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(Dedicated to Professor D. S. Hooda on His 80^{th} Birth Anniversary Celebrations)

PERCEIVING SOLUTIONS FOR AN EXPONENTIAL DIOPHANTINE EQUATION LINKING SAFE AND SOPHIE GERMAIN PRIMES $q^x + p^y = z^2$

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Abstract

In this article, an exponential Diophantine equation $q^x + p^y = z^2$ where p , q are Safe primes and q Sophie Germain primes respectively and x, y, z are positive integers is measured for all the opportunities of $x+y = 0, 1, 2, 3$ and showed that all conceivable integer solutions are $(p, q, x, y, z) = (7, 3, 1, 0, 2), (11, 5, 1, 1, 4), (5, 2, 3, 0, 3), (2q + 1, q, 2, 1, q + 1)$ by retaining basic rules of Mathematics.
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1. Introduction

The study of Diophantine equations is a huge piece of speculation in Number theory [3, 5]. In recent years, many researchers showed their interest to work on the Diophantine equation in the form $p^x + q^y = z^2$ where p, q are distinct primes and x, y, z are non-negative integers [1, 6]. In [2], Burshtein proved that the Diophantine equation $p^x+(p+4)^y = z^2$ where x, y, z are positive integers and p, $p+4$ are primes with $p > 3$ has no solution. In [4], the authors found all the solutions of the Diophantine equation $p^x+(p+6)^y = z^2$, where x, y, z are non-negative integers such that $x+y = 2, 3, 4$ and $p, p + 6$ are primes. For further analysis, one can refer [7]. In this paper, an exponential Diophantine equation $q^x+p^y = z^2$ where p is a Safe prime, q is a Sophie Germain prime and x, y, z are non- negative integers is studied when $x + y = 0, 1, 2, 3$ and all credible integer solutions are symbolized by the following set
$(p, q, x, y, z) = (7, 3, 1, 0, 2), (11, 5, 1, 1, 4), (5, 2, 3, 0, 3), (2q + 1, q, 2, 1, q + 1)$.

2. Basic definition

Definition 2.1. *A safe prime is a prime p of the form $p = 2q + 1$ where q is a prime as well. In such instances, q is referred to be a Sophie Germain prime.*

3. Attaining solutions to an exponential Diophantine equation

In this section, the possible solution to an exponential Diophantine equation $q^x + p^y = z^2$ where p and q are safe prime and Sophie Germain prime such that $x + y = 0, 1, 2, 3$ is analysed by considering various cases in the following theorem.

Theorem 3.1. *Let p, q be Safe primes and Sophie Germain primes respectively. If $x + y = 0, 1, 2, 3$, then an exponential Diophantine equation $q^x + p^y = z^2$ where x, y, z are positive integers has solutions $(p, q, x, y, z) = (7, 3, 1, 0, 2), (11, 5, 1, 1, 4), (5, 2, 3, 0, 3), (2q + 1, q, 2, 1, q + 1)$.*

Proof. The stated Diophantine equation with exponents x and y is

$$q^x + p^y = z^2, \tag{3.1}$$

where x, y, z are integers with positive values, $p = 2q + 1$ is a Safe prime such that q is a Sophie Germain prime.

Now, all the selections of $x + y = 0, 1, 2, 3$ are examined as follows.

Case 1. $x + y = 0$.

The unique possibility of each exponent $x = 0$ and $y = 0$ describes (3.1) as

$$z^2 = 2. \tag{3.2}$$

This postulation is impossible for any integer.

As an effect, equation (3.2) and hence equation (3.1) does not have any solution.

Case 2. $x + y = 1$

Subcase 2(i). Consider $x = 1, y = 0$.

These two values of x and y condenses (3.1) as

$$1 + q = z^2. \tag{3.3}$$

The only guaranteed value of q nourishing (3.3) is pointed out by $q = 3$.

Then, $p = 7$ and $z = 2$.

Thus, the unique feasible solution of (3.1) is $(p, q, x, y, z) = (7, 3, 1, 0, 2)$.

Subcase 2(ii). Allocate $x = 0$,$y = 1$

These inferences modernized (3.1) to the equation with degree two in terms of two variables as $1 + p = z^2$.

Corresponding formation of the above equation is defined by

$$2(1 + q) = z^2. \tag{3.4}$$

From (3.4), it is effortlessly detected that the left-hand side is a multiple of 2 however the right-hand is of the form either $4k$ or $4k + 1$ where $k \in N$.

Hence, the above hypothesis is constantly not possible. Consequently, equation (3.4) and hence equation (3.1) does not acquire any solution.

Case 3. $x + y = 2$

Subcase 3(i). Let $x = 2$,$y = 0$.

These propositions make things easier to (3.1) as the resultant equation

$$q^2 = z^2 - 1. \tag{3.5}$$

Since, the square of an integer minus one can never be a square, the above supposition is always impracticable.

As a result, equation (3.5) and hence equation (3.1) does not own any solution.

Subcase 3(ii).Opt $x = 1$, $y = 1$

Replacing the overhead values of x and y well-found (3.1) as

$$3q = (z - 1)(z + 1). \tag{3.6}$$

If $q|(z - 1)$, then $(z - 1) = Aq$, A is any positive integer and $(z + 1) = Aq + 2$.

Consequently (3.6) turned out to be $3q = Aq(Aq + 2)$ which is not possible for any values of q and A and hence $q \nmid (z - 1)$.

If $q \mid (z + 1)$ then $(z + 1) = Bq$, B is any positive integer and $(z - 1) = Bq - 2$.

Accordingly, equation (3.6) is converted into $3 = B(Bq - 2)$ which is possible only when $q = 5$ and $B = 1$.

This will lead the choices of p and z as $p = 11, z = 4$.

Thus, the solution to (3.1) is indicated by $(p, q, x, y, z) = (11, 5, 1, 1, 4)$.

Subcase 3(iii). Select $x = 0$, $y = 2$.

These predilections of x and y enhance (3.1) to the equation affianced with q and z as

$$2(2q^2 + 2q + 1) = z^2. \tag{3.7}$$

According to an amplification given in Subcase 2(ii), the statement fabricated above does not hold. As an outcome, equation (3.7) and hence equation (3.1) has no solution in integer.

Case 4. $x + y = 3$

Subcase 4(i). Permit $x = 3, y = 0$.

Replacements of these predispositions trim down (3.1) as

$$1 + q^3 = z^2. \tag{3.8}$$

The credible choice of $q = 2$ in (3.8) offered the optimal values of p and z as $p = 5, z = 3$ and there is no other probable solution for any additional choice of q.

Thus, the assured solution of (3.1) is $(p, q, x, y, z) = (5, 2, 3, 0, 3)$.

Subcase 4(ii). Let $x = 2, y = 1$.

These two values of x and y express (3.1) to the successive equation in two unknowns

$$(1 + q)^2 = z^2. \tag{3.9}$$

In view of (3.9), it is visible that for all Sophie Germain prime q , (3.1) has solutions belong to the set of all non-negative integers which is denoted by $(p, q, x, y, z) = (2q + 1, q, 2, 1, q + 1)$.

Subcase 4(iii). Admit $x = 1, y = 2$.

Under these assumptions, the subsequent form of equation (3.1) is evaluated by

$$q + p^2 = z^2. \tag{3.10}$$

An equivalent structure of (3.10) is precised as below

$$q(4q + 5) = (z - 1)(z + 1). \tag{3.11}$$

Suppose $q \mid (z - 1)$, then $(z - 1) = Cq$, C is any positive integer and $(z + 1) = Cq + 2$.

Thus, the equation (3.11) is articulated into

$$4q + 5 = C(Cq + 2) \Rightarrow q = \frac{C(Cq + 2) - 5}{4}. \tag{3.12}$$

In the vision of (3.12), it is apparent that the right-hand side of (3.12) not at all equal to q for any value of the parameter C.

This confirms that (3.12) is not possible and hence $q \nmid (z - 1)$.

If $q \mid (z + 1)$, then $(z + 1) = Dq$, D is any positive integer and $(z - 1) = Dq - 2$.

From (3.11), it is monitored by

$$4q + 5 = D(Dq - 2) \Rightarrow q = \frac{D(Dq - 2) - 5}{4}. \tag{3.13}$$

It is intensely experimental that (3.13) is not conceivable for any numerical value of D.

This shows that $q \nmid (z + 1)$.

The conclusion is (3.1) does not offer a solution.

Subcase 4(iv). State $x = 0, y = 3$.

Manipulation of these alternatives abbreviated (3.1) to the cubic equation as

$$1 + p^3 = z^2. \tag{3.14}$$

This is true only for $p = 2$ which is not a safe prime because the least safe prime is 5.

Hence, there exists no integer solution for (3.1).

4. Conclusion

In this manuscript, it is accredited that positive integer solutions to an exponential equation $q^x + p^y = z^2$ such that $x + y = 0, 1, 2, 3$ where, q are Safe primes and Sophie Germain primes respectively and x, y, z are positive integers. It is accomplished that one can also examine solutions of the specified equation for$x + y > 3$ and p, q are some other prime numbers.

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References

[1] N. Burshtein, On solutions of the Diophantine equation $p^x + (q)^y = z^2$, *Annals of Pure and Applied Mathematics*, **13**(1)(2017), 143-149.

[2] Bushtein, All the solutions of the Diophantine equation $p^x + (p + 4)^y = z^2$ where $p, (p + 4)$ are primes and$x + y = 2, 3, 4$, *Annals of Pure and Applied Mathematics*, **16**(1)(2018),241-244.

[3] L.E. Dickson, *History of theory of numbers*, **2**, Chelsea publishing company, New York, 1952.

[4] Sani Gupta, Satish Kumar and Hari Kishan, On the Non-Linear Diophantine Equation $p^x + (p + 6)^y = z^2$, *Annals of Pure and Applied Mathematics*, **18**(1) (2018), 125-128.

[5] L. J. Mordell, *Diophantine Equations*, Academic press, New York, 1969.

[6] B. Poonen, Some diophantine equations of the form $x^n + y^n = z^m$, *Acta Arith.*, **86** (1998), 193-205.

[7] A. Suvarnamani, On the Diophantine equation $p^x + (p + 1)^y = z^2$, *International Journal Pure and Applied Mathematics*, **94**(5)(2014),689-692 .