1. Introduction

Following Zemanian [6], we denote by $\mathcal{H}_{\alpha, \beta}$ the space of Hankel transformable functions, $(\alpha - \beta) \in \mathbb{R}$. $\mathcal{H}_{\alpha, \beta}$ consists of all those infinitely differentiable functions $\phi = \phi(x)$ defined on $I = (0, \infty)$ such that

$$p_{m,k}^{\alpha,\beta} = \sup_{x \in I} \left| (1 + x^2)^m (x^{-1} D)^k x^{2\beta-1} \phi(x) \right| < \infty, \quad m, k \in \mathbb{N}.$$

$\mathcal{H}_{\alpha, \beta}$ being a Frechet Space when endowed with the topology generated by the family of seminorms $\left\{ p_{m,k}^{\alpha,\beta} \right\}_{(m,k) \in \mathbb{N} \times \mathbb{N}}$.

The Hankel type transformation

$$\left(h_{\alpha,\beta}\phi\right)(t) = \int_0^\infty \phi(x) (xt)^{\alpha+\beta} J_{\alpha-\beta}(xt) \, dx$$

is an automorphism of $\mathcal{H}_{\alpha, \beta}$; provided $(\alpha - \beta) \geq -\frac{1}{2}$, where $J_{\alpha-\beta}$ denotes the Bessel type function of first kind and of order $(\alpha - \beta)$. If $(\alpha - \beta) \geq -\frac{1}{2}$, the generalized Hankel type transformation $h_{\alpha,\beta}$ is defined on $\mathcal{H}_{\alpha, \beta}$, the dual space of $\mathcal{H}_{\alpha, \beta}$ as the adjoint of $h_{\alpha,\beta}$. Then $h_{\alpha,\beta}$ is an automorphism of $\mathcal{H}_{\alpha, \beta}$.

Following [1],[2] and [4], for $(\alpha - \beta) \geq -\frac{1}{2}$, we introduce the subspace $O_{\alpha,\beta}^*$ of $\mathcal{H}_{\alpha, \beta}$ as the space of all those $T \in \mathcal{H}_{\alpha, \beta}$ such that $\theta(x) = x^{2\beta-1} \left(h_{\alpha,\beta} T\right)(x)$ is a smooth function on $I$ with the property that for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ satisfying

$$\sup_{x \in I} \left| (1 + x^2)^{n_k} (x^{-1} D)^k \phi(x) \right| < \infty.$$

Clearly, $\mathcal{H}_{\alpha, \beta}$ is a subspace of $O_{\alpha,\beta}^*$. The space $O$ of all those smooth functions $\theta = \theta(x)$ on $I$ possessing the above property turns out to be the space of multiplication operators on $\mathcal{H}_{\alpha, \beta}$ and on $\mathcal{H}_{\alpha, \beta}$ ($(\alpha - \beta) \in \mathbb{R}$), whereas $O_{\alpha,\beta}^*$ is the space of convolution operators on $\mathcal{H}_{\alpha, \beta}$ and on $\mathcal{H}_{\alpha, \beta}$ ($(\alpha - \beta) \geq -1/2$).

Throughout this paper we shall always assume that $(\alpha - \beta)$ is a real number $\geq -1/2$ and, unless otherwise stated, that $\mathcal{H}_{\alpha, \beta}$ is endowed with its weak topology.

2. Characterization of $O_{\alpha,\beta}^*$ in $\mathcal{L}(\mathcal{H}_{\alpha, \beta})$ and in $\mathcal{L}(\mathcal{H}_{\alpha, \beta}^*)$

Let $\mathcal{L}(\mathcal{H}_{\alpha, \beta})$ (respectively, $\mathcal{L}(\mathcal{H}_{\alpha, \beta}^*)$) denote the space of all linear continuous operator from $\mathcal{H}_{\alpha, \beta}$ (resp. $\mathcal{H}_{\alpha, \beta}^*$) into itself. The characterization of elements in $\mathcal{L}(\mathcal{H}_{\alpha, \beta})$ and in $\mathcal{L}(\mathcal{H}_{\alpha, \beta}^*)$ that commute with Hankel type translation is our first objective.
We recall that the Hankel type translation $\tau_x\phi$ of $\phi \in \mathcal{H}_{a,\beta}$ by $x \in I$ is defined as
\[
(\tau_x\phi)(y) = \int_0^\infty \phi(z) D_{a,\beta}(x, y, z) dz, \quad y \in I,
\]
where,
\[
D_{a,\beta}(x, y, z) = \int_0^\infty t^{2\beta-1} j_{a-\beta}(xt) j_{a-\beta}(yt) j_{a-\beta}(zt) dt, \quad x, y, z \in I,
\]
and $j_{a-\beta}(z) = z^{a-\beta} J_{a-\beta}(z)$, $(z \in I)$. The map $\phi \mapsto \tau_x\phi$ is a continuous endomorphism of $\mathcal{H}_{a,\beta}$.

Further
\[
\left(h_{a,\beta} \tau_x \phi\right)(t) = t^{2\beta-1} j_{a-\beta}(xt) \left(h_{a,\beta} \phi\right)(t), \quad t \in I
\]
whenever $\phi \in \mathcal{H}_{a,\beta}$ and $x \in I$.

If $u \in \mathcal{H}_{a,\beta}$ and $x \in I$, we define $\tau_x u \in \mathcal{H}_{a,\beta}$ by transposition:
\[
\langle \tau_x u, \phi \rangle = \langle u, \tau_x \phi \rangle, \quad \phi \in \mathcal{H}_{a,\beta}.
\]

The following analogue of (2.1) holds for the generalized translation (2.2).

**Lemma 2.1.** Let $u \in \mathcal{H}'_{a,\beta}$ and $x \in I$. Then
\[
\left(h_{a,\beta}' \tau_x u\right)(t) = t^{2\beta-1} j_{a-\beta}(xt) \left(h_{a,\beta}' u\right)(t), \quad t \in I.
\]

**Proof.** Let $u \in \mathcal{H}_{a,\beta}$, $x \in I$ and $\phi \in \mathcal{H}_{a,\beta}$. Then a combination of (2.1) and (2.2) gives
\[
\left(h_{a,\beta}' \tau_x u, h_{a,\beta} \phi\right) = \langle \tau_x u, \phi \rangle = \langle u, \tau_x \phi \rangle = \left(h_{a,\beta}' u, h_{a,\beta} \tau_x \phi\right)
\]
\[
= \left(t^{2\beta-1} j_{a-\beta}(xt) \left(h_{a,\beta} \phi\right)(t)\right) = \left(t^{2\beta-1} j_{a-\beta}(xt) \left(h_{a,\beta}' u\right)(t)\right)
\]
This completes the proof. The classical Hankel convolution $\phi * \psi$ of $\phi, \psi \in \mathcal{H}_{a,\beta}$ is the function
\[
\phi * \psi(x) = \int_0^\infty \phi(y) (\tau_x \psi)(y) dy, \quad x \in I.
\]
The map $(\phi, \psi) \mapsto \phi * \psi$ is continuous from $\mathcal{H}_{a,\beta} \times \mathcal{H}_{a,\beta}$ into $\mathcal{H}_{a,\beta}$. The generalized Hankel type convolution $u * \phi$ of $u \in \mathcal{H}_{a,\beta}$ and $\phi \in \mathcal{H}_{a,\beta}$ is the distribution given by
\[
\langle u * \phi, \psi \rangle = \langle u, \phi \ast \psi \rangle, \quad \psi \in \mathcal{H}_{a,\beta}.
\]
The map $(u, \phi) \mapsto u * \phi$ is separately continuous from $\mathcal{H}'_{a,\beta} \times \mathcal{H}_{a,\beta}$ into $\mathcal{H}'_{a,\beta}$, when $\mathcal{H}'_{a,\beta}$ is endowed either with its weak* or its strong topology.

Finally, for $u \in \mathcal{H}_{a,\beta}$ and $T \in \mathcal{O}_{a,\beta}^\ast$, the generalized function $u * T \in \mathcal{H}_{a,\beta}$ defined as
\[
\langle u * T, \phi \rangle = \langle u, T * \phi \rangle, \quad \phi \in \mathcal{H}_{a,\beta}.
\]
Note that each of these definitions, extends the previous one. Moreover,
\[
\left(h_{a,\beta}' u * T\right)(t) = t^{2\beta-1} \left(h_{a,\beta}' T\right)(t) \left(h_{a,\beta}' u\right)(t), t \in I
\]
whenever $u \in \mathcal{H}_{a,\beta}$ and $T \in \mathcal{O}_{a,\beta}^\ast$.

If $C_{a,\beta} = 2^{a-\beta} \Gamma(3\alpha + \beta)$ then the element $\delta_{a-\beta}$ of $\mathcal{O}_{a,\beta}^\ast$ given by
\[
\left(\delta_{a-\beta} \phi\right) = C_{a,\beta} \lim_{x \to 0} x^{2\beta-1} \phi(x), \quad \phi \in \mathcal{H}_{a,\beta}
\]
is an identity for (2.3).

The generalized *-convolution commutes with Hankel type translations:

**Lemma 2.2.** Let $u \in \mathcal{H}'_{a,\beta}$ and $x \in I$. If $T \in \mathcal{O}_{a,\beta}^\ast$, then
\[
\tau_x (u * T) = \left(\tau_x u\right) * T = u * \left(\tau_x T\right).
\]

**Proof.** Since $h_{a,\beta}'$ is an automorphism of $\mathcal{H}'_{a,\beta}$, we prove the lemma by fixing $t \in I$ and using Lemma 2.1, along with (2.4) to write,
\[
\left(h_{a,\beta}' \tau_x (u * T)\right)(t) = t^{2\beta-1} j_{a-\beta}(xt) \left(h_{a,\beta}' u * T\right)(t) = t^{2\beta-2} j_{a-\beta}(xt) \left(h_{a,\beta}' T\right)(t) \left(h_{a,\beta}' u\right)(t),
\]
\[
\left(h_{a,\beta}' (\tau_x u) * T\right)(t) = t^{2\beta-1} \left(h_{a,\beta}' T\right)(t) \left(h_{a,\beta}' u\right)(t) = t^{2\beta-2} j_{a-\beta}(xt) \left(h_{a,\beta}' T\right)(t) \left(h_{a,\beta}' u\right)(t),
\]
\[
\left(h_{a,\beta}' u * (\tau_x T)\right)(t) = t^{2\beta-1} \left(h_{a,\beta}' T\right)(t) \left(h_{a,\beta}' u\right)(t) = t^{2\beta-2} j_{a-\beta}(xt) \left(h_{a,\beta}' T\right)(t) \left(h_{a,\beta}' u\right)(t).
\]
Thus proof is completed. Now we are ready to prove the following theorem.
Theorem 2.1. If $T \in O'_{\alpha,\beta}$, and $L$ is the element of $L(H_{\alpha,\beta})$ defined by

$$L\phi = T*\phi, \quad \phi \in H_{\alpha,\beta},$$

(2.5)

then

$$\tau_x L = L \tau_x, \quad x \in I.$$  

(2.6)

conversely, if $L \in L(H_{\alpha,\beta})$ satisfies (2.6) then there exists a unique $T \in O'_{\alpha,\beta}$, for which (2.5) holds.

Proof. Let $T \in O'_{\alpha,\beta}$. The fact that $L \in L(H_{\alpha,\beta})$ defined by (2.5) satisfies (2.6) is contained in Lemma 2.2. On the other hand, assume that $L \in L(H_{\alpha,\beta})$ is such that (2.6) holds, and define $T \in H_{\alpha,\beta}$ by

$$\langle T, \phi \rangle = \langle \delta_{\alpha-\beta}, L\phi \rangle, \quad \phi \in H_{\alpha,\beta}. $$

Then

$$\langle T * \phi \rangle(x) = \langle T \tau_x \phi \rangle = \langle \delta_{\alpha-\beta}, L \tau_x \phi \rangle = \langle \delta_{\alpha-\beta}, \tau_x L \phi \rangle$$

whenever $\phi \in H_{\alpha,\beta}$, which proves (2.5). As $O'_{\alpha,\beta}$ is the space of convolution operators of $\mathcal{H}_{\alpha,\beta}$, it follows from (2.5) that $T \in O'_{\alpha,\beta}$. As to the uniqueness assertion, note that if $S \in O'_{\alpha,\beta}$ is such that $S * \phi = 0$ for every $\phi \in H_{\alpha,\beta}$, then $S = 0$. In fact, $S * \phi = 0 (\phi \in H_{\alpha,\beta})$ and (2.4) imply $\hat{t}^{\alpha-1} \langle h_{\alpha,\beta} S \rangle(t) = 0, (\psi \in H_{\alpha,\beta}, \ t \in I)$. By particularizing $\psi(t) = t^\alpha e^{-t^2}$ $(t \in I)$ we find that $\hat{t}^{\alpha-1} \langle h_{\alpha,\beta} S \rangle(t) = 0$, whence $\langle h_{\alpha,\beta} S \rangle = 0$ and $S = 0$. This completes the proof. The following result will help in characterising the elements of $O'_{\alpha,\beta}$ as those in $L(H_{\alpha,\beta})$ that commute with Hankel type translations.

Lemma 2.3. The linear hull of the set of generalized functions of the form $\tau_x \delta_{\alpha-\beta}$ ($x \in I$) is weakly* dense in $H'_{\alpha,\beta}$.

Proof. As $\langle h_{\alpha,\beta} \tau_x \delta_{\alpha-\beta} \rangle(t) = t^{\alpha} \langle \tau_x \delta_{\alpha-\beta} \rangle(t) = j_{\alpha-\beta}(xt), \quad x, t \in I$.

If $\phi \in H_{\alpha,\beta}$ does not vanish identically then there exists $x \in I$ such that $\phi(x) \neq 0$ and hence

$$\langle \tau_x \delta_{\alpha-\beta}, \phi \rangle = \langle h_{\alpha,\beta} \tau_x \delta_{\alpha-\beta}, h_{\alpha,\beta} \phi \rangle$$

$$= \int_0^\infty (h_{\alpha,\beta} \phi(t)) j_{\alpha-\beta}(xt) dt = \phi(x) \neq 0.$$

This shows that the subset $\langle \tau_x \delta_{\alpha-\beta} \rangle_{x \in I}$ of $H'_{\alpha,\beta}$ separates points in $H_{\alpha,\beta}$. By [3], problem $W(b)$, this family is total in $H'_{\alpha,\beta}$ with respect to the weak* topology.

Thus proof is completed.

Theorem 2.2. If $T \in O'_{\alpha,\beta}$, and $L \in L(H_{\alpha,\beta})$ is defined by

$$Lu = u * T, \quad u \in H'_{\alpha,\beta},$$

(2.7)

then

$$\tau_x L = L \tau_x, \quad x \in I,$$

(2.8)

and also

$$L \delta_{\alpha-\beta} \in O'_{\alpha,\beta},$$

(2.9)

Conversely, given $L \in L(H_{\alpha,\beta})$ satisfying (2.8) and (2.9), a unique $T \in O'_{\alpha,\beta}$ may be found so that (2.7) holds.

Proof. Note that $L$ given by (2.7) satisfies (2.8) is a consequence of Lemma 2.2. Clearly it also satisfies (2.9).

Conversely, let $L \in L(H_{\alpha,\beta})$ be such that both (2.8) and (2.9) hold. Then

$$L(u * \delta_{\alpha-\beta}) = u * (L \delta_{\alpha-\beta}), \quad u \in H_{\alpha,\beta}.$$  

(2.10)

To demonstrate (2.10), define from $H'_{\alpha,\beta}$ into $H'_{\alpha,\beta}$ the linear map

$$Mu = L(u * \delta_{\alpha-\beta}) - u * (L \delta_{\alpha-\beta}), \quad u \in H'_{\alpha,\beta}.$$
The definition of $\Lambda$ is consistent by virtue of (2.9). Since $\Lambda \in \mathcal{L}(\mathcal{H}_{a,\beta})$, its kernel is a closed subspace of $\mathcal{H}_{a,\beta}$.

In view of (2.8) this kernel contains $\tau_x \delta_{a-\beta}$ ($x \in I$), and hence (Lemma 2.3) it is also dense in $\mathcal{H}_{a,\beta}$. Therefore (2.10) holds.

Now, letting $T = L \delta_{a-\beta}$ we have
\[
u * T = \nu * (L \delta_{a-\beta}) = L(\nu * \delta_{a-\beta}) = Lu,
\]
which proves (2.7).

As to the uniqueness assertion, assume that $S \in \mathcal{O}_{a,\beta}$ is not the zero distribution, so that $\phi \in \mathcal{H}_{a,\beta}$ exists for which $S * \phi \neq 0$. Since $\mathcal{H}_{a,\beta}$ separates points in $\mathcal{H}_{a,\beta}$ we may find $\nu \in \mathcal{H}_{a,\beta}$ such that
\[\langle \nu * S, \phi \rangle = \langle \nu, S * \phi \rangle \neq 0.
\]
This completes the proof.

3. A property of convolution operators
Motivated by Theorem 2 in [5], our aim in this section is to prove the following theorem.

**Theorem 3.1.** Let $(\alpha - \beta) \geq -1/2$. For $S \in \mathcal{O}_{a,\beta}$, the following are equivalent:

(i) To every $k \in \mathbb{N}$ there correspond $m, n \in \mathbb{N}$ and a positive constant $M$, such that
\[
\max_{0 \leq j \leq m} \sup_{t \in I} \left\{ \left| (t^{-1} D)^j t^{2\beta - 1} \left( h_{a,\beta} S \right) (t) \right| : t \in I, |x - t| \leq (1 + x^2)^{-k} \right\} \geq (1 + x^2)^{-n},
\]
whenever $x \in I, x \geq M$.

(ii) If $\nu \in \mathcal{O}_{a,\beta}$ and $S * \nu \in \mathcal{H}_{a,\beta}$, then $\nu \in \mathcal{H}_{a,\beta}$.

**Proof.** Suppose that (ii) is not satisfied. Then there exist $\nu \in \mathcal{O}_{a,\beta}$ such that $S * \nu \in \mathcal{H}_{a,\beta}$, but $\nu \notin \mathcal{H}_{a,\beta}$. This shows that $\mathcal{T}^{2\beta - 1} [h_{a,\beta} S] (t) \notin O$, $\mathcal{T}^{2\beta - 1} [h_{a,\beta} S] (t) \mathcal{T}^{\alpha,\beta} (t) \in \mathcal{H}_{a,\beta}$, and $h_{a,\beta} \mathcal{T} \notin \mathcal{H}_{a,\beta}$.

As both $\mathcal{T}^{2\beta - 1} [h_{a,\beta} S] (t)$ and $\mathcal{T}^{2\beta - 1} [h_{a,\beta} \mathcal{T}] (t)$ lie in $O$, to every $t \in \mathbb{N}$ there correspond $r_i \in \mathbb{N}$, $M_i > 0$ satisfying
\[
\left| (t^{-1} D)^j t^{2\beta - 1} \left( h_{a,\beta} S \right) (t) \right| \leq M_i (1 + \mathcal{T})^r_i, \quad t \in I,
\]
and $s_i \in \mathbb{N}$, $N_i > 0$ satisfying
\[
\left| (t^{-1} D)^j t^{2\beta - 1} \left( h_{a,\beta} \mathcal{T} \right) (t) \right| \leq N_i (1 + \mathcal{T})^s_i, \quad t \in I.
\]

Moreover, as $h_{a,\beta} \mathcal{T} \notin \mathcal{H}_{a,\beta}$, there are $l_0, n_0 \in \mathbb{N}$ and a sequence $\{t_j\}_{j \in \mathbb{N}}$ in $I$, such that $t_j \to \infty$ as $j \to \infty$ and
\[
\left| (t^{-1} D)^j t^{2\beta - 1} \left( h_{a,\beta} \mathcal{T} \right) (t) \right| \geq (1 + \mathcal{T})^{-n_0}, \quad j \in \mathbb{N}.
\]

Set $k = s_{k+1} + n_0 + 2$, and define
\[
B_{j,k} = \{ t \in I : |t - j| \leq (1 + \mathcal{T})^{-k} \}, \quad j \in \mathbb{N}.
\]

From (3.2) and (3.3) we can infer that, for sufficiently large $j$,
\[
\inf_{t \in B_{j,k}} \left| (t^{-1} D)^j t^{2\beta - 1} \left( h_{a,\beta} \mathcal{T} \right) (t) \right| \geq \frac{1}{2} (1 + \mathcal{T})^{-n_0} > 0.
\]

Indeed, if $j$ is large enough and if $t \in B_{j,k}$, then
\[
\left| (t^{-1} D)^j t^{2\beta - 1} \left( h_{a,\beta} \mathcal{T} \right) (t) \right|
\geq \left| (t^{-1} D)^j t^{2\beta - 1} \left( h_{a,\beta} \mathcal{T} \right) (t) \right| \geq \left| (t_j + (1 + \mathcal{T})^{-k}) \left(1 + \mathcal{T} \right)^{-n_0} \right|
\geq (1 + \mathcal{T})^{-n_0} - C (1 + \mathcal{T})^{s_{k+1} + j - 1} - (1 + \mathcal{T})^{-n_0} - C (1 + \mathcal{T})^{s_{k+1} + \mathcal{T}^{-n_0} - 1},
\]
where $C > 0$ is a constant independent from $j$. This proves (3.5).

Now $t^{2\beta - 1} \left( h_{a,\beta} S \right) (t) \mathcal{T} (t) \in \mathcal{H}_{a,\beta}$, and therefore
\[
\sup_{t \in B_{j,k}} \left| (t^{-1} D)^j t^{2\beta - 2} \left( h_{a,\beta} S \right) (t) \mathcal{T} (t) \right| = O \left( (1 + \mathcal{T})^{-n} \right), \quad l, n \in \mathbb{N}, \quad j \to \infty.
\]
Clearly, for fixed \( l, n \in \mathbb{N} \) we may write 
\[
\sup_{t \in B_{\beta}} \left| (t^{-1} D)^j \left( \beta S \right)(t) \right| \leq C_{n,l} \left( 1 + \frac{t_l}{t} \right)^n, \quad t \in I.
\]

where \( C_{n,l} > 0 \) is a constant, and the right hand side of this inequality is clearly \( O \left( \left( 1 + \frac{t_l}{t} \right)^n \right) \) as \( j \to \infty \).

Next we prove that 
\[
\text{max} \sup_{0 \leq m \leq s} \left| (t^{-1} D)^j \left( \beta S \right)(t) \right| = O \left( \left( 1 + \frac{t_l}{t} \right)^n \right), \quad m, n \in \mathbb{N}, \quad j \to \infty, \quad (3.7)
\]
a contradiction to (i). In the sequel, \( n \) will denote an arbitrary positive integer.

We first assume that \( l_0 = 0 \) and proceed by induction on \( m \). In view of (3.5) and (3.6), we have 
\[
\sup_{t \in B_{\beta}} \left| (t^{-1} D)^j \left( \beta S \right)(t) \right| \leq 2 \left( 1 + \frac{t_l}{t} \right)^{-n_0} \sup_{t \in B_{\beta}} \left| (t^{-1} D)^j \left( \beta T \right)(t) \right| = O \left( \left( 1 + \frac{t_l}{t} \right)^{-n} \right), \quad j \to \infty.
\]

Thus, condition (3.7) is satisfied for \( m = 0 \).

Now suppose that (3.7) holds for some \( m \). We must prove that it also holds for \( m + 1 \).

By Leibnitz’s rule, 
\[
(t^{-1} D)^{m+1} \left( \beta S \right)(t) = \sum_{i=0}^{m+1} (-1)^i \begin{pmatrix} m+1 \end{pmatrix} t^{-1} D^{m+1-i} \left( \beta S \right)(t) \left( t^{-1} D \right)^i (t^{-1} D)^j \left( \beta T \right)(t) \quad t \in I.
\]

Bearing in mind (3.2), (3.6) and the induction hypothesis, we find that 
\[
\sup_{t \in B_{\beta}} \left| (t^{-1} D)^{m+1-i} \left( \beta S \right)(t) \left( t^{-1} D \right)^i \left( \beta T \right)(t) \right| = O \left( \left( 1 + \frac{t_l}{t} \right)^{-n} \right), \quad j \to \infty,
\]
as \( j \to \infty \), whenever \( 0 \leq i \leq m + 1 \). Consequently 
\[
(t^{-1} D)^{m+1} \left( \beta T \right)(t) = O \left( \left( 1 + \frac{t_l}{t} \right)^{-n} \right), \quad j \to \infty.
\]

This shows that (3.7) holds when \( l_0 = 0 \). Next, assume that \( l_0 \neq 0 \) and \( l_0 \) is the smallest positive integer for which \( n_0 \in \mathbb{N} \) and a sequence \( \{t_j\}_{j \in \mathbb{N}} \) in \( I \) may be found so that (3.3) (and hence (3.5), with large enough \( j \)) is satisfied.

This means that 
\[
(t^{-1} D)^j \left( \beta T \right)(t) = O \left( \left( 1 + \frac{t_l}{t} \right)^{-n} \right), \quad l < l_0, \quad t \to \infty.
\]

Arguing as in the proof of (3.6) we are led to 
\[
\sup_{t \in B_{\beta}} \left| (t^{-1} D)^j \left( \beta T \right)(t) \right| = O \left( \left( 1 + \frac{t_l}{t} \right)^{-n} \right), \quad l < l_0, \quad j \to \infty. \quad (3.8)
\]

By virtue of Leibnitz’s rule, 
\[
(t^{-1} D)^j \left( \beta S \right)(t) \left( t^{-1} D \right)^h \left( \beta T \right)(t) \quad t \in I.
\]
Then, from (3.1), (3.6) and (3.8) it follows that
\[
\sup_{t \in B_{jk}} \left| t^{2\beta-1} \left( h_{a,\beta} S \right)(t) - (t^{-1} D)^l t^{2\beta-1} \left( h_{a,\beta} T \right)(t) \right| = O \left( (1 + t^2) \right)^n, \quad j \to \infty.
\] (3.9)

Finally, using (3.5), (3.6) and (3.9) we obtain (3.7) by an argument similar to that employed in the case \( l_0 = 0 \). This completes the proof that (i) implies (ii).

Conversely, suppose that (i) does not hold. Then there exist \( k \in \mathbb{N} \) and a sequence \( \{t_j\}_{j \in \mathbb{N}} \) in \( I \), with \( t_j \to \infty \) as \( j \to \infty \) such that
\[
\max_{0 \leq f \leq j} \sup_{t \in B_{jk}} \left| (t^{-1} D)^l t^{2\beta-1} \left( h_{a,\beta} S \right)(t) \right| < (1 + t^2)^{-l}, \quad j \in \mathbb{N};
\] (3.10)
where the sets \( B_{jk} \) are given by (3.4). There is no loss of generality in assuming that \( t_0 > 1 \) and \( t_{j+1} > t_j + 1 \). Let \( a \in \mathcal{D}(I) \) be such that \( 0 \leq a \leq 1 \), supp \( a = [1/2, 3/2] \) and \( a(1) = 1 \), and set
\[
\theta(t) = a \left( 1 + \frac{1}{2}(t-t_j) \right) (1 + t^2)^k, \quad \theta(t) = \sum_{j=0}^{\infty} \theta_j(t), \quad t \in I.
\]
The sum defining \( \theta \) is finite, because supp \( \theta_j = B_{kk}(j \in \mathbb{N}) \) and \( B_{kk} \cap B_{kk} = \phi \ (i, j \in \mathbb{N}, i \neq j) \). If \( l, j \in \mathbb{N} \) and \( t \in B_{kk} \) then for some \( a_m \in \mathbb{R} \ (0 \leq m \leq l) \), we have
\[
\left| (t^{-1} D)^l \theta(t) \right| = \left| (t^{-1} D)^l \theta_j(t) \right|
\]
\[
= \sum_{m=0}^{l} |a_m t^{-l-m} D^m \theta_j(t)|
\]
\[
\leq 2^{l+m} \sum_{m=0}^{l} |a_m D^m \theta_j(t)|
\]
\[
\leq C_l 2^{-kl} (1 + t^2)^l \sum_{m=0}^{l} |D^m \theta_j(y)| |y^{l+1}_{j+1}(t^{-l-j})(1+t^2)|
\]
\[
\leq C_l (1 + t^2)^k \leq C_l (1 + t^2)^k,
\]
where \( C_l > 0 \) denotes an appropriate constant (not necessarily the same in each occurrence). Then
\[
\left| (t^{-1} D)^l \theta(t) \right| \leq C_l (1 + t^2)^k, \quad t \in I,
\] (3.11)
thus proving that \( \theta \in O \). Thus, there exist \( T \in O_{a,\beta} \) such that \( \left( h_{a,\beta}' \right)(t) = \tilde{r}(\theta)(t), \ t \in I \).

Let \( n, l \in \mathbb{N} \). Thus function
\[
(1 + t^2)^n (t^{-1} D)^l t^{2\beta-2} \left( h_{a,\beta} S \right)(t) \left( h_{a,\beta} T \right)(t), \quad t \in I
\]
is bounded on the interval \( 0 < t < t_{n+l} - (1 + t^2)^{-l} \). Letting \( j = n + kl + r \ (r \in \mathbb{N}) \) and \( t \in B_{jk} \), Leibnitz rule, along with (3.10) and (3.11), implies
\[
\left| (1 + t^2)^n (t^{-1} D)^l t^{2\beta-2} \left( h_{a,\beta} S \right)(t) \left( h_{a,\beta} T \right)(t) \right|
\]
\[
= \left| (1 + t^2)^n (t^{-1} D)^l t^{2\beta-2} \left( h_{a,\beta} S \right)(t) \theta(t) \right|
\]
\[
\leq C (1 + t^2)^{n+kl} (1 + t^2)^{-n-kl} \leq C,
\]
where \( C > 0 \) is a suitable constant (concerning the value of \( C \), we make the same convolution as before). This shows that
\[
(1 + t^2)^n \left( h_{a,\beta}' S \right)(t) \left( h_{a,\beta}' T \right)(t) \in \mathcal{H}_{a,\beta}. \quad \text{But } \left( h_{a,\beta}' T \right) \in \mathcal{H}_{a,\beta}.
\]

Since
\[
\tilde{r}_t^\beta \left( h_{a,\beta}' T \right)(t) = a(1) = 1; \quad as \ t_j \to \infty \ (j \to \infty).
\]

We conclude that \( T \in O_{a,\beta} \) and that \( S \ast T \in \mathcal{H}_{a,\beta} \) although \( T \notin \mathcal{H}_{a,\beta} \), which contradict (ii) and completes the proof.
4. Conclusion

In the present research article, we have accomplished two major objectives regarding Hankel type convolution operator. Firstly, $O'_{\alpha,\beta,*}$ elements are defined as those in $L(H_{\alpha,\beta})$ and in $L(H'_{\alpha,\beta})$ that commute with Hankel translations. Furthermore, we obtain certain results that aid in proving the objective.

Secondly, To ensure that every $T \in O'_{\alpha,\beta,*}$ such that $S \ast T \in H_{\alpha,\beta}$ lie in $H_{\alpha,\beta}$, necessary and sufficient conditions on the generalised Hankel type transform $h'_{\alpha,\beta}S$ of $S \in O'_{\alpha,\beta,*}$ are established.

The findings of this research may be significant for areas in engineering, physics and mathematics.

References


