ISSN 0304-9892 (Print) ISSN 2455-7463 (Online)
www.vijnanaparishadofindia.org/jnanabha
***Jñānābha*, Vol. 52(2) (2022), 158-164**
(Dedicated to Professor D. S. Hooda on His 80^{th} Birth Anniversary Celebrations)

ON HANKEL TYPE CONVOLUTION OPERATORS

B. B. Waphare[1] , R. Z. Shaikh[1] and N. M. Rane[2]
[1]Department of Mathematics, MAEER's MIT Arts, Commerec & Science College
Alandi(D), Pune-412105, Maharastra, India.
[2] Avantika University, Ujjain-456006, Madhya Pradesh, India.
Email: balasahebwaphare@gmail.com, shaikhrahilanaz@gmail.com, nitin@avantika.edu.in
(Received : May 31, 2022; In front : July 30, 2022; Accepted : October 17, 2022)

DOI: https://doi.org/10.58250/jnanabha.2022.52218

Abstract

Let $\mathcal{H}'_{\alpha,\beta}$ be the Zemanian type space of Hankel transformable generalised functions and let $O'_{\alpha,\beta,*}$ be the space of Hankel convolution operators for $\mathcal{H}'_{\alpha,\beta}$. This $\mathcal{H}'_{\alpha,\beta}$ is the dual of a subspace $\mathcal{H}_{\alpha,\beta}$ of $O'_{\alpha,\beta,*}$ for which $O'_{\alpha,\beta,*}$ is also the space of Hankel convolution. In this paper the elements of $O'_{\alpha,\beta,*}$ are characterised as those in $\mathcal{L}\left(\mathcal{H}_{\alpha,\beta}\right)$ and in $\mathcal{L}\left(\mathcal{H}'_{\alpha,\beta}\right)$ that commute with Hankel translations. Moreover, necessary and sufficient condition on the generalised Hankel type transform $h'_{\alpha,\beta}S$ of $S \in O'_{\alpha,\beta,*}$ are established in order that every $T \in O'_{\alpha,\beta,*}$ such that $S * T \in \mathcal{H}_{\alpha,\beta}$ lie in $\mathcal{H}_{\alpha,\beta}$.
2020 Mathematical Sciences Classification: 46F12
Keywords and Phrases: Generalized functions, Hankel type transformation, Hankel type translation, Hankel type convolution.

1. Introduction

Following Zemanian [6], we denote by $\mathcal{H}_{\alpha,\beta}$ the space of Hankel transformable functions, $(\alpha-\beta) \in \mathbb{R}$. $\mathcal{H}_{\alpha,\beta}$ consists of all those infinitely differentiable functions $\phi = \phi(x)$ defined on $I = (0, \infty)$ such that

$$\rho^{\alpha,\beta}_{m,k} = \sup_{x \in I} \left|(1+x^2)^m \,(x^{-1}D)^k \, x^{2\beta-1}\, \phi(x)\right| < \infty, \qquad m,k \in \mathbb{N}.$$

$\mathcal{H}_{\alpha,\beta}$ being a Frechet Space when endowed with the topology generated by the family of seminorms $\left\{\rho^{\alpha,\beta}_{m,k}\right\}_{(m,k)\in\mathbb{N}\times\mathbb{N}}$.
The Hankel type transformation

$$\left(h_{\alpha,\beta}\phi\right)(t) = \int_0^\infty \phi(x)\,(xt)^{\alpha+\beta}\, J_{\alpha-\beta}(xt)\,dx$$

is an automorphism of $\mathcal{H}_{\alpha,\beta}$; provided $(\alpha-\beta) \geq -\frac{1}{2}$, where $J_{\alpha-\beta}$ denotes the Bessel type function of first kind and of order $(\alpha-\beta)$. If $(\alpha-\beta) \geq -\frac{1}{2}$, the generalized Hankel type transformation $h'_{\alpha,\beta}$ is defined on $\mathcal{H}'_{\alpha,\beta}$, the dual space of $\mathcal{H}_{\alpha,\beta}$ as the adjoint of $h_{\alpha,\beta}$. Then $h'_{\alpha,\beta}$ is an automorphism of $\mathcal{H}'_{\alpha,\beta}$.
Following [1],[2] and [4], for $(\alpha-\beta) \geq -\frac{1}{2}$, we introduce the subspace $O'_{\alpha,\beta,*}$ of $\mathcal{H}'_{\alpha,\beta}$ as the space of all those $T \in \mathcal{H}'_{\alpha,\beta}$ such that $\theta(x) = x^{2\beta-1}\left(h'_{\alpha,\beta}T\right)(x)$ is a smooth function on I with the property that for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ satisfying

$$\sup_{x\in I}\left|(1+x^2)^{-n_k}\,(x^{-1}D)^k\,\phi(x)\right| < \infty.$$

Clearly, $\mathcal{H}_{\alpha,\beta}$ is a subspace of $O'_{\alpha,\beta,*}$.The space O of all those smooth functions $\theta = \theta(x)$ on I possessing the above property turns out to be the space of multiplication operators on $\mathcal{H}_{\alpha,\beta}$ and on $\mathcal{H}'_{\alpha,\beta}$ ($(\alpha-\beta)\in\mathbb{R}$), whereas $O'_{\alpha,\beta,*}$ is the space of convolution operators on $\mathcal{H}_{\alpha,\beta}$ and on $\mathcal{H}'_{\alpha,\beta}$ ($(\alpha-\beta) \geq -1/2$).
Throughout this paper we shall always assume that $(\alpha-\beta)$ is a real number $\geq -1/2$ and, unless otherwise stated, that $\mathcal{H}'_{\alpha,\beta}$ is endowed with its weak* topology.

2. Characterization of $O'_{\alpha,\beta,*}$ in $\mathcal{L}\left(\mathcal{H}_{\alpha,\beta}\right)$ and in $\mathcal{L}\left(\mathcal{H}'_{\alpha,\beta}\right)$

Let $\mathcal{L}\left(\mathcal{H}_{\alpha,\beta}\right)$ (respectively, $\mathcal{L}\left(\mathcal{H}'_{\alpha,\beta}\right)$) denote the space of all linear continuous operator from $\mathcal{H}_{\alpha,\beta}$ (resp. $\mathcal{H}'_{\alpha,\beta}$) into itself. The characterization of elements in $\mathcal{L}\left(\mathcal{H}_{\alpha,\beta}\right)$ and in $\mathcal{L}\left(\mathcal{H}'_{\alpha,\beta}\right)$ that commute with Hankel type translation is our first objective.

We recall that the Hankel type translation $\tau_x\phi$ of $\phi \in \mathcal{H}_{\alpha,\beta}$ by $x \in I$ is defined as

$$(\tau_x\phi)(y) = \int_0^\infty \phi(z)\, D_{\alpha,\beta}(x,y,z)\, dz, \quad y \in I,$$

where,

$$D_{\alpha,\beta}(x,y,z) = \int_0^\infty t^{2\beta-1}\ j_{\alpha-\beta}(xt)\ j_{\alpha-\beta}(yt)\ j_{\alpha-\beta}(zt)dt, \quad x,y,z \in I,$$

and $j_{\alpha-\beta}(z) = z^{\alpha+\beta}\, J_{\alpha-\beta}(z), \quad (z \in I)$. The map $\phi \mapsto \tau_x\phi$ is a continuous endomorphism of $\mathcal{H}_{\alpha,\beta}$.

Further

$$\left(h_{\alpha,\beta}\, \tau_x\, \phi\right)(t) = t^{2\beta-1}\ j_{\alpha-\beta}(xt)\ \left(h_{\alpha,\beta}\, \phi\right)(t), \quad t \in I \tag{2.1}$$

whenever $\phi \in \mathcal{H}_{\alpha,\beta}$ and $x \in I$.

If $u \in \mathcal{H}'_{\alpha,\beta}$ and $x \in I$, we define $\tau_x u \in \mathcal{H}'_{\alpha,\beta}$ by traposition:

$$\langle \tau_x u, \phi\rangle = \langle u, \tau_x\phi\rangle, \quad \phi \in \mathcal{H}_{\alpha,\beta}. \tag{2.2}$$

The following analogue of (2.1) holds for the generalized translation (2.2).

Lemma 2.1. *Let $u \in \mathcal{H}'_{\alpha,\beta}$ and $x \in I$. Then*

$$\left(h'_{\alpha,\beta}\, \tau_x\, u\right)(t) = t^{2\beta-1}\ j_{\alpha-\beta}(xt)\ \left(h'_{\alpha,\beta}\, u\right)(t), \quad t \in I.$$

Proof. Let $u \in \mathcal{H}'_{\alpha,\beta}$, $x \in I$ and $\phi \in \mathcal{H}_{\alpha,\beta}$. Then a combination of (2.1) and (2.2) gives

$$\begin{aligned}\left\langle h'_{\alpha,\beta}\, \tau_x\, u,\ h_{\alpha,\beta}\, \phi\right\rangle = \langle \tau_x u,\ \phi\rangle = \langle u, \tau_x\phi\rangle &= \left\langle h'_{\alpha,\beta}\, u,\ h_{\alpha,\beta}\, \tau_x\, \phi\right\rangle \\ &= \left\langle \left(h'_{\alpha,\beta}\, u\right)(t),\ t^{2\beta-1}\ j_{\alpha-\beta}(xt)\ \left(h_{\alpha,\beta}\, \phi\right)(t)\right\rangle \\ &= \left\langle t^{2\beta-1}\ j_{\alpha-\beta}(xt)\ \left(h'_{\alpha,\beta}\, u\right)(t),\ \left(h'_{\alpha,\beta}\, \phi\right)(t)\right\rangle.\end{aligned}$$

This completes the proof. The classical Hankel convolution $\phi * \psi$ of $\phi,\ \psi \in \mathcal{H}_{\alpha,\beta}$ is the function

$$\phi * \psi(x) = \int_0^\infty \phi(y)\ (\tau_x\, \psi)(y)dy, \quad x \in I.$$

The map $(\phi,\ \psi) \mapsto \phi * \psi$ is continuous from $\mathcal{H}_{\alpha,\beta} \times \mathcal{H}_{\alpha,\beta}$ into $\mathcal{H}_{\alpha,\beta}$. The generalized Hankel type convolution $u * \phi$ of $u \in \mathcal{H}'_{\alpha,\beta}$ and $\phi \in \mathcal{H}_{\alpha,\beta}$ is the distribution given by

$$\langle u * \phi,\ \psi\rangle = \langle u, \phi * \psi\rangle, \quad \psi \in \mathcal{H}_{\alpha,\beta}.$$

The map $(u,\ \phi) \mapsto u * \phi$ is separately continuous from $\mathcal{H}'_{\alpha,\beta} \times \mathcal{H}_{\alpha,\beta}$ into $\mathcal{H}'_{\alpha,\beta}$, when $\mathcal{H}'_{\alpha,\beta}$ is endowed either with its weak* or its strong topology.

Finally, for $u \in \mathcal{H}'_{\alpha,\beta}$ and $T \in O'_{\alpha,\beta,*}$, the generalized function $u * T \in \mathcal{H}'_{\alpha,\beta}$ defined as

$$\langle u * T,\ \phi\rangle = \langle u, T * \phi\rangle, \quad \phi \in \mathcal{H}_{\alpha,\beta}. \tag{2.3}$$

Note that each of these definitions, extends the previous one. Moreover,

$$\left(h'_{\alpha,\beta}\, u * T\right)(t) = t^{2\beta-1}\ \left(h'_{\alpha,\beta}\, T\right)(t)\ \left(h'_{\alpha,\beta}\, u\right)(t), t \in I \tag{2.4}$$

whenever $u \in \mathcal{H}'_{\alpha,\beta}$ and $T \in O'_{\alpha,\beta,*}$.

If $C_{\alpha,\beta} = 2^{\alpha-\beta}\, \Gamma(3\alpha+\beta)$ then the element $\delta_{\alpha-\beta}$ of $O'_{\alpha,\beta,*}$ given by

$$\left\langle \delta_{\alpha-\beta},\ \phi\right\rangle = C_{\alpha,\beta} \lim_{x\to 0^+} x^{2\beta-1}\, \phi(x), \quad \phi \in \mathcal{H}_{\alpha,\beta}$$

is an identity for (2.3).

The generalized *-convolution commutes with Hankel type translations:

Lemma 2.2. *Let $u \in \mathcal{H}'_{\alpha,\beta}$ and $x \in I$. If $T \in O'_{\alpha,\beta,*}$, then*

$$\tau_x\, (u * T) = (\tau_x\, u) * T = u * (\tau_x\, T)\,.$$

Proof. Since $h'_{\alpha,\beta}$ is an automorphism of $\mathcal{H}'_{\alpha,\beta}$, we prove the lemma by fixing $t \in I$ and using Lemma 2.1, along with (2.4) to write,

$$\begin{aligned}\left(h'_{\alpha,\beta}\, \tau_x\, (u * T)\right)(t) &= t^{2\beta-1}\ j_{\alpha-\beta}(xt)\ \left(h'_{\alpha,\beta}\, u * T\right)(t) &&= t^{4\beta-2}\ j_{\alpha-\beta}(xt)\ \left(h'_{\alpha,\beta}\, T\right)(t)\ \left(h'_{\alpha,\beta}\, u\right)(t),\\ \left(h'_{\alpha,\beta}\, (\tau_x u) * T\right)(t) &= t^{2\beta-1}\ \left(h'_{\alpha,\beta}\, T\right)(t)\ \left(h'_{\alpha,\beta}\, \tau_x u\right)(t) &&= t^{4\beta-2}\ j_{\alpha-\beta}(xt)\ \left(h'_{\alpha,\beta}\, T\right)(t)\ \left(h'_{\alpha,\beta}\, u\right)(t),\\ \left(h'_{\alpha,\beta}\, u * (\tau_x T)\right)(t) &= t^{2\beta-1}\ \left(h'_{\alpha,\beta}\, \tau_x T\right)(t)\ \left(h'_{\alpha,\beta}\, u\right)(t) &&= t^{4\beta-2}\ j_{\alpha-\beta}(xt)\ \left(h'_{\alpha,\beta}\, T\right)(t)\ \left(h'_{\alpha,\beta}\, u\right)(t).\end{aligned}$$

Thus proof is completed. Now we are ready to prove the following theorem.

Theorem 2.1. *If $T \in O^{'}_{\alpha,\beta,*}$ and L is the element of $\mathcal{L}\left(\mathcal{H}_{\alpha,\beta}\right)$ defined by*

$$L\phi = T * \phi, \quad \phi \in \mathcal{H}_{\alpha,\beta}, \tag{2.5}$$

then

$$\tau_x L = L\tau_x, \quad x \in I. \tag{2.6}$$

conversely, if $L \in \mathcal{L}\left(\mathcal{H}_{\alpha,\beta}\right)$ satisfies (2.6) *then there exists a unique $T \in O^{'}_{\alpha,\beta,*}$ for which* (2.5) *holds.*

Proof. Let $T \in O^{'}_{\alpha,\beta,*}$. The fact that $L \in \mathcal{L}\left(\mathcal{H}_{\alpha,\beta}\right)$ defined by (2.5) satisfies (2.6) is contained in Lemma 2.2.On the other hand, assume that $L \in \mathcal{L}\left(\mathcal{H}_{\alpha,\beta}\right)$ is such that (2.6) holds, and define $T \in \mathcal{H}^{'}_{\alpha,\beta}$ by

$$\langle T, \phi\rangle = \left\langle \delta_{\alpha-\beta}, L\phi\right\rangle, \quad \phi \in \mathcal{H}_{\alpha,\beta}.$$

Then

$$\begin{aligned}(T * \phi)(x) = \langle T, \tau_x \phi\rangle &= \left\langle \delta_{\alpha-\beta}, L\tau_x\phi\right\rangle = \left\langle \delta_{\alpha-\beta}, \tau_x L\phi\right\rangle \\ &= \left(\delta_{\alpha-\beta} * L\phi\right)(x) = (L\phi)(x), \quad x \in I,\end{aligned}$$

whenever $\phi \in \mathcal{H}_{\alpha,\beta}$, which proves (2.5). As $O^{'}_{\alpha,\beta,*}$ is the space of convolution operators of $\mathcal{H}_{\alpha,\beta}$, it follows from (2.5) that $T \in O^{'}_{\alpha,\beta,*}$. As to the uniqueness assertion, note that if $S \in O^{'}_{\alpha,\beta,*}$ is such that $S * \phi = 0$ for every $\phi \in \mathcal{H}_{\alpha,\beta}$, then $S = 0$. In fact, $S * \phi = 0$ ($\phi \in \mathcal{H}_{\alpha,\beta}$) and (2.4) imply $t^{2\beta-1}\left(h^{'}_{\alpha,\beta}S\right)\psi(t) = 0$, ($\psi \in \mathcal{H}_{\alpha,\beta},\ t \in I$). By particularizing $\psi(t) = t^{2\alpha}e^{-t^2}$ ($t \in I$) we find that $t^{2\beta-1}\left(h^{'}_{\alpha,\beta}S\right)(t) = 0$, whence $\left(h^{'}_{\alpha,\beta}S\right) = 0$ and $S = 0$. This completes the proof. The following result will help in characterising the elements of $O^{'}_{\alpha,\beta,*}$ as those in $\mathcal{L}\left(\mathcal{H}_{\alpha,\beta}\right)$ that commute with Hankel type translations.

Lemma 2.3. *The linear hull of the set of generalized functions of the form $\tau_x\,\delta_{\alpha-\beta}$ ($x \in I$) is weakly* dense in $\mathcal{H}^{'}_{\alpha,\beta}$.*

Proof. As $\left(h^{'}_{\alpha,\beta}\,\delta_{\alpha-\beta}\right)(t) = t^{2\alpha}$ ($t \in I$) by Lemma 2.1, we have

$$\left(h^{'}_{\alpha,\beta}\,\tau_x\,\delta_{\alpha-\beta}\right)(t) = j_{\alpha-\beta}(xt), \quad x, t \in I.$$

If $\phi \in \mathcal{H}_{\alpha,\beta}$ does not vanish identically then there exists $x \in I$ such that $\phi(x) \neq 0$ and hence

$$\begin{aligned}\left\langle \tau_x\,\delta_{\alpha-\beta}, \phi\right\rangle &= \left\langle h^{'}_{\alpha,\beta}\,\tau_x\,\delta_{\alpha-\beta}, h_{\alpha,\beta}\,\phi\right\rangle \\ &= \int_0^\infty (h_{\alpha,\beta}\,\phi)(t)\; j_{\alpha-\beta}(xt)dt = \phi(x) \neq 0.\end{aligned}$$

This shows that the subset $\{\tau_x\,\delta_{\alpha-\beta}\}_{x\in I}$ of $\mathcal{H}^{'}_{\alpha,\beta}$ separates points in $\mathcal{H}_{\alpha,\beta}$. By [3], problem $W(b)$, this family is total in $\mathcal{H}^{'}_{\alpha,\beta}$ with respect to the weak* topology.
Thus proof is completed.

Theorem 2.2. *If $T \in O^{'}_{\alpha,\beta,*}$ and $L \in \mathcal{L}\left(\mathcal{H}^{'}_{\alpha,\beta}\right)$ is defined by*

$$Lu = u * T, \quad u \in \mathcal{H}^{'}_{\alpha,\beta}, \tag{2.7}$$

then

$$\tau_x L = L\tau_x, \quad x \in I, \tag{2.8}$$

and also

$$L\,\delta_{\alpha-\beta} \in O^{'}_{\alpha,\beta,*}. \tag{2.9}$$

Conversely, given $L \in \mathcal{L}\left(\mathcal{H}^{'}_{\alpha,\beta}\right)$ satisfying (2.8) *and* (2.9)*, a unique $T \in O^{'}_{\alpha,\beta,*}$ may be found so that* (2.7) *holds.*

Proof. Note that L given by (2.7) satisfies (2.8) is a consequence of Lemma2.2. Clearly it also satisfies (2.9).
Conversely, Let $L \in \mathcal{L}\left(\mathcal{H}_{\alpha,\beta}\right)$ be such that both (2.8) and (2.9) hold. Then

$$L\left(u * \delta_{\alpha-\beta}\right) = u * \left(L\,\delta_{\alpha-\beta}\right), \quad u \in \mathcal{H}^{'}_{\alpha,\beta}. \tag{2.10}$$

To demonstrate (2.10), define from $\mathcal{H}^{'}_{\alpha,\beta}$ into $\mathcal{H}^{'}_{\alpha,\beta}$ the linear map

$$\Lambda u = L\left(u * \delta_{\alpha-\beta}\right) - u * \left(L\delta_{\alpha-\beta}\right), \quad u \in \mathcal{H}^{'}_{\alpha,\beta}.$$

The definition of Λ is consistent by virtue of (2.9). Since $\Lambda \in \mathcal{L}\left(\mathcal{H}'_{\alpha,\beta}\right)$, its kernel is a closed subspace of $\mathcal{H}'_{\alpha,\beta}$. In view of (2.8) this kernel contains $\tau_x\, \delta_{\alpha-\beta}$ $(x \in I)$, and hence (Lemma 2.3) it is also dense in $\mathcal{H}'_{\alpha,\beta}$. Therefore (2.10) holds.

Now, letting $T = L\,\delta_{\alpha-\beta}$ we have

$$u * T = u * (L\,\delta_{\alpha-\beta}) = L(u * \delta_{\alpha-\beta}) = Lu,$$

which proves (2.7).

As to the uniqueness assertion, assume that $S \in O'_{\alpha,\beta,*}$ is not the zero distribution, so that $\phi \in \mathcal{H}_{\alpha,\beta}$ exists for which $S * \phi \neq 0$. Since $\mathcal{H}'_{\alpha,\beta}$ separates points in $\mathcal{H}_{\alpha,\beta}$ we may find $u \in \mathcal{H}'_{\alpha,\beta}$ such that

$$\langle u * S,\, \phi\rangle = \langle u,\, S * \phi\rangle \neq 0.$$

This completes the proof.

3. A property of convolution operators

Motivated by Theorem 2 in [5], our aim in this section is to prove the following theorem.

Theorem 3.1. *Let $(\alpha-\beta) \geq -1/2$. For $S \in O'_{\alpha,\beta,*}$, the following are euivalent:*

(i) To every $k \in \mathbb{N}$ there correspond $m, n \in \mathbb{N}$ and a positive constant M, such that

$$\max_{0\leq l\leq m} \sup\left\{\left|(t^{-1}\,D)^l\, t^{2\beta-1}\, \left(h'_{\alpha,\beta}\, S\right)(t)\right| : t \in I,\ |x-t| \leq (1+x^2)^{-k}\right\} \geq (1+x^2)^{-n},$$

whenever $x \in I, x \geq M$.

(ii) If $T \in O'_{\alpha,\beta,*}$ and $S * T \in \mathcal{H}_{\alpha,\beta}$, then $T \in \mathcal{H}_{\alpha,\beta}$.

Proof. Suppose that (ii) is not satisfied. Then there exist $T \in O'_{\alpha,\beta,*}$ such that $S * T \in \mathcal{H}_{\alpha,\beta}$, but $T \notin \mathcal{H}_{\alpha,\beta}$. This shows that $t^{2\beta-1}\left(h'_{\alpha,\beta}\, S\right)(t) \in O$, $t^{2\beta-1}\left(h'_{\alpha,\beta}\, S\right)(t)\left(h'_{\alpha,\beta}\, T\right)(t) \in \mathcal{H}_{\alpha,\beta}$, and $h'_{\alpha,\beta}\, T \notin \mathcal{H}_{\alpha,\beta}$.
As both $t^{2\beta-1}\left(h'_{\alpha,\beta}\, S\right)(t)$ and $t^{2\beta-1}\left(h'_{\alpha,\beta}\, T\right)(t)$ lie in O, to every $t \in \mathbb{N}$ there correspond $r_l \in \mathbb{N}$, $M_l > 0$ satisfying

$$\left|(t^{-1}\,D)^l\, t^{2\beta-1}\, \left(h'_{\alpha,\beta}\, S\right)(t)\right| \leq M_l\,(1+t^2)^{r_l}, \quad t \in I, \tag{3.1}$$

and $s_l \in \mathbb{N}$, $N_l > 0$ satisfying

$$\left|(t^{-1}\,D)^l\, t^{2\beta-1}\, \left(h'_{\alpha,\beta}\, T\right)(t)\right| \leq N_l\,(1+t^2)^{s_l}, \quad t \in I. \tag{3.2}$$

Moreover, as $h'_{\alpha,\beta}\, T \notin \mathcal{H}_{\alpha,\beta}$, there are $l_0,\ n_0 \in \mathbb{N}$ and a sequence $\left\{t_j\right\}_{j\in\mathbb{N}}$ in I, such that $t_j \to \infty$ as $j \to \infty$ and

$$\left|(t^{-1}\,D)^{l_0}\, t^{2\beta-1}\, \left(h'_{\alpha,\beta}\, T\right)(t)|_{t=t_j}\right| \geq (1+t_j^2)^{-n_0}, \quad j \in \mathbb{N}. \tag{3.3}$$

Set $k = s_{l_0+1} + n_0 + 2$, and define

$$B_{j,k} = \left\{t \in I : \left|t - t_j\right| \leq (1+t_j^2)^{-k}\right\}, \quad j \in \mathbb{N}. \tag{3.4}$$

From (3.2) and (3.3) we can infer that, for sufficiently large j,

$$\inf_{t\in B_{j,k}} \left|(t^{-1}\,D)^{l_0}\, t^{2\beta-1}\, \left(h'_{\alpha,\beta}\, T\right)(t)\right| \geq \frac{1}{2}(1+t_j^2)^{-n_0} > 0. \tag{3.5}$$

Indeed, if j is large enough and if $t \in B_{j,k}$, then

$$\begin{aligned}
&\left|(t^{-1}\,D)^{l_0}\, t^{2\beta-1}\, \left(h'_{\alpha,\beta}\, T\right)(t)\right| \\
&\geq \left|(y^{-1}\,D)^{l_0}\, y^{2\beta-1}\, \left(h'_{\alpha,\beta}\, T\right)(y)|_{y=t_j}\right| - \left(t_j + (1+t_j^2)^{-k}\right)(1+t_j^2)^{-k} \sup_{y\in B_{j,k}} \left|(y^{-1}\,D)^{l_0+1}\, y^{2\beta-1}\, \left(h'_{\alpha,\beta}\, T\right)(y)\right| \\
&\geq (1+t_j^2)^{-n_0} - C\,(1+t_j^2)^{s_{l_0+1}-k+1} \\
&= (1+t_j^2)^{-n_0} - C\,(1+t_j^2)^{-n_0-1},
\end{aligned}$$

where $C > 0$ is a constant independent from j.This proves (3.5).

Now $t^{2\beta-1}\left(h'_{\alpha,\beta}\, S\right)(t)\left(h'_{\alpha,\beta}\, T\right)(t) \in \mathcal{H}_{\alpha,\beta}$, and therefore

$$\sup_{t\in B_{j,k}} \left|(t^{-1}\,D)^l\, t^{4\beta-2}\, \left(h'_{\alpha,\beta}\, S\right)(t)\, \left(h'_{\alpha,\beta}\, T\right)(t)\right| = O\left((1+t_j^2)^{-n}\right), \quad l, n \in \mathbb{N},\ j \to \infty. \tag{3.6}$$

Clearly, For fixed $l, n \in \mathbb{N}$ we may write

$$\sup_{t \in B_{j,k}} \left| (t^{-1} D)^l \, t^{4\beta-2} \left(h'_{\alpha,\beta} S \right)(t) \left(h'_{\alpha,\beta} T \right)(t) \right|$$

$$= \sup_{|t| \le (1+t_j^2)^{-k}} \left| (y^{-1} D)^l \, y^{4\beta-2} \left(h'_{\alpha,\beta} S \right)(y) \left(h'_{\alpha,\beta} T \right)(y)|_{y=t+t_j} \right|$$

$$\le C_{n,l} \sup_{|t| \le (1+t_j^2)^{-k}} \left| (1 + (t+t_j)^2) \right|^{-n} \le C_{n,l} \left| (1 + t_j^2 - (1+t_j^2)^{-k}) \right|^{-n},$$

where $C_{n,l} > 0$ is a constant, and the right hand side of this inequality is clearly $O\left((1+t_j^2)^{-n}\right)$ as $j \to \infty$.

Next we prove that

$$\max_{0 \le l \le m} \sup_{t \in B_{j,k}} \left| (t^{-1} D)^l \, t^{2\beta-1} \left(h'_{\alpha,\beta} S \right)(t) \right| = O\left((1+t_j^2)^{-n}\right), \quad m, n \in \mathbb{N},\ j \to \infty, \tag{3.7}$$

a contradiction to (i). In the sequal, n will denote an arbitrary positive integer.

We first assume that $l_0 = 0$ and proceed by induction on m.In view of (3.5) and (3.6), we have

$$\sup_{t \in B_{j,k}} \left| t^{2\beta-1} \left(h'_{\alpha,\beta} S \right)(t) \right| \le 2(1+t_j^2)^{-n_0} \sup_{t \in B_{j,k}} \left| t^{4\beta-2} \left(h'_{\alpha,\beta} S \right)(t) \left(h'_{\alpha,\beta} T \right)(t) \right|$$

$$= O\left((1+t_j^2)^{-n}\right), \quad j \to \infty.$$

Thus, condition (3.7) is satisfied for $m = 0$.

Now suppose that (3.7) holds for some m. We must prove that it also holds for $m+1$.

By Leibnitz's rule,

$$t^{2\beta-1} \left(h'_{\alpha,\beta} T \right)(t) \, (t^{-1} D)^{m+1} \, t^{2\beta-1} \left(h'_{\alpha,\beta} S \right)(t)$$

$$= \sum_{i=0}^{m+1} (-1)^i \binom{m+1}{i} (t^{-1} D)^{m+1-i} \, t^{2\beta-1} \left(h'_{\alpha,\beta} S \right)(t) \, (t^{-1} D)^i \, t^{2\beta-1} \left(h'_{\alpha,\beta} T \right)(t), \quad t \in I.$$

Bearing in mind (3.2), (3.6) and the induction hypothesis, we find that

$$\sup_{t \in B_{j,k}} \left| (t^{-1} D)^{m+1-i} \, t^{2\beta-1} \left(h'_{\alpha,\beta} S \right)(t) \, (t^{-1} D)^i \, t^{2\beta-1} \left(h'_{\alpha,\beta} T \right)(t) \right| = O\left((1+t_j^2)^{-n}\right),$$

as $j \to \infty$, whenever $0 \le i \le m+1$. Consequently

$$t^{2\beta-1} \left(h'_{\alpha,\beta} T \right)(t) \, (t^{-1} D)^{m+1} \, t^{2\beta-1} \left(h'_{\alpha,\beta} S \right)(t)$$

satisfies this very estimate, and from (3.5) we conclude

$$\sup_{t \in B_{j,k}} \left| (t^{-1} D)^{m+1} \, t^{2\beta-1} \left(h'_{\alpha,\beta} S \right)(t) \right|$$

$$\le 2\,(1+t_j^2)^{-n_0} \sup_{t \in B_{j,k}} \left| t^{2\beta-1} \left(h'_{\alpha,\beta} T \right)(t) \, (t^{-1} D)^{m-1} \, t^{2\beta-1} \left(h'_{\alpha,\beta} S \right)(t) \right|$$

$$= O\left((1+t_j^2)^{-n}\right), \quad j \to \infty.$$

This shows that (3.7) holds when $l_0 = 0$ Next, assume that $l_0 \neq 0$ and l_0 is the smallest positive integer for which $n_0 \in \mathbb{N}$ and a sequence $\left\{ t_j \right\}_{j \in \mathbb{N}}$ in I may be found so that (3.3) (and hence (3.5), with large enough j) is satisfied.

This means that

$$(t^{-1} D)^l \, t^{2\beta-1} \left(h'_{\alpha,\beta} T \right)(t) = O\left((1+t_j^2)^{-n}\right),\ l < l_0,\ t \to \infty.$$

Arguing as in the proof of (3.6) we are led to

$$\sup_{t \in B_{j,k}} \left| (t^{-1} D)^l \, t^{2\beta-1} \left(h'_{\alpha,\beta} T \right)(t) \right| = O\left((1+t_j^2)^{-n}\right),\ l < l_0,\ j \to \infty. \tag{3.8}$$

By virtue of Leibnitz's rule,

$$t^{2\beta-1} \left(h'_{\alpha,\beta} S \right)(t) \, (t^{-1} D)^{l_0} \, t^{2\beta-1} \left(h'_{\alpha,\beta} T \right)(t)$$

$$= \sum_{l=0}^{l_0} (-1)^l \binom{l_0}{l} (t^{-1} D)^{l_0-l} \left(t^{2\beta-1} \left(h'_{\alpha,\beta} T \right) (t^{-1} D)^l \, t^{2\beta-1} \left(h'_{\alpha,\beta} S \right) \right)(t), \quad t \in I.$$

Then, from (3.1), (3.6) and (3.8) it follows that

$$\sup_{t\in B_{j,k}} \left| t^{2\beta-1} \left(h'_{\alpha,\beta} S\right)(t)\,(t^{-1} D)^{l_0}\, t^{2\beta-1} \left(h'_{\alpha,\beta} T\right)(t) \right| = O\left((1+t_j^2)^{-n}\right),\ j\to\infty. \tag{3.9}$$

Finally, using (3.5), (3.6) and (3.9) we obtain (3.7) by an argument similar to that employed in the case $l_0 = 0$. This completes the proof that (i) implies (ii).

Conversely, suppose that (i) does not hold. Then there exist $k\in\mathbb{N}$ and a sequence $\left\{t_j\right\}_{j\in\mathbb{N}}$ in I, with $t_j\to\infty$ as $j\to\infty$ such that

$$\max_{0\le l\le j}\ \sup_{t\in B_{j,k}} \left|(t^{-1} D)^l\, t^{2\beta-1} \left(h'_{\alpha,\beta} S\right)(t)\right| < (1+t_j^2)^{-j},\ j\in\mathbb{N}; \tag{3.10}$$

where the sets $B_{j,k}$ are given by (3.4). There is no loss of generality in assuming that $t_0 > 1$ and $t_{j+1} > t_j + 1$. Let $a\in\mathcal{D}(I)$ be such that $0\le a\le 1$, $\operatorname{supp} a = [1/2,\ 3/2]$ and $a(1) = 1$, and set

$$\theta_j(t) = a\left(1+\frac{1}{2}(t-t_j)\,(1+t_j^2)^k\right),\ \theta(t) = \sum_{j=0}^{\infty}\theta_j(t),\quad t\in I.$$

The sum defining θ is finite, because $\operatorname{supp}\theta_j = B_{j,k}(j\in\mathbb{N})$ and $B_{i,k}\cap B_{j,k} = \phi$ $(i, j\in\mathbb{N},\ i\ne j)$. If $l, j\in\mathbb{N}$ and $t\in B_{j,k}$ then for some $a_m\in\mathbb{R}$ $(0\le m\le l)$, we have

$$\begin{aligned}
\left|(t^{-1} D)^l\,\theta(t)\right| &= \left|(t^{-1} D)^l\,\theta_j(t)\right| \\
&= \sum_{m=0}^{l}\left|a_m\, t^{-l-m}\, D^m\,\theta_j(t)\right| \\
&\le 2^{l+m}\sum_{m=0}^{l}\left|a_m\, D^m\,\theta_j(t)\right| \\
&\le C_l\, 2^{-kl}\,(1+t_j^2)^{kl}\sum_{m=0}^{l}\left|D^m\,\theta_j(y)|_{y=1+\frac{1}{2}(t-t_j)\,(1+t_j^2)^k}\right| \\
&\le C_l\,(1+t_j^2)^{kl}\le C_l\,(1+t^2)^{kl},
\end{aligned}$$

where $C_l > 0$ denotes an appropriate constant (not necessarily the same in each occurrence). Then

$$\left|(t^{-1} D)^l\,\theta(t)\right|\le C_l\,(1+t^2)^{kl},\quad t\in I, \tag{3.11}$$

thus proving that $\theta\in O$. Thus, there exist $T\in O'_{\alpha,\beta,*}$ such that $\left(h'_{\alpha,\beta}\right)(t) = t^{2\alpha}\,\theta(t),\ t\in I.$

Let $n, l\in\mathbb{N}$. Thus function

$$(1+t^2)^n\,(t^{-1} D)^l\, t^{4\beta-2}\left(h'_{\alpha,\beta} S\right)(t)\left(h'_{\alpha,\beta} T\right)(t),\quad t\in I$$

is bounded on the interval $0 < t < t_{n+kl} - (1+t_{n+kl}^2)^{-k}$. Letting $j = n + kl + r$ $(r\in\mathbb{N})$ and $t\in B_{j,k}$, Leibnitz rule, along with (3.10) and (3.11), implies

$$\begin{aligned}
&\left|(1+t^2)^n\,(t^{-1} D)^l\, t^{4\beta-2}\left(h'_{\alpha,\beta} S\right)(t)\left(h'_{\alpha,\beta} T\right)(t)\right| \\
&= \left|(1+t^2)^n\,(t^{-1} D)^l\, t^{2\beta-1}\left(h'_{\alpha,\beta} S\right)(t)\,\theta(t)\right| \\
&\le C\,(1+t^2)^{n+kl}\,(1+t_j^2)^{-n-kl}\le C,
\end{aligned}$$

where $C > 0$ is a suitable constant (concerning the value of C, we make the same convolution as before). This shows that

$$t^{2\beta-1}\left(h'_{\alpha,\beta} S\right)(t)\left(h'_{\alpha,\beta} T\right)(t)\in\mathcal{H}_{\alpha,\beta}.\qquad \text{But } h'_{\alpha,\beta} T\in\mathcal{H}_{\alpha,\beta},$$

Since

$$t_j^{2\beta-1}\left(h'_{\alpha,\beta} T\right)(t_j) = a(1) = 1;\quad as\ t_j\to\infty\ (j\to\infty).$$

We conclude that $T\in O'_{\alpha,\beta,*}$ and that $S * T\in\mathcal{H}_{\alpha,\beta}$ although $T\notin\mathcal{H}_{\alpha,\beta}$ which contradict (ii) and completes the proof.

4. Conclusion

In the present research article, we have accomplished two major objectives regarding Hankel type convolution operator. Firstly, $O'_{\alpha,\beta,*}$ elements are defined as those in $\mathcal{L}\left(\mathcal{H}_{\alpha,\beta}\right)$ and in $\mathcal{L}\left(\mathcal{H}'_{\alpha,\beta}\right)$ that commute with Hankel translations. Furthermore, we obtain certain results that aid in proving the objective.

Secondly, To ensure that every $T \in O'_{\alpha,\beta,*}$ such that $S * T \in \mathcal{H}_{\alpha,\beta}$ lie in $\mathcal{H}_{\alpha,\beta}$, necessary and sufficient conditions on the generalised Hankel type transform $h'_{\alpha,\beta}S$ of $S \in O'_{\alpha,\beta,*}$ are established.

The findings of this research may be significant for areas in engineering, physics and mathematics.

References

[1] M. Belhadj and J. J. Betancor, Hankel convolution operators on entire functions and distributions, *J. Math. Anal. Appl.*, **276**(1) (2002), 40-63.

[2] J. J. Betancor and I. Marrero, Structure and convergence in certain spaces of distributions and the generalized Hankel convolution, *Math. Japon.,* **38**(6) (1993), 1141-1155.

[3] J. L. Kelley, *General Topology*, D.van Nostrand, Princeton, New Jersey, 1968.

[4] I. Marrero and J. J. Betancor, Hankel Convolution of generalized functions, *Rendiconti di Matematica e delle sue Applicazioni. Serie VII,* **15**(3-4) (1995), 351.

[5] S. Sznajder and Z. Zielezny, On some properties of convolution operators in $\mathcal{K}'_1$ and S', *J. Math. Anal. Appl.,* **65** (1978), 543-554.

[6] A. H. Zemanian, *Generalized Integral Transformations*, Interscience,New York, 1968.