceq, d)$ is a regular partially ordered metric space, we have $x_n \to \xi^*$ and that $x_n \preceq \xi^*$ for all $n \in \mathbb{N}$. Finally, from partial continuity of $\mathcal{T}$, it follows that

$$\xi^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \mathcal{T}x_n = \mathcal{T}\left(\lim_{n\to\infty} x_n\right) = \mathcal{T}\xi^*.$$

Similarly, if $x_0 \succeq \mathcal{T}x_0$, it can be shown using analogous arguments that $\mathcal{T}$ has a fixed point $\xi^*$ and the sequence $\{x_n\}_{n=0}^\infty$ of successive iterations converges monotone nonincreasingly to $\xi^*$ Thus, in both the cases $\mathcal{T}$ has a fixed point $\xi^*$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^\infty$ of successive iterations converges monotonically to $\xi^*$. This completes the proof.

We note that Theorem 2.1 generalizes the hybrid fixed point theorem of Dhage *et al.* [12] proved in a partially ordered Banach space. When $S = E$ in above Theorem 2.1, we obtain the following fixed point result in the hybrid fixed point theory.

**Corollary 2.1.** *Let $(E, \preceq, d)$ be a regular partially compact Janhavi metric space and let $\mathcal{T} : E \to E$ be a monotone nondecreasing, partially continuous mapping. If there exists an element $x_0 \in E$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x = x$ has a solution $\xi^*$ in $E$ and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to $\xi^*$.*

**Definition 2.1.** *An operator $\mathcal{T}$ on a partially ordered metric space $(E, \preceq, d)$ is called* ***nonlinear partial contraction*** *if there is an upper semi-continuous nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

$$d(\mathcal{T}x, \mathcal{T}y) \leq \psi(d(x, y)) \tag{2.3}$$

*for all comparable elements $x, y \in E$, where $\psi(r) < r,\ r > 0$.*

A few examples of the nonlinear functions $\psi$ along with some applications appear in Dhage and Dhage [11]. If $\psi(r) = q\,r$, $0 \leq q < 1$, then $\mathcal{T}$ is called a **partial contraction** on $E$ with the contraction constant $q$.

**Theorem 2.2.** *Let $S$ be a non-empty partially closed subset of a regular partially ordered complete metric space $(E, \preceq, d)$ and let $\mathcal{T} : S \to S$ be a monotone nondecreasing nonlinear partial contraction. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x = x$ has a unique comparable solution $\xi^*$ in $S$ and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to $\xi^*$. Moreover, $\xi^*$ is unique provided every pair of elements in $E$ has a lower bound or an upper bound.*

*Proof.* The proof is obtained by giving arguments similar to those given in Dhage [1, 2] with appropriate modifications. Hence we omit the details.

**Corollary 2.2.** *Let $S$ be a non-empty partially closed subset of a regular partially ordered complete metric space $(E, \preceq, d)$ and let $\mathcal{T} : S \to S$ be a monotone nondecreasing partial contraction. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x = x$ has a unique comparable solution $\xi^*$ in $S$ and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to $\xi^*$. Moreover, $\xi^*$ is unique provided every pair of elements in $E$ has a lower bound or an upper bound.*

*Remark* 2.1. When $S = X$ in Theorem 2.2, we obtain the hybrid fixed point theorems for contraction mappings proved in Dhage [1, 2] as the special cases.

**Theorem 2.3.** *Let $S$ be a non-empty partially closed subset of a regular partially ordered complete metric space $(E, \preceq, d)$ and let $\mathcal{T} : S \to S$ be a monotone nondecreasing mapping with the property that there exists a positive integer $p$ such that $T^p$ is nonlinear partial contraction. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x = x$ has a unique comparable solution $\xi^*$ in $S$ and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to $\xi^*$. Moreover, $\xi^*$ is unique provided every pair of elements in $E$ has a lower bound or an upper bound.*

*Proof.* By Theorem 2.3, $\mathcal{T}^p$ has a unique comparable fixed point $\xi^*$ which is unique provided every pair of elements in $E$ has a lower bound or an upper bound. Then, we have $\mathcal{T}^p\xi^* = \xi^*$ which further gives that $\mathcal{T}(\mathcal{T}^p)(\xi^*) = \mathcal{T}^p(\mathcal{T}\xi^*) = \mathcal{T}\xi^*$. This shows that $\mathcal{T}\xi^*$ is again a fixed point of $\mathcal{T}^p$. By uniqueness of the fixed point $\xi^*$, we get $\mathcal{T}x^* = x^*$ showing that $\mathcal{T}$ has a unique fixed point $\xi^*$ and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to $\xi^*$.

**Theorem 2.4.** *Let $B_r[x]$ denote the partially closed ball centered at $x$ of radius $r$, for some real number $r > 0$, in a regular partially ordered metric space $(E, \preceq, d)$ and let $\mathcal{T} : (E, \preceq, d) \to (E, \preceq, d)$ be a monotone nondecreasing and partial contraction operator with contraction constant $q$. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying*

$$d(x_0, \mathcal{T}x_0) \leq (1 - q)r, \tag{2.4}$$

*then $\mathcal{T}$ has a unique comparable fixed point $x^*$ in $B_r[x_0]$ and the sequence $\{\{\mathcal{T}^n x_0\}\}_{n=0}^\infty\}$ of successive iterations converges monotonically to $x^*$. Furthermore, if every pair of elements in $X$ has a lower or upper bound, then $x^*$ is unique.*

*Proof.* Assume first that we have an element $x_0 \in B_r[x_0]$ such that $x_0 \preceq \mathcal{T}x_0$ and define a sequence $\{x_n\}_{n=0}^\infty$ of points in $S$ by

$$x_{n+1} = \mathcal{T}x_n,\ n = 0, 1, 2, \ldots. \tag{2.5}$$

Because $\mathcal{T}$ is monotone nondecreasing, $\{x_n\}_{n=0}^\infty$ is monotone nondecreasing sequence in $X$. We show that $\{x_n\} \subset B_r[x_0]$. From the contraction of $\mathcal{T}$, it follows that

$$d(x_n, x_{n+1}) \leq q^n d(x_0, x_1) \tag{2.6}$$

for all $n \in \mathbb{N}$. Therefore, we have

$$d(x_0, x_n) \leq d(x_0, x_1) + \cdots + d(x_{n-1}, x_n)$$

$$\begin{aligned}&\leq (1+q+q^2+\cdots+q^n)d(x_0,x_1)\\&\leq \frac{1-q^n}{2-q}\cdot(1-q)r\\&\leq r.\end{aligned}$$

This shows that $x_n \in B_r[x_0]$ for all $n \in \mathbb{N}$. Now, from the inequality (2.6) it foolws that $\{x_n\}$ is Cauchy sequence in $B_r[x_0]$. The rest of the proof is similar to that given in Dhage [1, 2] and so, we omit the details.

Next, we obtain a Banach space version of above Theorem 2.1. Let $X$ denote a real Banach space with norm $\|\cdot\|$. A non-empty closed subset $K$ of $X$ is called a cone if i) $K+K \subseteq K$, ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}$, $\lambda > 0$ and iii) $(-K)\cap K = \{\theta\}$, where $\theta$ is a zero element of $X$. We introduce an order relation $\preceq$ in $X$ as follows. Let $x, y \in X$ be arbitrary. Then, we define

$$x \preceq y \iff y - x \in K. \tag{2.7}$$

Now the Banach space $X$ together with the order relation $\preceq$ becomes a partially ordered Banach space and we denote it by $(X, K)$. Then the following lemmas are immediate which are proved in Dhage [8, 9].

**Lemma 2.1** (Dhage [8, 9])**.** *Every partially ordered Banach space $(X, K)$ is regular.*

**Lemma 2.2** (Dhage [8, 9])**.** *Every partially ordered subset $S$ of a ordered Banach space $(X, K)$ is a Janhavi set in $X$.*

Now, we are equipped well to state our interesting applicable hybrid fixed point theorem in the subject of hybrid fixed point theory and applications. See Dhage *et al.* [12] and the references therein.

**Theorem 2.5** (Dhage et al. [12])**.** *Let $S$ be a non-empty partially compact subset of a partially ordered Banach space $(X, K)$ and let $\mathcal{T} : S \to S$ be a partially continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid operator equation $\mathcal{T}x = x$ has a solution $\xi^*$ in $S$ and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to $\xi^*$.*

*Proof.* From Lemmas 2.1 and 2.2 it follows that $(X, K)$ is a regular partially ordered Banach space and every partially compact subset $S$ of $X$ is a Janhavi set in $X$, i.e., every chain $C$ in $S$ is Janhavi set. Now the desired conclusion follows by an application of Theorem 2.1.

Again, as a consequence of Theorem 2.3, we obtain

**Corollary 2.3.** *Let $(X, K)$ be a partially compact normed linear space and let $\mathcal{T} : X \to X$ be a monotone nondecreasing, partially continuous operator. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid operator equation $\mathcal{T}x = x$ has a solution $\xi^*$ in $X$ and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to $\xi^*$.*

**Theorem 2.6.** *Let $S$ be a non-empty partially closed subset of a partially ordered complete normed linear space $(X, K)$ and let $\mathcal{T} : S \to S$ be a monotone nondecreasing nonlinear partial contraction. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x = x$ has a unique comparable solution $\xi^*$ in $S$ and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to $\xi^*$. Moreover, $x^*$ is unique provided every pair of elements in $X$ has a lower bound or an upper bound.*

*Proof.* The proof is similar to Theorem 2.2 and Hence we omit the details.

**Corollary 2.4.** *Let $S$ be a non-empty partially closed subset of a partially ordered complete normed linear space $(X, K)$ and let $\mathcal{T} : S \to S$ be a monotone nondecreasing partial contraction. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x = x$ has a unique comparable solution $\xi^*$ in $S$ and the sequence $\{\mathcal{T}^n x_0\}_0^\infty$ of successive iterations converges monotonically to $\xi^*$. Moreover, $\xi^*$ is unique provided every pair of elements in $X$ has a lower bound or an upper bound.*

**Theorem 2.7.** *Let $B_r[x]$ denote the partial closed ball centered at $x$ of radius $r$, for some real number $r > 0$, in a partially ordered Banach space $(X, K)$ and let $\mathcal{T} : (X, K) \to (X, K)$ be a monotone nondecreasing and partial contraction operator with contraction constant $q$. If there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$ satisfying*

$$\|x_0 - \mathcal{T}x_0\| \leq (1-q)r \tag{2.8}$$

*for some real number $r > 0$, then $\mathcal{T}$ has a unique comparable fixed point $x^*$ in $B_r[x_0]$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^\infty\}$ of successive iterations converges monotonically to $x^*$. Furthermore, if every pair of elements in $X$ has a lower or upper bound, then $x^*$ is unique.*

*Remark* 2.2. If the initial approximation $x_0$ in Theoems 2.5, 2.6 and 2.7 is positive, that is, $x_0 \in K$, then the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of iterations of the operator $\mathcal{T}$ converges to the positie fixed point $\xi^*$ monotonically.

If we compare our hybrid fixed point result Theorem 2.3 with the classical Schauder's fixed point theorem in a Banach space, we find that the convexity argument which otherwise needed in Schauder fixed point theorem is altogether omitted from the statement, however it is compensated with the monotonicity of the operator in question. But instead of mere existence of fixed point, we obtain an algorithm to calculate the approximation of fixed point by our fixed point result. Therefore, in view of above observation, our hybrid fixed point theorems are a better achievement over the famous Schauder and Banach fixed point theorems of nonlinear functional analysis.

## 3. Functional Integral Equations

Given the real numbers $r > 0$ and $T > 0$, consider the closed and bounded intervals $I_0 = [-r, 0]$ and $I = [0, T]$ in $\mathbb{R}$ and let $J = [-r, T]$. By $\mathcal{C} = C(I_0, \mathbb{R})$ we denote the space of continuous real-valued functions defined on $I_0$. We equip the space $\mathcal{C}$ with he norm $\|\cdot\|_{\mathcal{C}}$ defined by

$$\|x\|_{\mathcal{C}} = \sup_{-r\leq\theta\leq 0} |x(\theta)|. \tag{3.1}$$

Clearly, $\mathcal{C}$ is a Banach space with this supremum norm and it is called the history space of the functional differential equation in question.

For any continuous function $x : J \to \mathbb{R}$ and for any $t \in I$, we denote by $x_t$ the element of the space $\mathcal{C}$ defined by

$$x_t(\theta) = x(t+\theta), \ -r \leq \theta \leq 0. \tag{3.2}$$

As an application of our newly developed hybrid fixed point theorem, we consider the following nonlinear Fredholm type functional integral equation (in short FIE)

$$\left.\begin{aligned} x(t) &= \int_0^T k(t,s) f(s, x(s), x_s)\, ds, \quad t \in I, \\ x_0 &= \phi, \end{aligned}\right\} \tag{3.3}$$

where $k : I \times I \to \mathbb{R}$ and $f : I \times \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ are continuous real-valued functions.

**Definition 3.1.** *A function $x \in C(J, \mathbb{R})$ is said to be a* solution *of the Fredholm type FIE (3.3) defined on $J$ if*

*(i)* $x_0 = \phi$,
*(ii)* $x_t \in \mathcal{C}$ *for each $t \in I$, and*
*(iii)* *$x$ is satisfies the equations in (3.3) on $I$,*

*where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on $J$.*

The significance of the functional integral equation (3.3) with some delay lies in their occurrence in some natural physical sciences (see Hale [15] and the references therein) and studied existence and uniqueness theorems under some standard compactness and Lipschitz type conditions. But the study of the numerical aspect of the solution is very rare in the literature and in this paper we prove an approximate solution of the FIE (3.3) under usual standard conditions on the nonlinearity $f$. Below we obtain an algorithm which goes monotonically to the solution of the functional integral equation (3.3) on $J$. This method of tackling the nonlinear problem is initiated by the present author in Dhage [1, 2] and Dhage and Dhage [10] and commonly known as **Dhage iteration method** in the subject of nonlinear analysis.

We equip the Banach space $C(J, \mathbb{R})$ with the norm $\|\cdot\|$ and the order relation $\leq$ defined by

$$\|x\| = \sup_{t\in J} |x(t)| \tag{3.4}$$

and

$$x \leq y \iff y - x \in K, \tag{3.5}$$

where $K$ is a cone in $C(J, \mathbb{R})$ given by

$$K = \{x \in C(J, \mathbb{R}) \mid x(t) \geq 0 \ \forall\, t \in J\}. \tag{3.6}$$

It is known that the partially ordered Banach space $C(J, \mathbb{R})$ is regular and lattice so that every pair of elements of $E$ has a lower and an upper bound. See Dhage [6, 8, 9] and the references therein. The following useful lemma concerning the Janhavi subsets of $C(J, \mathbb{R})$ follows immediately from the Arzelá-Ascoli theorem for compactness in a Banach space $C(J, \mathbb{R})$.

**Lemma 3.1.** *Let $(C(J, \mathbb{R}), K)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3.1) and (3.2) respectively. Then every partially compact subset $S$ of $C(J, \mathbb{R})$ is Janhavi set.*

We introduce an order relation $\preceq_C$ in $C$ induced by the order relation $\preceq$ defined in $C(J,\mathbb{R})$. Thus, for any $x, y \in C$, $x \preceq_C y$ implies $x(\theta) \leq y(\theta)$ for all $\theta \in I_0$. Note that if $x, y \in C(J,\mathbb{R})$ and $x \preceq y$, then $x_t \preceq_C y_t$ for all $t \in I$.

Let $C_{eq}(J,\mathbb{R})$ denote the subset of all equicontinuous functions in $C(J,\mathbb{R})$. Then for a constant $M > 0$, by $C_{eq}^M(J,\mathbb{R})$ we denote the class of equicontinuous functions in $C(J,\mathbb{R})$ defined by

$$C_{eq}^M(J,\mathbb{R}) = \{x \in C_{eq}(J,\mathbb{R}) \mid \|x\| \leq M\}.$$

Clearly, $C_{eq}^M(J,\mathbb{R})$ is a closed and uniformly bounded subset of the set of equicontinuous functions of the Banach space $C(J,\mathbb{R})$ which is compact in view of Arzelá-Ascoli theorem.

We need the following definition in what follows.

**Definition 3.2.** *A function $u \in C_{eq}^M(J,\mathbb{R})$ is said to be a* lower solution *of the FIE (3.3) if the conditions (i) and (ii) of Definition 3.1 hold and satisfies*

$$\left.\begin{aligned} u(t) &\leq \int_0^T k(t,s) f(s, u(s), u_s)\, ds, \quad t \in I,\\ u_0 &\preceq_C \phi \end{aligned}\right\}.$$

*Similarly, a function $v \in C_{eq}^M(J,\mathbb{R})$ is called an upper solution of the FIE (3.3) if the above inequalities are satisfied with reverse sign. By a* solution *of the FIE (3.3) in a subset $C_{eq}^M(J,\mathbb{R})$ of the Banach space $C(J,\mathbb{R})$ we mean a function $x \in C_{eq}^M(J,\mathbb{R})$ which is both lower and upper solution of the FIE (3.3) defined on $J$.*

We consider the following set of hypotheses in what follows:

($H_1$) The function $t \to f(t,0,0) = F(t)$ is bounded on $J$ by $F_0$.

($H_2$) There exists constants $\ell_1 > 0$, $\ell_2 > 0$ such that

$$0 \leq f(t, x_1, x_2) - f(t, y_1, y_2) \leq \ell_1(x_1 - y_2) + \ell_2\|x_2 - y_2\|_C$$

for all $t \in J$, where $x_1, y_1 \in \mathbb{R}$ and $x_2, y_2 \in C$ with $x_1 \geq y_1$, $x_2 \geq y_2$.

($H_3$) The function $k$ is nonnegative and bounded on $I \times I$ with bound $M_k$.

($H_4$) The FIE (3.3) has a lower solution $u \in C_{eq}^M(J,\mathbb{R})$.

($H_5$) The FIE (3.3) has an upper solution $v \in C_{eq}^M(J,\mathbb{R})$.

**Theorem 3.1.** *Suppose that hypotheses ($H_1$) through ($H_4$) hold. Furthermore, if*

$$\frac{\|\phi\|_C + M_k T F_0}{1 - M_k T(\ell_1 + \ell_2)} \leq M, \quad M_k T(\ell_1 + \ell_2) < 1, \tag{3.7}$$

*then the FIE (3.3) has a solution $x^*$ defined on $J$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$\left.\begin{aligned} x_0(t) &= u(t), \quad t \in J,\\ x_{n+1}(t) &= \begin{cases} \displaystyle\int_0^T k(t,s) f(s, x_n(s), x_s^n)\, ds, & t \in I,\\ \phi(t), & t \in I_0, \end{cases} \end{aligned}\right\} \tag{3.8}$$

*where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in I_0$, converges monotone nondecreasingly to $x^*$.*

*Proof.* Set $S = C_{eq}^M(J,\mathbb{R})$. Then, $S$ is a uniformly bounded and equicontinuous subset of the ordered Banach space $(X, K)$. Hence $S$ is compact in view of Arzellá-Ascoli theorem. Consequently, $S$ is partially compact subset of $(X, K)$. Define an operator $\mathcal{T} : S \to C(J,\mathbb{R})$ by

$$\mathcal{T}x(t) = \begin{cases} \displaystyle\int_0^T k(t,s) f(s, x(s), x_s)\, ds, & t \in I,\\ \phi(t), & t \in I_0. \end{cases} \tag{3.9}$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.5 in a series of following steps.

**Step I:** *$\mathcal{T}$ is well defined and $\mathcal{T} : S \to S$*

Clearly, $\mathcal{T}$ is well defined in view of continuity of the functions $k$ and $f$ on $J \times J$ and $J \times \mathbb{R} \times C$ receptively. We show that $\mathcal{T}(S) \subset S$. Let $x \in S$ be arbitrary. Now by hypothesis ($H_1$),

$$\begin{aligned} \left|f(t, x(t), x_t)\right| &\leq \left|f(t, x(t), x_t) - f(t,0,0)\right| + |f(t,0,0)|\\ &\leq \ell_1|x(t)| + \ell_2\|x_t\|_C + F_0 \end{aligned}$$

$$\leq (\ell_1 + \ell_2)\|x\| + F_0$$
$$\leq (\ell_1 + \ell_2)M + F_0$$

for all $t \in J$. Hence, by definition of $\mathcal{T}$,

$$|\mathcal{T}x(t)| \leq \begin{cases} \displaystyle\int_0^T k(t,s)|f(s,x(s),x_s)|\,ds \\ |\phi(t)| \end{cases}$$
$$\leq \|\phi\|_C + TM_k\big[(\ell_1 + \ell_2)M + F_0\big]$$
$$\leq M$$

for all $t \in J$. Taking the supremum over $t$, we obtain $\|\mathcal{T}x\| \leq M$ for all $x \in C_{eq}^M(J,\mathbb{R})$.

Next, we prove that $\mathcal{T}(S) \subset S$. Let $y \in \mathcal{T}(S)$ be arbitrary. Then there is an $x \in S$ such that $y = \mathcal{T}x$. Now we consider the following three cases:

**Case I :** Suppose that $t_1, t_2 \in I$. Then, we have

$$|y(t_1) - y(t_2)| = \left|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)\right|$$
$$\leq \int_0^T \left|k(t_1,s) - k(t_2,s)\right| |f(s,x(s),x_s)|\,ds$$
$$\leq \big[(\ell_1 + \ell_2)L + F_0\big] \int_0^T |k(t_1,s) - k(t_2,s)|\,ds$$

Since $k$ is continuous on compact $J \times J$, it is uniformly continuous there. Therefore, for each fixed $s \in J$, we have

$$|k(t_1,s) - k(t_2,s)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly. This further in view of inequality (3.6) implies that

$$\left|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)\right| \to 0 \quad \text{as} \quad t_1 \to t_2 \tag{$i$}$$

uniformly for all $x \in S$.

**Case II :** Suppose that $t_1, t_2 \in I_0$. Then, we have

$$|y(t_1) - y(t_2)| = \left|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)\right|$$
$$= |\phi(t_1) - \phi(t_2)|$$
$$\to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for $x \in S$.

**Case III :** Let $t_1 \in I_)$ and $t_2 \in I$. Then we obtain

$$|y(t_1) - y(t_2)| = \left|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)\right|$$
$$\leq \left|\mathcal{T}x(t_1) - \mathcal{T}x(0)\right| + \left|\mathcal{T}x(0) - \mathcal{T}x(t_2)\right|$$

If $t_1 \to t_2$, that is, $|t_1 - t_2| \to 0$, then $t_1 \to 0$ and $t_2 \to 0$ which in view of inequalities (i) and (ii) implies that

$$|y(t_1) - y(t_2)| \to 0 \quad \text{as} \quad t_1 \to t_2 \tag{$iii$}$$

uniformly for all $y \in \mathcal{T}(S)$. From above three cases (i)-(iii) it follows that $\mathcal{T}x \in S$ for all $x \in S$. As a result $\mathcal{T}(S) \subseteq S$.

**Step II:** *$\mathcal{T}$ is a monotone nondereasing operator on $S$.*

Let $x, y \in S$ be such that $x \geq y$. Then, $x_t \geq y_t$ for each $t \in I$. Therefore, by hypothesis (H$_2$), we get

$$\mathcal{T}x(t) = \begin{cases} \displaystyle\int_0^T k(t,s)f(s,x(s),x_s)\,ds, \ , \ t \in I, \\ \phi(t), \ \ t \in I_0, \end{cases}$$
$$\geq \begin{cases} \displaystyle\int_0^T k(t,s)f(s,y(s),y_s)\,ds, \ , \ t \in I, \\ \phi(t), \ \ t \in I_0, \end{cases}$$
$$= \mathcal{T}y(t)$$

for all $t \in J$. This shows that $\mathcal{T}x \geq \mathcal{T}y$ and consequently the operator $\mathcal{T}$ is monotone nondecreasing on $S$.

**Step III:** *$\mathcal{T}$ is partially continuous on $S$.*

Let $C$ be a chain in the closed and bounded subset $C_{eq}^M(J,\mathbb{R})$ of the ordered Banach space $(C(J,\mathbb{R}),K)$ and let $\{x_n\}$ be a sequence of points in $C$ such that $x_n \to x$ as $n \to \infty$. Then, by definition of the operator $\mathcal{T}$,we obtain

$$
\begin{aligned}
\lim_{n\to\infty} \mathcal{T} x_n &= \lim_{n\to\infty} \begin{cases} \displaystyle\int_0^T k(t,s) f(s, x_n(s), x_s^n)\, ds, \ , \ t \in I, \\ \phi(t), \quad t \in I_0, \end{cases} \\
&= \begin{cases} \displaystyle\int_0^T k(t,s) \left[\lim_{n\to\infty} f(s, x_n(s), x_s^n)\right] ds, \ , \ t \in I, \\ \phi(t), \quad t \in I_0, \end{cases} \\
&= \begin{cases} \displaystyle\int_0^T k(t,s) f(s, x(s), x_s)\, ds, \ , \ t \in I, \\ \phi(t), \quad t \in I_0, \end{cases} \\
&= \mathcal{T} x(t)
\end{aligned}
$$

for all $t \in J$. This shows that $\mathcal{T} x_n \to \mathcal{T} x$ pointwise on $J$. Next, by following the arguments as in Step II, it is proved that $\{\mathcal{T} x_n\}$ is an equicontinuous sequence of points in $S$. This shows that $\mathcal{T} x_n \to \mathcal{T} x$ uniformly on $J$. Consequently $\mathcal{T}$ is a partially continuous operator on $S$ into itself.

Thus $\mathcal{T}$ satisfies all the conditions of Theorem 2.5 on a partially compact subset $S$ of the Banach space $C(J,\mathbb{R})$. Hence $\mathcal{T}$ has a fixed point $x^* \in S$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^\infty$ of successive iterations converges monotone nondecreasingly to $x^*$. This further implies that the FIE (3.3) has a solution $x^*$ on $J$ and the sequence $\{x_n\}_{n=0}^\infty$ successive approximations defined by (3.8) converges monotone nondecreasingly to $x^*$. This completes the proof.

*Remark* 3.1. Theorem 3.1 also remains true if we replace the hypotheses ($H_1$) and ($H_2$) with the following ones:
($H_6$) The function $f$ is bounded on $I \times \mathbb{R} \times C$ with bound $M_f$.
($H_7$) The function $f(t,x,y)$ is nnondeceasing in $x$ and $y$ for each $t \in I$.
In this case the condition (3.7) is replaced by $\|\phi\|_C + M_k M_f T \le M$.

**Theorem 3.2.** *Suppose that the hypotheses ($H_2$), ($H_3$) and ($H_4$) hold with $C_{eq}^M(J,\mathbb{R})$ replaced by $C(J,\mathbb{R})$. Furthermore, if $M_k T(\ell_1 + \ell_2) < 1$, then the FIE* (3.3) *has a unique solution $x^*$ defined on $J$ and the sequence $\{x_n\}_{n=0}^\infty$ of successive approximations defined by (3.8) converges monotone nondecreasingly to $x^*$.*

*Proof.* Set $X = C(J,\mathbb{R})$. Then $(X,K)$ is an ordered Banach space which is also a lattice w.r.t. the lattice operations $x \vee y = \max\{x\,,\,y\}$ and $x \wedge y = \min\{x\,,\,y\}$. Therefore, for any pair of elements $\{x,y\}$ there exist a lower and an upper bound. Define an operator $\mathcal{T}$ on $(X,K)$ by (3.9). Then $\mathcal{T}$ is well defined. We shall show that $\mathcal{T}$ is a partial contraction on $(X,K)$.

Let $x,y \in S$ be such that $x \ge y$. Then, by hypothesis ($H_2$), we have

$$
\begin{aligned}
\left|\mathcal{T} x(t) - \mathcal{T} y(t)\right| &= \left| \int_0^T k((t,s)[f(s,x(s),x_s) - f(s,y(s),y_s)]\, ds \right| \\
&\le \int_0^T k((t,s)\left|f(s,x(s),x_s) - f(s,y(s),y_s)\right| ds \\
&\le \int_0^T k((t,s)[f(s,x(s),x_s) - f(s,y(s),y_s)]\, ds \\
&\le \int_0^T k(t,s)[\ell_1|x(s) - y(s))| + \ell_2\|x_s - y_s\|_C]\, ds \\
&\le \int_0^T k(t,s)(\ell_1 + \ell_2)\|x - y\|\, ds \\
&\le M_k T(\ell_1 + \ell_2)\|x - y\|
\end{aligned}
$$

for all $t \in J$. Taking the supremum over $t$, we obtain

$$\|\mathcal{T} x - \mathcal{T} y\| \le M_k T(\ell_1 + \ell_2)\|x - y\|$$

for all comparable elements $x,y \in S$. This shows that $\mathcal{T}$ is a partial contraction on $C(J,\mathbb{R})$. We know that every partially Lipschitz operator is partially continuous, so $\mathcal{T}$ is a partially continuous operator on $S$. Now, we apply Theorem 2.6

to the operator $\mathcal{T}$ and conclude that $\mathcal{T}$ has a unique fixed point $x^*$ and the sequence $\{\mathcal{T}^n u\}_{n=0}^{\infty}$ of successive iterations converges to $x^*$. This further implies that FIE (3.3) has a unique solution $x^*$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.8) converges monotone nondecreasingly to $x^*$.

*Remark* 3.2. The conclusion of existence and uniqueness theorems, Theorems 3.1 and 3.2 for the problem (3.3) also remains true if we replace the hypothesis ($H_4$) by ($H_5$). In this case the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (3.8) converges monotone nonincreasingly to the solution $x^*$ of the functional FIE (3.3) defined on $J$.

## 4. Functional Boundary Value Problem

With the usual notations of section 3, we consider the following two pint boundary value problem (in short BVP) of nonlinear ordinary second order functional differential equation,

$$\left.\begin{aligned} -x''(t) &= f(t, x(t), x_t), \quad t \in I, \\ x(0) &= 0 = x(T), \\ x_0 &= \phi, \end{aligned}\right\} \tag{4.1}$$

where $f : I \times \mathbb{R} \times C \to \mathbb{R}$ is a nonnegative continuous real-valued function.

When $f(t, x, y) = f(t, x)$, the reduced nonlinear BVP without delay is common in the literature which is discussed extensively for existence of the solution. It has also been discussed in Dhage [6] for existence and approximation of the solution via a new Dhage iteration method.

Obviously, the functional BVP (4.1) is equivalent to the nonlinear functional integral equation,

$$x(t) = \begin{cases} \displaystyle\int_0^T G(t, s) f(s, x(s), x_t)\, ds, & t \in I, \\ \phi(t), & t \in I_0, \end{cases} \tag{4.2}$$

where $G(t, s)$ is the Green's function associated with the linear homogeneous BVP

$$\left.\begin{aligned} -x''(t) &= 0, \quad t \in I, \\ x(0) &= 0 = x(T). \end{aligned}\right\} \tag{4.3}$$

It is known that the Green's function $G$ continuous and nonnegative on $J \times J$ satisfying the inequality

$$0 \le G(t, s) \le \frac{T}{4} = M_G \tag{$*$}$$

for all $t, s \in J$.

We consider the following definition in what follows.

**Definition 4.1.** *A function $u \in C^1(J, \mathbb{R})$ is said to be a* lower solution *of the functional BVP (4.1) if the conditions (i) and (ii) of Definition 3.1 hold and*

$$\left.\begin{aligned} -u''(t) &\le f(t, u(t), u_t), \quad t \in I, \\ u(0) &\le 0 \ge u(T), \\ x_0 &\preceq_C \phi, \end{aligned}\right\} \tag{4.4}$$

*where $C^1(J, \mathbb{R})$ is the space of continuously differential real-valued functions defined on $J$. Similarly, an upper solution $v \in C^1(J, \mathbb{R})$ for the functional BVP (4.1) is defined by reversing the inequality signs. By a* solution *of the BVP (4.1) we mean a function $x \in C^1(J, \mathbb{R})$ which is both lower and upper solution of the BVP (4.1) defined on $J$.*

The following lemma easily follows by an application of maximal principle (see Protter and Weinberger [16]).

**Lemma 4.1.** *If a function $u \in C^1(J, \mathbb{R})$ is a lower solution of the functional BVP (4.1) defined on $J$, then by maximum principle,*

$$u(t) \le \begin{cases} \displaystyle\int_0^T G(t, s) f(s, u(s), u_t)\, ds, & t \in I, \\ \phi(t), & t \in I_0, \end{cases} \tag{4.5}$$

We need the following hypotheses in what follows.

($H_8$) There exists a lower solution $u \in C^1(J, \mathbb{R}) \bigcap C_{eq}^M(J, \mathbb{R})$ of the functional BVP (4.1) defined on $J$.

($H_9$) There exists an upper solution $v \in C^1(J, \mathbb{R}) \bigcap C_{eq}^M(J, \mathbb{R})$ of the functional BVP (4.1) defined on $J$.

**Theorem 4.1.** *Suppose that the hypotheses ($H_1$)-($H_3$) and ($H_8$) hold. Furthermore, if*

$$\frac{4\|\phi\|_C + T^2F_0}{4 - T^2(\ell_1 + \ell_2)} \le M, \quad T^2(\ell_1 + \ell_2) < 4, \tag{4.6}$$

*then the functional BVP (4.1) has a solution $x^*$ defined on $J$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by*

$$\left.\begin{aligned} x_0(t) &= u(t), \quad t \in J,\\ x_{n+1}(t) &= \begin{cases} \displaystyle\int_0^T G(t,s) f(s, x_n(s), x_s^n)\, ds, & t \in I,\\ \phi(t), & t \in I_0, \end{cases} \end{aligned}\right\} \tag{4.7}$$

*where $x_s^n(\theta) = x_n(s+\theta)$, converges monotone nondecrasingly to $x^*$.*

*Proof.* Set $S = C_{eq}^M(J,\mathbb{R})$ and we place the problem of FIE (4.2) in the function space $S$. As mentioned earlier, the functional BVP (4.1) is equivalent to the functional integral equation (4.2) on $J$. Notice that the functions involved in the functional integral equation (4.2) satisfy all the conditions of Theorem 3.1 in view of inequality (*) and the Lemma 4.1. Therefore, the desired result follows by an application of Theorem 3.1. This completes the proof.

**Theorem 4.2.** *Suppose that the hypotheses ($H_2$), ($H_3$) and ($H_8$) hold with $C_{eq}^M(J,\mathbb{R})$ replaced by $C(J,\mathbb{R})$. Furthermore, if $T^2(\ell_1 + \ell_2) < 4$, then the functional BVP (4.1) has a unique solution $x^*$ defined on $J$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (4.7) converges monotone nondecreasingly to $x^*$.*

*Proof.* The proof is similar to Theorem 3.2, so we omit the details.

*Remark* 4.1. The conclusion of existence and uniqueness theorems, Theorems 4.1 and 4.2 for the problem (4.1) also remains true if we replace the hypothesis ($H_8$) by ($H_9$). In this case the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (4.7) converges monotone nonincreasingly to the solution $x^*$ of the functional BVP (4.1) defined on $J$. We also note that our theorems, Theorems 4.1 and 4.2 include the existence and approximation results of Dhage [6] for the BVP (4.1) without functional argument and with a different method as the special cases.

*Remark* 4.2. We remark that the results of Section 4 may be extended with appropriate modifications to the nonlinear BVP of second type of ordinary second order functional functional differential equation,

$$\left.\begin{aligned} -x''(t) &= f(t, x(t), x_t), \quad t \in I,\\ x(0) &= 0 = x'(T)\\ x_0 &= \phi. \end{aligned}\right\} \tag{4.8}$$

Notice that the functional BVP is equivalent to the following nonlinear functional integral equation

$$x(t) = \begin{cases} \displaystyle\int_0^T H(t,s) f(s, x(s), x_t)\, ds, & t \in I,\\ \phi(t), & t \in I_0, \end{cases} \tag{4.9}$$

where $H(t,s)$ is the Green's function associated with the linear homogeneous BVP

$$\left.\begin{aligned} -x''(t) &= 0, \quad t \in I,\\ x(0) &= 0 = x'(T), \end{aligned}\right\} \tag{4.10}$$

is continuous and nonnegaive on $[0,T] \times [0,T]$ satisfying the boundedness inequality $0 \le H(t,s) \le T$ for all $t, s \in I$. Therefore, the existence and uniqueness theorems for the nonlinear BVP (4.8) similar to Theorems 4.1 and 4.2 may also be proved by using the analogous arguments with appropriate changes in the hypotheses and conditions.

*Remark* 4.3. We note that if the functional FIE (3.3) and functional BVP (4.1) have a lower solution $u \in C_{eq}^M(J,\mathbb{R})$ as well as an upper solution $v \in C_{eq}^M(J,\mathbb{R})$ such that $u \le v$, then under the given conditions of Theorems 3.1 and 4.1, they have corresponding solutions $x_*$ and $y^*$ and these solutions satisfy the inequality

$$u = x_0 \le x_1 \le \cdots \le x_n \le x_* \le y^* \le y_n \le \cdots \le y_1 \le y_0 = v.$$

Hence $x_*$ and $y^*$ are respectively the minimal and maximal solutions of the FIE (3.3) and functional BVP (4.1) in the vector segment $[u,v]$ of a subset $S = C_{eq}^M(J,\mathbb{R})$ of the Banach space $C(J,\mathbb{R})$, where the vector segment $[u,v]$ is a set of elements in $C_{eq}^M(J,\mathbb{R})$ defined by

$$[u,v] = \{x \in C_{eq}^M(J,\mathbb{R}) \mid u \le x \le v\}.$$

This is because of the order cone $K$ defined by (3.6) is a non-empty closed convex subset of $C(J,\mathbb{R})$. However, we have not used any property of the cone $K$ in the main existence results of this paper. A few details concerning the order relation by the order cones and the Janhavi sets in an ordered Banach space are given in Dhage [8].

## 5. The Examples

In this section we give a couple of examples for illustrating the hypotheses and abstract ideas involved in Theorems 3.1 and 4.1.

**Example 5.1.** Let $I_0 = \left[-\frac{\pi}{2}, 0\right]$ and $I == \left[0, \frac{\pi}{2}\right]$ be two closed and bounded intervals in $\mathbb{R}$, the set of real number and let $J = \left[-\frac{\pi}{2}, 0\right] \bigcup \left[0, \frac{\pi}{2}\right]$. Given a history function $\phi(t) = \sin t$, $t \in \left[-\frac{\pi}{2}, 0\right]$, consider the functional integral equation,

$$x(t) = \begin{cases} \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{8}(t-s)\, f_1(s, x(s), x_s)\, ds, & t \in \left[0, \frac{\pi}{2}\right], \\ \sin t, & t \in \left[-\frac{\pi}{2}, 0\right], \end{cases} \tag{5.1}$$

for all $t, s \in \left[0, \frac{\pi}{2}\right]$ with $t \geq s$ and $x_s(\theta) = x(t+\theta)$, $\theta \in \left[-\frac{\pi}{2}, 0\right]$, where

$$f_1(t, x, y) = \begin{cases} 0, & \text{if} \quad x \leq 0,\ y \preceq_C 0, \\ \frac{1}{8} \tanh x, & \text{if} \quad x > 0,\ y \preceq_C 0, \\ \frac{1}{8} \tanh(\|y\|_C), & \text{if} \quad x \leq 0,\ y \succeq_C 0,\ y \neq 0, \\ \frac{1}{8} \left[\tanh x + \tanh(\|y\|_C)\right], & \text{if} \quad x > 0,\ y \succeq_C 0,\ y \neq 0. \end{cases}$$

Here, $T = \frac{\pi}{2}$, $k(t, s) = t - s$ which is continuous, nonnegative and bounded function on $\left[0, \frac{\pi}{2}\right] \times \left[0, \frac{\pi}{2}\right]$ and $f : \left[0, \frac{\pi}{2}\right] \times \mathbb{R} \times C \to \mathbb{R}$. Clearly, the functions $k(t, s)$ and $f(t, x, y)$ satisfy the hypotheses ($H_3$) and ($H_6$) with $M_k = \frac{\pi}{2}$ and $M_f = 2$. Next, we show that the function $f(t, x, y)$ is nondecreasing in $x$ and $y$ for each $t \in \left[0, \frac{\pi}{2}\right]$. Let $x, y \in C(J, \mathbb{R})$ be any two elements such that $x \geq y$. Then there are following four cases, namely, (i) $x \geq y \geq 0$, (ii) $x \geq 0 \geq y$, (iii) $0 \geq x \geq y$ and (iv) $x \leq 0 \leq y$. We treat the case (i) only and other cases can be treated similarly. Now suppose that $x \geq y \geq 0$. Then, $x(t) \geq y(t) \geq 0$ and $x_t \succeq_C y_t \succeq_C 0$ for all $t \in \left[0, \frac{\pi}{2}\right]$. This further implies that $\|x_t\|_C \geq \|y\|_C \geq 0$. Therefore, by nondecreasing nature of hyperbolic function tanh, we obtain

$$f(t, x(t), x_t) = \tanh x(t) + \tanh \|x_t\|_C \geq \tanh y(t) + \tanh \|y_t\|_C = f(t, y(t)_1, y_t)$$

for all $t \in \left[0, \frac{\pi}{2}\right]$. This shows that $f(t, x, y)$ is monotone nondecreasing in $x$ and $y$ for each $t \in \left[0, \frac{\pi}{2}\right]$ and so hypothesis ($H_7$) is satisfied. . Again, the function $u : J \to \mathbb{R}$ defined by

$$u(t) = \begin{cases} 0, & t \in \left[0, \frac{\pi}{2}\right], \\ \sin t, & t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}$$

serves as a lower solution of the FIE (5.1) defined on $J$. Furthermore, the condition (3.7) is satisfied with $M = \frac{317}{196}$ in view of Remark 3.1. Hence, by a direct application of Theorem 3.1 yields that the FIE (5.1) has a solution $x^* \in C_{eq}^{317/196}(J, \mathbb{R})$ and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$x_0 = \begin{cases} 0, & t \in \left[0, \frac{\pi}{2}\right], \\ \sin t, & t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}$$

$$x_{n+1}(t) = \begin{cases} \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{8}(t-s)\left[\tanh x_n(s) + \tanh \|x_s^n\|_C\right] ds, & t \in \left[0, \frac{\pi}{2}\right], \\ \sin t, & t \in \left[-\frac{\pi}{2}, 0\right], \end{cases}$$

where $x_s^n(\theta) = x_n(s+\theta)$, $\theta \in \left[-\frac{\pi}{2}, 0\right]$, converges monotone nondecreasingly to $x^*$.

**Example 5.2.** Let $I_0 = \left[-\frac{\pi}{2}, 0\right]$ and $I == [0,1]$ be two closed and bounded intervals in $\mathbb{R}$, the set of real number and let $J = \left[-\frac{\pi}{2}, 0\right] \bigcup [0,1] = \left[-\frac{\pi}{2}, 1\right]$. Given a history function $\phi(t) = \sin t$, $t \in \left[-\frac{\pi}{2}, 0\right]$, consider the nonlinear two point functional BVP

$$\left.\begin{aligned} -x''(t) &= \frac{1}{5} + f_2(t, x(t), x_t), \quad t \in [0,1], \\ x(0) &= \phi(0) = 0 = x(1), \\ x_0 &= \phi, \end{aligned}\right\} \tag{5.2}$$

for all $t \in [0,1]$, where $x_t(\theta) = x(t+\theta)$, $\theta \in \left[-\frac{\pi}{2}, 0\right]$ and the function $f_2$ is given by

$$f_2(t,x,y) = \begin{cases} 0, & \text{if} \quad x \le 0,\ y \preceq_C 0, \\ \dfrac{1}{8}\tan^{-1} x, & \text{if} \quad x > 0,\ y \preceq_C 0, \\ \dfrac{1}{8}\tan^{-1}(\|y\|_C), & \text{if} \quad x \le 0,\ y \succeq_C 0,\ y \neq 0, \\ \dfrac{1}{8}\left[\tan^{-1} x + \tan^{-1}(\|y\|_C)\right], & \text{if} \quad x > 0,\ y \succeq_C 0,\ y \neq 0, \end{cases}$$

for all $t \in [0,1]$.

Here, $T = 1$ and $f_2$ defines a function $f : [0,1] \times \mathbb{R} \times C \to \mathbb{R}$. We shall show that $f_2$ satisfies all the conditions of Theorem 4.2. Now for any $t \in [0,1]$, one has $F(t) = f(t,0,0) = \dfrac{1}{5} = F_0$ and so, hypothesis $(\mathrm{H}_1)$ holds. Next, let $x_1, y_1 \in \mathbb{R}$ and $x_2, y_2 \in C$ be such that $x_1 \ge y_1 \ge 0$ and $x_2 \succeq_C y_2 \succeq 0$. Then there exist constants $\xi_1$ and $\xi_2$ with $y_1 < \xi_1 < x_1$ and $\|y_2\|_C < \xi_2 < \|x_2\|_C$ satisfying

$$\begin{aligned} 0 &\le f(t,x_1,x_2) - f(t,y_1,y_2) \\ &\le \frac{1}{8}\cdot[\tan^{-1} x_1 - \tan^{-1} y_1] \\ &\quad + \frac{1}{8}\cdot[\tan^{-1}\|x_2\|_C - \tan^{-1}\|y_2\|_C] \\ &\le \frac{1}{8}\cdot\frac{1}{1+\xi_1^2}(x_1 - y_1) \\ &\quad + \frac{1}{8}\cdot\frac{1}{1+\xi_2^2}(\|x_2\|_C - \|y_2\|_C) \\ &\le \frac{1}{8}\cdot(x_1 - y_1) + \frac{1}{8}\cdot(\|x_2 - y_2\|_C) \end{aligned}$$

for all $t \in [0,1]$. Similarly, we get the same estimate for other values of the function $f_2$. So the hypothesis $(\mathrm{H}_2)$ holds with $\ell_1 = \frac{1}{8}$ and $\ell_2 = \frac{1}{8}$. Again, the Green's function $G$ is continuous and nonnegative on $[0,1] \times [0,1]$ with bound $M_G = \dfrac{\pi}{8}$, so that the hypothesis $(\mathrm{H}_3)$ holds. Moreover, here we have

$$\frac{4\|\phi\|_C + T^2 F_0}{4 - T^2(\ell_1 + \ell_2)} = \frac{4 + \frac{1}{5}}{4 - \frac{1}{4}} = \frac{84}{75},$$

and so the condition (4.6) of Theorem 4.1 is satisfied for $M = \frac{84}{75}$. Finally, the functions $u$ and $v$ defined by

$$u(t) = \begin{cases} \dfrac{1}{5}\displaystyle\int_0^1 G(t,s)\,ds, & t \in [0,1], \\ \sin t, & t \in \left[-\dfrac{\pi}{2}, 0\right], \end{cases}$$

and

$$v(t) = \begin{cases} \left(\pi + \dfrac{1}{5}\right)\displaystyle\int_0^1 G(t,s)\,ds, & t \in [0,1], \\ \sin t, & t \in \left[-\dfrac{\pi}{2}, 0\right], \end{cases}$$

satisfy respectively the inequalities of the lower solution and upper solution of the BVP (5.2) with $u \le v$ on $J$. Hence the functional BVP (5.2) has a unique solution $x^* \in C_{eq}^{84/75}(J, \mathbb{R})$ defined on $J = \left[-\frac{\pi}{2}, 1\right]$. Moreover, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_0(t) = u(t), \quad t \in \left[-\frac{\pi}{2}, 1\right],$$

$$x_{n+1}(t) = \begin{cases} \displaystyle\int_0^1 G(t,s) f(s, x_n(s), x_s^n)\, ds, & t \in [0,1], \\ \sin t, \quad t \in \left[-\dfrac{\pi}{2}, 0\right], \end{cases}$$

is monotone nondecreasing and converges to $x^*$, that is, we have

$$x_0 \le x_1 \le \cdots \le x_n \le x^*.$$

*Remark* 5.1. We note that the well-known method of upper and lower solution may not be applied to the BVP (5.2) in the above Example 5.2, because here, we are working in a smaller subset $C_{eq}^{84/75}(J, \mathbb{R})$ of the Banach space $C(J, \mathbb{R})$ which may not contain an upper solution of the problem in question. This clearly shows the advantage of the present monotone iteration method of this paper over that of well-known upper and lower solution method for nonlinear differential and integral equations. If at all, both the lower solution $u$ and an upper solution $v$ to the BVP (5.2) exist and satisfy the comparison condition $u \le v$, then we might apply Remark 4.3 for desired conclusion.

## 6. Conclusion

We remark that the nonlinear functional equations considered here in this paper for applications of our new hybrid fixed point results, Theorems 2.3 and 2.4 are of simple nature, however other complex different types of nonlinear functional differential and integral equations may also be considered for applications. We also remark that the use of above hybrid fixed point results not only yield the existence of solution, but also gives an algorithm to obtain approximation of the solution and so it is more powerful than Schauder fixed point theorem. Furthermore, we also improve in some sense the celebrated hybrid fixed point theorems of Dhage [1, 2] having numerous applications. Hence we claim that the hybrid fixed point results of this paper are very fundamental and useful contribution to the subject of nonlinear analysis and applications. In a forthcoming paper, we propose to discuss other IVPs and BVPs of nonlinear ordinary first order functional differential equations for applications of the abstract hybrid fixed point theorems developed in this paper.

**Acknowledgment**

The author is thankful to the referee for pointing out some misprints and corrections for the improvement of this paper.

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