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k-TRIANGULAR PRIME CORDIAL LABELING OF MAXIMAL OUTERPLANAR GRAPHS

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Abstract

In this paper, we study graph labeling, namely, k- triangular prime cordial labeling for $k = 1, 2, 3, 4, 5, 6$. This is a simple extension of prime cordial labeling where the vertex labels are defined as the higher order triangular numbers. Also we show that the maximal outerplanar graphs are k- triangular prime cordial under certain conditions.

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1. Introduction

A labeling of a graph G is a mapping that carries a set of graph elements, usually the vertices and edges into a set of numbers, usually real numbers or subsets of a set. For detailed study on different types of labelings we refer to [2,5,16,18].

Rosa [15] introduced a labeling of G called β-valuation, later on Golomb [4] called as "graceful labeling" which is an injection f from the set of vertices $V(G)$ to the set $\{0, 1, 2, ..., q\}$ such that when each edge $e = uv$ is assigned the label $|f(u) - f(v)|$, the resulting edge labels are distinct. A graph which admits a graceful labeling is called a graceful graph.

In this paper, for a graph $G = (V, E)$ we introduce the k-triangular prime cordial labeling for $k = 1, 2, 3, 4, 5, 6$ and study on maximal outerplanar graph structure. We consider only finite simple undirected graphs. The set of vertices and edges of a graph G will be denoted by $V(G)$ and $E(G)$ respectively, where $|V(G)| = p$ and $|E(G)| = q$. For graph theoretic notations, we follow Bondy and Murthy [1].

2. Triangular labelings of graphs

In this section first we discuss the triangular numbers and related labelings of graphs.

For any integer k, the k-th order triangular number is a number obtained by adding all the k-th powers of positive integers less than or equal to a given positive integer n. That is, the n-th term of k-th order triangular number is $1^k + 2^k + ... + n^k$, and is denoted by T_n^k.

A triangular graceful labeling of a graph G with q edges is an injection map t from the set of vertices $V(G)$ to the set $\{0, 1, 2, ..., T_q^1\}$ such that when each edge $e = uv$ is assigned the label $|t(u) - t(v)|$, the resulting edge labels are a sequence of distinct consecutive triangular numbers say $\{T_1^1, T_2^1, ..., T_q^1\}$. Here T_q^1 is the q-th triangular number of the triangular series $T_1^1 = 1, T_2^1 = 3, T_3^1 = 6, ..., T_n^1 = \frac{1}{2}n(n + 1)$. A graph which admits a triangular graceful labeling is called a triangular graceful graph.

Hegde and Shankaran [6] introduced a labeling of G called triangular sum labeling. This labeling is an injection f from the set of vertices $V(G)$ to the set of non-negative integers such that when each edge $e = uv$ is assigned the label $f(u) + f(v)$, the resulting edge labels are a sequence of distinct consecutive triangular numbers say $\{T_1^1, T_2^1, ..., T_q^1\}$. A graph which admits a triangular sum labeling is called a triangular sum graph.

Murugesan et al. [10] introduced centered triangular sum labeling of graphs. This labeling is an injection f from the set of vertices $V(G)$ to the set of non-negative integers such that when each edge $e = uv$ is assigned the label $f(u) + f(v)$, the resulting edge labels are a sequence of distinct consecutive centered triangular numbers say $\{C_1^2, C_2^2, ..., C_q^2\}$. Here C_i^2 is the i-th centered triangular number of the centered triangular series $C_1^2 = 1, C_2^2 = 4, C_3^2 =$

$10, ..., C_n^2 = \frac{1}{2}(3n^2 - 3n + 2)$. A graph which admits a centered triangular sum labeling is called a centered triangular sum graph.

Murugesan et al. [11] introduced second order triangular sum labeling of graphs. This labeling is an injection f from the set of vertices $V(G)$ to the set of non-negative integers such that when each edge $e = uv$ is assigned the label $f(u) + f(v)$, the resulting edge labels are a sequence of distinct consecutive second order triangular numbers say $\{T_1^2, T_2^2, ..., T_q^2\}$. Here T_i^2 is the i-th second order triangular number of the second order triangular series $T_1^2 = 1, T_2^2 = 5, T_3^2 = 14, ..., T_n^2 = \frac{1}{6}n(n+1)(2n+1)$. A graph which admits a second order triangular sum labeling is called a second order triangular sum graph. They also introduced third order triangular sum labeling which is an injection f from the set of vertices $V(G)$ to the set of non-negative integers such that when each edge $e = uv$ is assigned the label $f(u)+f(v)$, the resulting edge labels are a sequence of distinct consecutive third order triangular numbers say $\{T_1^3, T_2^3, ..., T_q^3\}$. Here T_i^3 is the i-th third order triangular number of the third order triangular series $T_1^3 = 1, T_2^3 = 9, T_3^3 = 36, ..., T_n^3 = \frac{1}{4}n^2(n+1)^2$. A graph which admits a third order triangular sum labeling is called a third order triangular sum graph.

Again, Murugesan et al. [12] introduced fourth order triangular sum labeling of graphs. This labeling is an injection f from the set of vertices $V(G)$ to the set of nonnegative integers such that when each edge $e = uv$ is assigned the label $f(u) + f(v)$, the resulting edge labels are a sequence of distinct consecutive fourth order triangular numbers say $\{T_1^4, T_2^4, ..., T_q^4\}$. Here T_i^4 is the i-th second order triangular number of the second order triangular series $T_1^4 = 1$, $T_2^4 = 17, T_3^4 = 98, ..., T_n^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$. A graph which admits a fourth order triangular sum labeling is called a fourth order triangular sum graph. They also introduced fifth order triangular sum labeling which is an injection f from the set of vertices $V(G)$ to the set of non-negative integers such that when each edge $e = uv$ is assigned the label $f(u) + f(v)$, the resulting edge labels are a sequence of distinct consecutive fifth order triangular numbers say $\{T_1^5, T_2^5, ..., T_q^5\}$. Here T_i^5 is the i-th fifth order triangular number of the fifth order triangular series $T_1^5 = 1, T_2^5 = 33, T_3^5 = 276, ..., T_n^5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$. A graph which admits a fifth order triangular sum labeling is called a fifth order triangular sum graph. Similarly, they defined a sixth order triangular sum labeling. This labeling is an injection f from the set of vertices $V(G)$ to the set of non-negative integers such that when each edge $e = uv$ is assigned the label $f(u)+f(v)$, the resulting edge labels are a sequence of distinct consecutive sixth order triangular numbers say $\{T_1^6, T_2^6, ..., T_q^6\}$. Here T_i^6 is the i-th sixth order triangular number of the sixth order triangular series $T_1^6 = 1, T_2^6 = 65, T_3^6 = 794, ..., T_n^6 = \frac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1)$. A graph which admits a sixth order triangular sum labeling is called a sixth order triangular sum graph.

3. Cordial Labelings of Graphs

A cordial labeling is a map f from the set of vertices V(G) to the set $\{0, 1\}$ such that when each edge $e = uv$ is assigned the label $|f(u) - f(v)|$ and satisfies the condition that the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. A graph which admits a cordial labeling is called a cordial graph. For various types of cordial labelings we refer to [13,14,17].

3.1. Prime cordial labelings of graphs

A prime cordial labeling is a map f from the set of vertices V(G) to the set $\{1, 2, ..., p\}$ such that each edge $e = uv$ is assigned the label 1 if $gcd(f(u), f(v)) = 1$ and 0 if $gcd(f(u), f(v)) > 1$ and satisfies the condition that the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. A graph which admits a prime cordial labeling is called a prime cordial graph.

4. Maximal outerplanar graph

A planar graph G is outer-planar if and only if there is an embedding of G on the plane in which every vertex lies on the exterior face. If we consider a planar graph with no loops or faces bounded by two edges (digons), it may be possible to add a new edge to the presentation of G such that these properties are preserved. When no such adjunction can be made, the graph is called a maximal outerplanar graph since any additional edge will destroy its outerplanar property. A maximal outerplanar graph can be viewed as a triangulation of a convex polygon.

Chartrand and Harary [3] showed that a graph is outerplanar if and only if it does not contain K_4 or $K_{2,3}$ minor. Kumar and Madhavan [7] gave a characterization of maximal outerplanar graphs, in the context of planar chordal graphs. In this paper we give some facts about maximal outerplanar graphs in the following Lemma:

Lemma 4.1 ([8]). *Let G be a maximal outerplanar graph with $n \geq 3$ vertices. Then G has*

(a) $2n - 3$ *edges, of which there are* $n - 3$ *chords;*

(b) $n - 2$ *inner faces. Each inner face is a triangle;*

(c) at least two vertices with degree 2.

5. k- triangular prime cordial labeling of maximal outerplanar graphs of small order

In our study, we extend the prime cordial labeling into a new kind of labeling known as k- triangular prime cordial labeling for $k = 1, 2, 3, 4, 5, 6$. We introduce a k- triangular prime cordial graphs which are the graphs labeled with triangular number in which every vertex and its incident edges labeled with numbers 0 and 1 satisfy the prime cordial labeling. That is, a prime cordial labeling is k-triangular if $f(V(G)) \to \{T_1^k, T_2^k, ..., T_p^k\}, k = 1, 2, 3, 4, 5, 6$.

Let $G = (V, E)$ be a graph with p vertices and q edges. For any integer $k = 1, 2, 3, 4, 5, 6$, we define a one-to-one map f from the set of vertices $V(G)$ to $\{T_1^k, T_2^k, ...T_p^k\}$ where T_i^k is the i-th term of k-th order triangular number. For each edge $e = uv$ we assign the label 1 if $gcd\,(f(u), f(v)) = 1$ and 0 if $gcd\,(f(u), f(v)) > 1$. If $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ and $e_f(1)$ respectively denote the number of edges labeled with 0 and the number of edges labeled with 1. Then G is said to be k- triangular prime cordial (k-*TPC*) and such a labeling f is called k- triangular prime cordial labeling of G.

Here, we note that 1- triangular prime cordial labeling is simply a triangular prime cordial labeling.

Definition 5.1. *Let $G = (V, E)$ be a (p, q)-graph. For each $k = 1, 2, 3, 4, 5, 6$, let $f : V(G) \to \{T_1^k, T_2^k, ..., T_p^k\}$ be an injective map where T_i^k is the i-th term of k-th order triangular number. For each edge uv assign the label 1 or 0 according as $gcd\,(f(u), f(v)) = 1$ or $gcd\,(f(u), f(v)) > 1$. Then f is called a k-triangular prime cordial labeling of G if $|e_f(0) - e_f(1)| \leq 1$ where $e_f(0)$ and $e_f(1)$ respectively denote the number of edges labeled with 0 and number of edges labeled with 1. A graph with k-triangular prime cordial labeling is called a k-triangular prime cordial.*

Example 5.1. A Peterson graph is k-TPC for $k = 1, 2, 3, 4, 5, 6$.

Theorem 5.1. *The maximal outerplanar graph of order 4 is k-TPC for* $k = 1, 2, 3, 5, 6$.

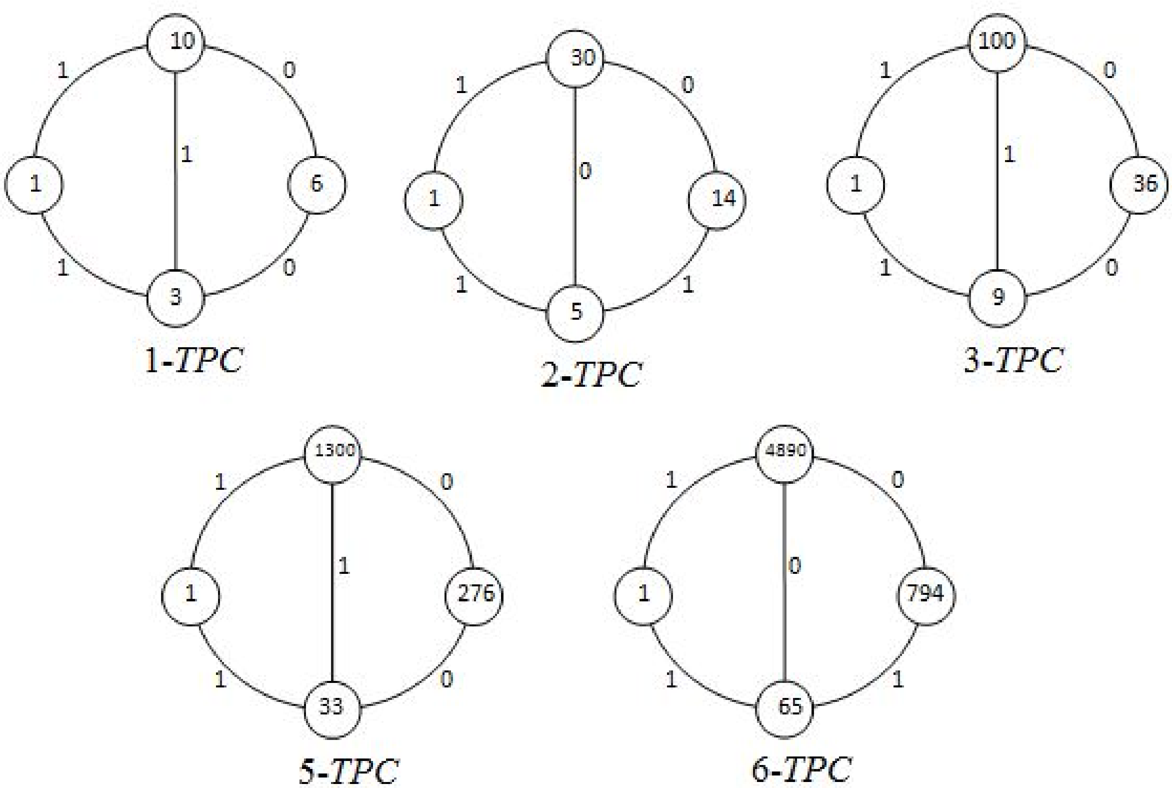

Figure 5.1: The maximal outerplanar graph of order 4 is k-*TPC* for $k = 1, 2, 3, 5, 6$.

Proof. Let G be a maximal outerplanar graph of order 4. By Lemma 4.1, we know that G has 5 edges, two vertices of degree 2 and two vertices of degree 3. Suppose $k = 1$, then assign the labels T_1^1 and T_3^1 to any vertices of degree 2 of G and then assign the labels T_2^1 and T_4^1 to any vertices of degree 3 of G. Figure 1 shows that G is 1-*TPC*. Thus, the theorem holds for $k = 1$. Suppose $k = 2, 3, 5$ *or* 6. By similar arguments the graphs of order 4 as shown in Figure 1 with their respective *k-TPC* labeling for $k = 2, 3, 5$ *or* 6. Thus the theorem holds.

Lemma 5.1. *A maximal outerplanar (MOP) graph of order 4 is not 4-TPC.*

Proof. Suppose that *MOP* graph G of order 4 is 4-*TPC*. Then $|e_f(0) - e_f(1)| \leq 1$. Therefore, $e_f(0) = 2$ and $e_f(1) = 3$ (or) $e_f(0) = 3$ and $e_f(1) = 2$ because G has 5 edges. Without loss of generality, we may assume that $e_f(0) = 2$. That means, number of edges labeled with 0 is 2. Since $T_1^4 = 1$, $T_2^4 = 17$, $T_3^4 = 98$ and $T_4^4 = 354$ are first four 4^{th} order triangular numbers, only one pair of gcd of T_3^4 and T_4^4 is not equal to one, otherwise, is equal one. Therefore, at most one edge label is 0. That is, $e_f(0) < 2$, which is a contradiction. Hence G is not 4-*TPC*.

Theorem 5.2. *The MOP graph of order 5 is k-TPC for* $k = 1, 2, 3, 5, 6$.

Proof. Let G be a *MOP* graph of order 5. Then G has outer cycle $v_1v_2v_3v_4v_5v_1$ with two chords v_1v_4 and v_2v_4. Suppose $k = 1$. We assign the labels $T_1^1, T_2^1, T_3^1, T_4^1, T_5^1$ to the consecutive vertices v_1, v_2, v_3, v_4, v_5 respectively. Figure 2 shows that G is 1-*TPC*. Thus, the theorem holds for $k = 1$. Suppose $k = 2, 3, 5$ *or* 6. By similar arguments the graphs of order 5 as shown in Figure 2 with their respective *k-TPC* labeling for $k = 2, 3, 5$ *or* 6. Thus the theorem holds.

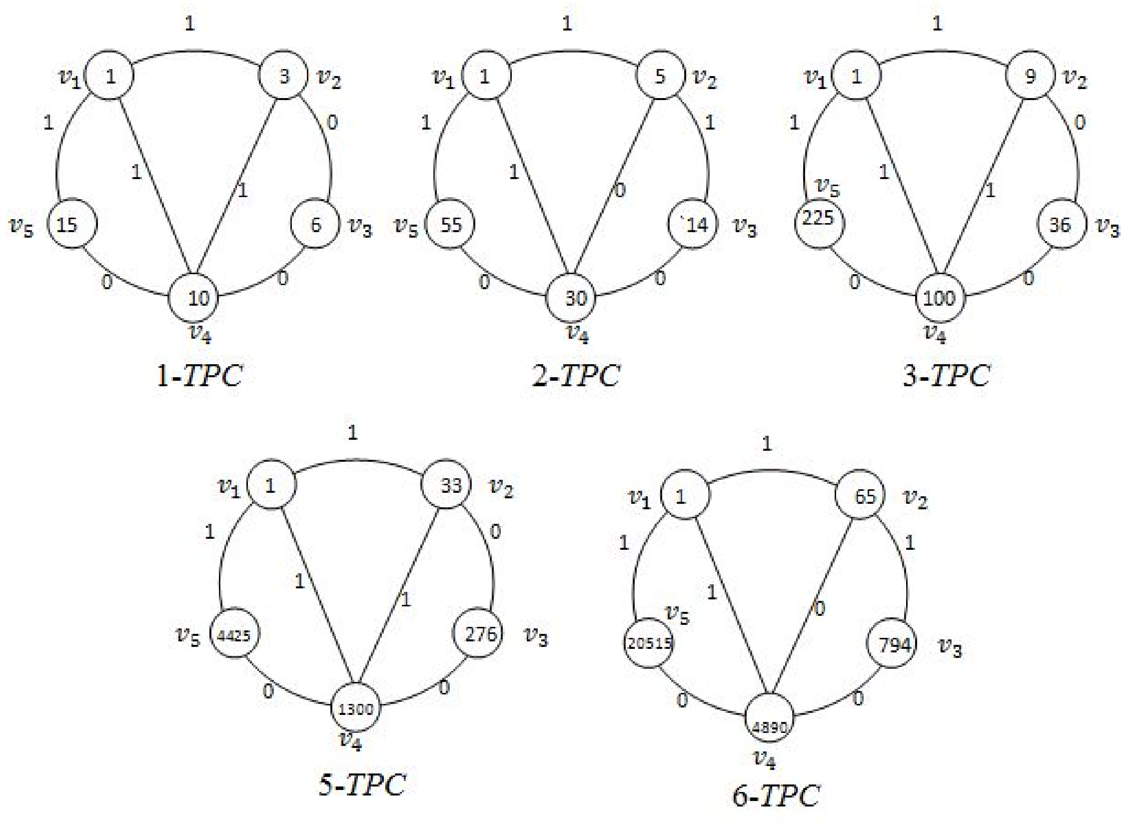

***Figure* 5.2:** The MOP graph of order 5 is *k-TPC* for $k = 1, 2, 3, 5, 6$.

Lemma 5.2. *A MOP graph of order 5 is not 4-TPC.*

Theorem 5.3. *For* $k = 1, 2, 3, 5, 6$*, all MOP graphs of order 6 are k-TPC.*

Proof. Up to isomorphism there are three maximal outerplanar graphs of order 6 (See Figure 3). By definition 5.1 and Tables 1-3, it is easy to see that they are *k-TPC* for $k = 1, 2, 3, 5, 6$.

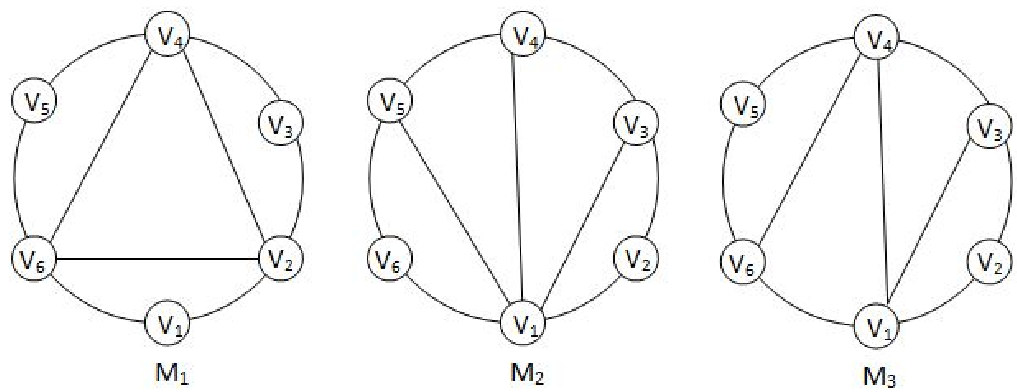

***Figure* 5.3:** Three non-isomorphic *MOP* graphs of order 6

***Table* 5.1:** Table for Theorem 5.3, M_1 is *k-TPC*, $k = 1, 2, 3, 5, 6$

$G = M_1$	**Vertex labeling**	**Edge condition**
$k = 1$	$f(v_i) = T_i^1, 1 \leq i \leq 6$	$e_f(0) = e_f(1) + 1$
$k = 2$	$f(v_i) = T_i^2, 1 \leq i \leq 4$ $f(v_5) = T_6^2$ $f(v_6) = T_5^2$	$e_f(1) = e_f(0) + 1$
$k = 3$	$f(v_i) = T_i^3, 1 \leq i \leq 6$	$e_f(0) = e_f(1) + 1$
$k = 5$	$f(v_i) = T_i^5, 1 \leq i \leq 6$	$e_f(0) = e_f(1) + 1$
$k = 6$	$f(v_i) = T_i^6, 1 \leq i \leq 6$	$e_f(1) = e_f(0) + 1$

We know that $T_1^1, T_2^1, T_3^1, T_4^1, T_5^1, T_6^1 : 1, 3, 6, 10, 15, 21,$
$T_1^2, T_2^2, T_3^2, T_4^2, T_5^2, T_6^2 : 1, 5, 14, 30, 55, 91,$
$T_1^3, T_2^3, T_3^3, T_4^3, T_5^3, T_6^3 : 1, 9, 36, 100, 225, 441,$
$T_1^4, T_2^4, T_3^4, T_4^4, T_5^4, T_6^4 : 1, 17, 98, 354, 979, 2275,$
$T_1^5, T_2^5, T_3^5, T_4^5, T_5^5, T_6^5 : 1, 33, 276, 1300, 4425, 12201,$
$T_1^6, T_2^6, T_3^6, T_4^6, T_5^6, T_6^6 : 1, 65, 794, 4890, 20515, 67171.$

***Table* 5.2:** Table for Theorem 5.3, M_2 is *k-TPC*, $k = 1, 2, 3, 5, 6$

$G = M_2$	**Vertex labeling**	**Edge condition**
$k = 1$	$f(v_i) = T_i^1, 1 \leq i \leq 6$	$e_f(1) = e_f(0) + 1$
$k = 2$	$f(v_i) = T_{i+1}^2, 1 \leq i \leq 5$ $f(v_6) = T_1^2$	$e_f(1) = e_f(0) + 1$
$k = 3$	$f(v_i) = T_i^3, 1 \leq i \leq 6$	$e_f(1) = e_f(0) + 1$
$k = 5$	$f(v_i) = T_i^5, 1 \leq i \leq 6$	$e_f(1) = e_f(0) + 1$
$k = 6$	$f(v_i) = T_{i+1}^6, 1 \leq i \leq 4$ $f(v_5) = T_1^6$ $f(v_6) = T_6^6$	$e_f(0) = e_f(1) + 1$

We know that $T_1^1, T_2^1, T_3^1, T_4^1, T_5^1, T_6^1 : 1, 3, 6, 10, 15, 21,$
$T_1^2, T_2^2, T_3^2, T_4^2, T_5^2, T_6^2 : 1, 5, 14, 30, 55, 91,$
$T_1^3, T_2^3, T_3^3, T_4^3, T_5^3, T_6^3 : 1, 9, 36, 100, 225, 441,$
$T_1^4, T_2^4, T_3^4, T_4^4, T_5^4, T_6^4 : 1, 17, 98, 354, 979, 2275,$
$T_1^5, T_2^5, T_3^5, T_4^5, T_5^5, T_6^5 : 1, 33, 276, 1300, 4425, 12201,$
$T_1^6, T_2^6, T_3^6, T_4^6, T_5^6, T_6^6 : 1, 65, 794, 4890, 20515, 67171.$

***Table* 5.3:** Table for Theorem 5.3, M_3 is *k-TPC*, $k = 1, 2, 3, 5, 6$

$G = M_3$	**Vertex labeling**	**Edge condition**
$k = 1$	$f(v_i) = T_i^1, 1 \le i \le 6$	$e_f(1) = e_f(0) + 1$
$k = 2$	$f(v_i) = T_{i+1}^2, 1 \le i \le 5$ $f(v_6) = T_1^2$	$e_f(1) = e_f(0) + 1$
$k = 3$	$f(v_i) = T_i^3, 1 \le i \le 4$ $f(v_5) = T_6^3$ $f(v_6) = T_5^3$	$e_f(1) = e_f(0) + 1$
$k = 5$	$f(v_i) = T_i^5, 1 \le i \le 6$	$e_f(1) = e_f(0) + 1$
$k = 6$	$f(v_i) = T_{i+1}^6, 1 \le i \le 5$ $f(v_6) = T_1^6$	$e_f(1) = e_f(0) + 1$

We know that $T_1^1, T_2^1, T_3^1, T_4^1, T_5^1, T_6^1 : 1, 3, 6, 10, 15, 21,$
$T_1^2, T_2^2, T_3^2, T_4^2, T_5^2, T_6^2 : 1, 5, 14, 30, 55, 91,$
$T_1^3, T_2^3, T_3^3, T_4^3, T_5^3, T_6^3 : 1, 9, 36, 100, 225, 441,$
$T_1^4, T_2^4, T_3^4, T_4^4, T_5^4, T_6^4 : 1, 17, 98, 354, 979, 2275,$
$T_1^5, T_2^5, T_3^5, T_4^5, T_5^5, T_6^5 : 1, 33, 276, 1300, 4425, 12201,$
$T_1^6, T_2^6, T_3^6, T_4^6, T_5^6, T_6^6 : 1, 65, 794, 4890, 20515, 67171.$

Theorem 5.4. *All MOP graphs of order 6 are not 4-TPC.*

Proof. Suppose that *MOP* graph G of order 6 is 4-*TPC*. Then $|e_f(0) - e_f(1)| \le 1$. Therefore, $e_f(0) = 4$ and $e_f(1) = 5$ (or) $e_f(0) = 5$ and $e_f(1) = 4$ because G has 9 edges. Without loss of generality, we may assume that $e_f(0) = 4$. That means, number of edges labeled with 0 is 4. Since $T_1^4 = 1$, $T_2^4 = 17$, $T_3^4 = 98$, $T_4^4 = 354$, $T_5^4 = 979$ and $T_6^4 = 2275$ are first six 4^{th} order triangular numbers, only two pairs $gcd\left(T_3^4, T_4^4\right) = 0$ and $gcd\left(T_3^4, T_6^4\right) = 0$, otherwise, is equal one. Therefore, at most two edge label is 0. That is, $e_f(0) < 3$, which is a contradiction. Hence G is not 4-*TPC*.

Theorem 5.5. *All MOP graphs of order 7 are k-TPC for* $k = 1, 2, 3, 4, 5, 6$.

Proof. Up to isomorphism there are four maximal outerplanar graphs of order 7 (See Figure 4). By definition 5.1 and Tables 4-7, it is easy to see that they are k *-TPC* for $k = 1, 2, 3, 4, 5, 6$.

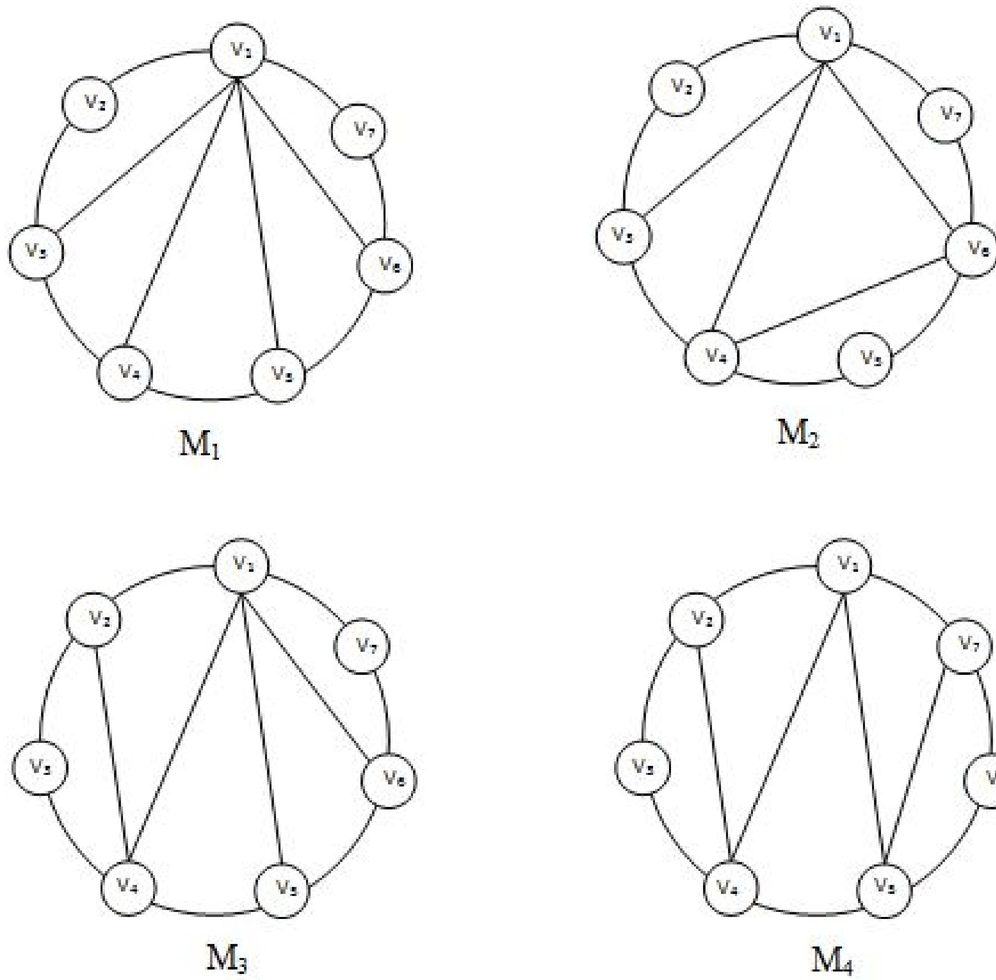

***Figure* 5.4:** Four non-isomorphic *MOP* graphs of order 7

***Table* 5.4:** Table for Theorem 5.5, M_1 is k-*TPC*, $k = 1, 2, 3, 4, 5, 6$

$G = M_1$	**Vertex labeling**	**Edge condition**
$k = 1$	$f(v_i) = T_i^1, 1 \leq i \leq 7$	$e_f(1) = e_f(0) + 1$
$k = 2$	$f(v_i) = T_{i+1}^2, 1 \leq i \leq 6$ $f(v_7) = T_1^2$	$e_f(0) = e_f(1) + 1$
$k = 3$	$f(v_i) = T_i^3, 1 \leq i \leq 7$	$e_f(1) = e_f(0) + 1$
$k = 4$	$f(v_i) = T_{i+2}^4, i = 1, 2, 4$ $f(v_3) = T_7^4$ $f(v_5) = T_5^4$ $f(v_j) = T_{j-5}^4, j = 6, 7$	$e_f(1) = e_f(0) + 1$
$k = 5$	$f(v_i) = T_i^5, 1 \leq i \leq 7$	$e_f(1) = e_f(0) + 1$
$k = 6$	$f(v_i) = T_{i+1}^6, 1 \leq i \leq 6$ $f(v_7) = T_1^6$	$e_f(0) = e_f(1) + 1$

Table **5.5:** Table for Theorem 5.5, M_2 is *k-TPC*, $k = 1, 2, 3, 4, 5, 6$

$G = M_2$	**Vertex labeling**	**Edge condition**
$k = 1$	$f(v_i) = T_i^1, 1 \le i \le 7$	$e_f(1) = e_f(0) + 1$
$k = 2$	$f(v_1) = T_7^2$ $f(v_i) = T_{i-1}^2, 2 \le i \le 7$	$e_f(0) = e_f(1) + 1$
$k = 3$	$f(v_i) = T_i^3, 1 \le i \le 7$	$e_f(1) = e_f(0) + 1$
$k = 4$	$f(v_i) = T_{i+2}^4, i = 1, 2, 4$ $f(v_3) = T_7^4$ $f(v_5) = T_5^4$ $f(v_j) = T_{j-5}^4, j = 6, 7$	$e_f(1) = e_f(0) + 1$
$k = 5$	$f(v_i) = T_i^5, 1 \le i \le 7$	$e_f(1) = e_f(0) + 1$
$k = 6$	$f(v_i) = T_{i+1}^6, 1 \le i \le 6$ $f(v_7) = T_1^6$	$e_f(0) = e_f(1) + 1$

Table **5.6:** Table for Theorem 5.5, M_3 is *k-TPC*, $k = 1, 2, 3, 4, 5, 6$

$G = M_3$	**Vertex labeling**	**Edge condition**
$k = 1$	$f(v_i) = T_i^1, 1 \le i \le 7$	$e_f(1) = e_f(0) + 1$
$k = 2$	$f(v_i) = T_{i+1}^2, 1 \le i \le 6$ $f(v_7) = T_1^2$	$e_f(1) = e_f(0) + 1$
$k = 3$	$f(v_i) = T_i^3, 1 \le i \le 7$	$e_f(1) = e_f(0) + 1$
$k = 4$	$f(v_i) = T_{i+2}^4, i = 1, 5$ $f(v_k) = T_{k-1}^4, k = 2, 3$ $f(v_j) = T_j^4, j = 4, 6$ $f(v_7) = T_5^4$	$e_f(1) = e_f(0) + 1$
$k = 5$	$f(v_i) = T_i^5, 1 \le i \le 7$	$e_f(1) = e_f(0) + 1$
$k = 6$	$f(v_i) = T_{i+1}^6, 1 \le i \le 6$ $f(v_7) = T_1^6$	$e_f(1) = e_f(0) + 1$

Table **5.7:** Table for Theorem 5.5, M_4 is *k-TPC*, $k = 1, 2, 3, 4, 5, 6$

$G = M_4$	**Vertex labeling**	**Edge condition**
$k = 1$	$f(v_i) = T_i^1, 1 \le i \le 7$	$e_f(1) = e_f(0) + 1$
$k = 2$	$f(v_i) = T_i^2, 1 \le i \le 7$	$e_f(1) = e_f(0) + 1$
$k = 3$	$f(v_i) = T_i^3, 1 \le i \le 7$	$e_f(1) = e_f(0) + 1$
$k = 4$	$f(v_i) = T_{i+2}^4, i = 1, 2, 3$ $f(v_j) = T_{j-5}^4, j = 6, 7$ $f(v_4) = T_7^4$ $f(v_5) = T_6^4$	$e_f(1) = e_f(0) + 1$
$k = 5$	$f(v_i) = T_i^5, 1 \le i \le 7$	$e_f(1) = e_f(0) + 1$
$k = 6$	$f(v_i) = T_{i+1}^6, 1 \le i \le 5$ $f(v_6) = T_1^6$ $f(v_7) = T_7^6$	$e_f(1) = e_f(0) + 1$

Theorem 5.6. *All MOP graphs of order* 8 *are k-TPC for* $k = 1, 2, 3, 4, 5, 6$.

Proof. According to Lee et al. [9], up to isomorphism there are 12 maximal outerplanar graphs of order 8. It is easy to verify that those graphs are $k - TPC$ for $k = 1, 2, 3, 4, 5, 6$.

Lee et al. [9] studied edge-graceful and edge-magic maximal outerplanar graphs. Here, we prove that number of maximal outerplanar graphs of order 8 is 12 (up to isomorphism). Also, we show that the above said graphs are k-Triangular prime cordial graphs. For this, we use the following theorems.

Theorem 5.7. *Let M_1 be a MOP graph of order 8 having outer cycle $v_1v_2v_3v_4v_5v_6v_7v_8v_1$ with 5 chords $v_1v_i, 3 \leq i \leq 7$. Then the graph M_1 admits a k-TPC labeling for $1 \leq k \leq 6$.*

Proof. Let M_1 be a *MOP* graph of order 8 having outer cycle $v_1v_2v_3v_4v_5v_6v_7v_8v_1$ with 5 chords v_1v_i, $3 \leq i \leq 7$. We claim that the graph M_1 is a *k-TPC* graph for $1 \leq k \leq 6$.

Case (1). If $k = 1$, we assign the first 8 triangular numbers $T_1^1, T_2^1, ..., T_8^1$ to consecutive vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$. It is easy to verify $e_f(1) = e_f(0) + 1$. Hence the graph M_1 is 1-*TPC*.

Case (2). If $k = 2$, we assign the labels $T_2^2, T_3^2, T_4^2, T_5^2, T_6^2, T_7^2, T_8^2, T_1^2$ to consecutive respectively, the vertices $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$. It is easy to verify $e_f(0) = e_f(1) + 1$. Hence the graph M_1 is 2-*TPC*.

Case (3). If $k = 3$, then the first 8 triangular numbers of order 3 are $T_1^3, T_2^3, ..., T_8^3$ to consecutive vertices$v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$. It is easy to verify $e_f(1) = e_f(0) + 1$. Thus, the graph M_1 admits 3-*TPC* labeling.

Case (4). If $k = 4$, then the vertex labels $T_3^4, T_4^4, T_5^4, T_6^4, T_7^4, T_8^4, T_2^4, T_1^4$ to consecutive vertices v_i, $1 \leq i \leq 8$. It is easy to verify $e_f(0) = e_f(1) + 1$. Therefore, the given graph M_1 is admitting a 4-*TPC* labeling.

Case (5). If $k = 5$, then we assign the first 8 triangular numbers of order $T_1^5, T_2^5, ..., T_8^5$ to consecutive vertices v_i, $1 \leq i \leq 8$. It is easy to verify $e_f(1) = e_f(0) + 1$. Hence the graph M_1 is 5-*TPC*.

Case (6). If $k = 6$, then the triangular labels $T_2^6, T_3^6, T_4^6, T_5^6, T_6^6, T_7^6, T_8^6, T_1^6$ to consecutive vertices v_i, $1 \leq i \leq 8$. It is easy to verify $e_f(0) = e_f(1) + 1$. Thus the graph M_1 is 6-*TPC*.

Hence, in all the cases, given maximal outerplanar graph M_1 of order 8 is a *k-TPC* graph for $k = 1, 2, 3, 4, 5, 6$.

Theorem 5.8. *If M_2 is a MOP graph of order 8 has outer cycle $v_1v_2v_3v_4v_5v_6v_7v_8v_1$ with 5 chords $v_1v_i, i = 3, 4, 7$ and $v_7v_j, j = 4, 5$, then M_2 admits a k-TPC labeling for $1 \leq k \leq 6$.*

Proof. Let us define a function f_k, $1 \leq k \leq 6$, from the vertex set of the given graph to the triangular number of order k as follows:

$$f_k(v_i) = T_i^k, \ \forall \ 1 \leq i \leq 8 \ \textit{and} \ \forall \ k = 1, 2, 3, 5,$$

$$f_4(v_i) = \begin{cases} T_8^4 & \text{if} \quad i = 1 \\ T_i^4 & \text{if} \quad 2 \leq i \leq 7 \\ T_1^4 & \text{if} \quad i = 8, \end{cases}$$

$$f_6(v_i) = \begin{cases} T_2^6 & \text{if} \quad i = 1 \\ T_1^6 & \text{if} \quad i = 2 \\ T_i^6 & \text{if} \quad 3 \leq i \leq 8. \end{cases}$$

It is easy to verify that all f_k's are bijective and $e_{f_k}(0) = e_{f_k}(1) + 1$ for $k = 1, 3, 4, 5, 6$, and $e_{f_k}(1) = e_{f_k}(0) + 1$ for $k = 2$. Hence, M_2 is *k-TPC* for every non-negative $k \leq 6$.

Remark 5.1.

(i) Consider another *MOP* graph of order 8 and denoted as M_3, having a cycle $v_1v_2v_3v_4v_5v_6v_7v_8v_1$ and five chords $v_2v_i, i = 7, 8$, $v_3v_j, j = 6, 7$ and v_4v_6. For every non-negative $k \leq 6$, we obtained that M_3 is k-TPC. The labelings of the vertices in the order of $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ are given below:

k is odd: $T_7^k, T_4^k, T_2^k, T_6^k, T_8^k, T_1^k, T_3^k, T_5^k$
$k = 2$ and 4: $T_1^k, T_2^k, T_3^k, T_4^k, T_5^k, T_6^k, T_7^k, T_8^k$
$k = 6$: $T_1^6, T_3^6, T_2^6, T_4^6, T_5^6, T_6^6, T_7^6, T_8^6$.

Edge condition:

$e_{f_k}(0) = e_{f_k}(1) + 1$ for $k = 1, 2, 3, 5, 6$
$e_{f_k}(1) = e_{f_k}(0) + 1$ for $k = 4$.

(ii) Consider another *MOP* graph of order 8 and denoted as M_4, having a cycle $v_1v_2v_3v_4v_5v_6v_7v_8v_1$ and five chords $v_1v_i, i = 3, 4, 6, 7$ and v_4v_6. For every non-negative $k \leq 6$, we obtained that M_4 is k-TPC. The labelings of the vertices in the order of $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ are given below:

k is odd: $T_1^k, T_2^k, T_3^k, T_4^k, T_5^k, T_6^k, T_7^k, T_8^k$
$k = 2$: $T_3^2, T_4^2, T_5^2, T_6^2, T_7^2, T_8^2, T_1^2, T_2^2$
$k = 4$: $T_8^4, T_2^4, T_3^4, T_4^4, T_5^4, T_6^4, T_7^4, T_1^4$
$k = 6$: $T_7^6, T_2^6, T_3^6, T_4^6, T_5^6, T_6^6, T_1^6, T_8^6$.

Edge condition:

$e_{f_k}(0) = e_{f_k}(1) + 1$ for $k = 3$

$e_{f_k}(1) = e_{f_k}(0) + 1$ for $k = 1, 2, 4, 5, 6$.

(iii) Consider another MOP graph of order 8 and denoted as M_5, having a cycle $v_1v_2v_3v_4v_5v_6v_7v_8v_1$ and five chords $v_1v_i, i = 3, 5, 7$ and $v_5v_j, j = 3, 7$. For every non-negative $k \leq 6$, we obtained that M_5 is k-TPC. The labelings of the vertices in the order of $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ are given below:

k is odd: $T^k_1, T^k_2, T^k_3, T^k_4, T^k_5, T^k_6, T^k_7, T^k_8$
$k = 2$: $T^2_2, T^2_1, T^2_3, T^2_4, T^2_5, T^2_6, T^2_7, T^2_8$
$k = 4$: $T^4_1, T^4_2, T^4_8, T^4_3, T^4_4, T^4_7, T^4_6, T^4_5$
$k = 6$: $T^6_1, T^6_2, T^6_4, T^6_3, T^6_7, T^6_5, T^6_8, T^6_6$.

Edge condition:
$e_{f_k}(0) = e_{f_k}(1) + 1$ for $k = 1, 2, 3, 5$
$e_{f_k}(1) = e_{f_k}(0) + 1$ for $k = 4, 6$.

(iv) Consider another MOP graph of order 8 and denoted as M_6, having a cycle $v_1v_2v_3v_4v_5v_6v_7v_8v_1$ and five chords $v_1v_i, i = 3, 4, 5, 7$ and v_5v_7. For every non-negative $k \leq 6$, we obtained that M_6 is k-TPC. The labelings of the vertices in the order of $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ are given below:

k is odd: $T^k_1, T^k_2, T^k_3, T^k_4, T^k_5, T^k_6, T^k_7, T^k_8$
$k = 2$: $T^2_6, T^2_2, T^2_3, T^2_4, T^2_5, T^2_1, T^2_7, T^2_8$
$k = 4$ and 6: $T^k_8, T^k_2, T^k_3, T^k_4, T^k_5, T^k_6, T^k_7, T^k_1$.

Edge condition:
$e_{f_k}(1) = e_{f_k}(0) + 1$ for $1 \leq k \leq 6$.

(v) Consider another MOP graph of order 8 and denoted as M_7, having a cycle $v_1v_2v_3v_4v_5v_6v_7v_8v_1$ and five chords $v_1v_i, i = 4, 6, 7$ and $v_4v_j, j = 2, 6$. For every non-negative $k \leq 6$, we obtained that M_7 is k-TPC. The labelings of the vertices in the order of $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ are given below:

k is odd: $T^k_1, T^k_2, T^k_3, T^k_4, T^k_5, T^k_6, T^k_7, T^k_8$
$k = 2$: $T^2_8, T^2_2, T^2_3, T^2_4, T^2_5, T^2_6, T^2_7, T^2_1$
$k = 4$: $T^4_8, T^4_2, T^4_1, T^4_4, T^4_5, T^4_6, T^4_7, T^4_3$
$k = 6$: $T^6_2, T^6_1, T^6_3, T^6_4, T^6_5, T^6_6, T^6_7, T^6_8$.

Edge condition:
$e_{f_k}(1) = e_{f_k}(0) + 1$ for $1 \leq k \leq 6$.

(vi) Consider another MOP graph of order 8 and denoted as M_8, having a cycle $v_1v_2v_3v_4v_5v_6v_7v_8v_1$ and five chords $v_1v_i, i = 3, 4, 5$ and $v_8v_j, j = 5, 6$. For every non-negative $k \leq 6$, we obtained that M_8 is k-TPC. The labelings of the vertices in the order of $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ are given below:

k is odd: $T^k_1, T^k_2, T^k_3, T^k_4, T^k_5, T^k_6, T^k_8, T^k_7$
$k = 2$ and 6: $T^k_7, T^k_2, T^k_3, T^k_4, T^k_5, T^k_6, T^k_1, T^k_8$
$k = 4$: $T^4_8, T^4_2, T^4_3, T^4_4, T^4_5, T^4_6, T^4_1, T^4_7$.

Edge condition:
$e_{f_k}(0) = e_{f_k}(1) + 1$ for $k = 1, 2, 3, 5, 6$
$e_{f_k}(1) = e_{f_k}(0) + 1$ for $k = 4$.

(vii) Consider another MOP graph of order 8 and denoted as M_9, having a cycle $v_1v_2v_3v_4v_5v_6v_7v_8v_1$ and five chords $v_1v_i, i = 3, 4, 5, 6$ and v_6v_8. For every non-negative $k \leq 6$, we obtained that M_9 is k-TPC. The labelings of the vertices in the order of $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ are given below:

k is odd: $T^k_1, T^k_2, T^k_3, T^k_4, T^k_5, T^k_6, T^k_7, T^k_8$
$k = 2$: $T^2_2, T^2_1, T^2_3, T^2_4, T^2_5, T^2_6, T^2_7, T^2_8$
$k = 4$: $T^4_8, T^4_2, T^4_3, T^4_4, T^4_5, T^4_6, T^4_1, T^4_7$
$k = 6$: $T^6_7, T^6_2, T^6_3, T^6_4, T^6_5, T^6_6, T^6_1, T^6_8$.

Edge condition:
$e_{f_k}(0) = e_{f_k}(1) + 1$ for $k = 1, 3, 5, 6$
$e_{f_k}(1) = e_{f_k}(0) + 1$ for $k = 2, 4$.

(viii) Consider another MOP graph of order 8 and denoted as M_{10}, having a cycle $v_1v_2v_3v_4v_5v_6v_7v_8v_1$ and five chords $v_1v_i, i = 3, 4, 5$ and $v_5v_j, j = 7, 8$. For every non-negative $k \leq 6$, we obtained that M_{10} is k-TPC. The labelings of the vertices in the order of $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ are given below:

k is odd: $T^k_1, T^k_2, T^k_3, T^k_4, T^k_5, T^k_6, T^k_7, T^k_8$
$k = 2$: $T^2_2, T^2_1, T^2_3, T^2_4, T^2_5, T^2_6, T^2_7, T^2_8$
$k = 4$: $T^4_1, T^4_2, T^4_5, T^4_4, T^4_3, T^4_6, T^4_7, T^4_8$
$k = 6$: $T^6_7, T^6_2, T^6_3, T^6_4, T^6_5, T^6_6, T^6_1, T^6_8$.

Edge condition:

$e_{f_k}(0) = e_{f_k}(1) + 1$ for $k = 1, 2, 3, 5, 6$

$e_{f_k}(1) = e_{f_k}(0) + 1$ for $k = 4$.

(ix) Consider another *MOP* graph of order 8 and denoted as M_{11}, having a cycle $v_1v_2v_3v_4v_5v_6v_7v_8v_1$ and five chords $v_1v_i, i = 3, 5, 6$ and v_3v_5 and v_6v_8. For every non-negative $k \leq 6$, we obtained that M_{11} is k-TPC. The labelings of the vertices in the order of $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ are given below:

k is odd: $T_1^k, T_2^k, T_3^k, T_4^k, T_5^k, T_7^k, T_6^k, T_8^k$

$k = 2$: $T_7^2, T_2^2, T_3^2, T_4^2, T_5^2, T_6^2, T_1^2, T_8^2$

$k = 4$: $T_8^4, T_2^4, T_3^4, T_1^4, T_4^4, T_6^4, T_5^4, T_7^4$

$k = 6$: $T_8^6, T_1^6, T_2^6, T_3^6, T_4^6, T_5^6, T_6^6, T_7^6$.

Edge condition:

$e_{f_k}(0) = e_{f_k}(1) + 1$ for $k = 1, 2, 3, 5$

$e_{f_k}(1) = e_{f_k}(0) + 1$ for $k = 4, 6$.

(x) Consider another *MOP* graph of order 8 and denoted as M_{12}, having a cycle $v_1v_2v_3v_4v_5v_6v_7v_8v_1$ and five chords $v_1v_i, i = 4, 5, 6$ and v_2v_4 and v_6v_8. For every non-negative $k \leq 6$, we obtained that M_{12} is k-TPC. The labelings of the vertices in the order of $v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8$ are given below:

k is odd: $T_1^k, T_2^k, T_3^k, T_4^k, T_5^k, T_6^k, T_7^k, T_8^k$

$k = 2$: $T_2^2, T_1^2, T_3^2, T_4^2, T_5^2, T_6^2, T_7^2, T_8^2$

$k = 4$: $T_8^4, T_1^4, T_5^4, T_3^4, T_4^4, T_2^4, T_6^4, T_7^4$

$k = 6$: $T_5^6, T_2^6, T_3^6, T_4^6, T_1^6, T_6^6, T_8^6, T_7^6$.

Edge condition:

$e_{f_k}(0) = e_{f_k}(1) + 1$ for $k = 1, 3, 5$

$e_{f_k}(1) = e_{f_k}(0) + 1$ for $k = 2, 4, 6$.

6. Conclusion

Study of higher order triangular numbers is very interesting in the theory of numbers. According to existing literature, much works have been done in triangular related labelings and cordial related labelings. In our study, a new labeling called *k*- triangular prime cordial labeling was introduced. This will add new dimension to the research work in the area connecting two branches, namely, graph labeling and number theory. It is challenging to investigate *k*- triangular prime cordial labeling of *MOP* graphs of small order. We shown that all *MOP* graphs of order 4,5 and 6 admit *k-TPC* labeling for $k = 1, 2, 3, 5, 6$. But these graphs are not 4-*TPC* graphs. We also proved that all *MOP* graphs of order 7 admit *k-TPC* labeling for $k = 1, 2, 3, 4, 5, 6$. Moreover, it is verified that the *MOP* graphs of order 8 are *k-TPC* graphs for $k = 1, 2, 3, 4, 5, 6$. In future, this work may be extended for higher order *MOP* graphs.

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References

[1] J. A. Bondy and U. S. R. Murthy, *Graph Theory and Applications*, North-Holland, New York, 1976.

[2] I. Cahit, Cordial Graphs: A Weaker Version of Graceful and Harmonious Graphs, *ARS Combinatotia*, **23** (1987), 201-207.

[3] G. Chartrand and F. Harary, Planar Permutation Graphs, *Annales de 1'Institut Henri Poincare C, Analyse Non Lineaire*, **B3** (1967), 433-438.

[4] S. W. Golomb, How to Number a Graph, in: R. C. Read (Ed.), *Graph Theory and Computing*, Academic Press, New York, 1972, 23-37.

[5] J. A. Gallian, A Dynamic Survey of Graph Labeling, *Electronic Journal of Combinations*, DS6, 2017.

[6] S. M. Hegde and P. Shankaran, On Triangular Sum Labelings of Graphs, in: B. D. Acharya, S. Arumugam, and A. Rosa (Ed.), *Labeling of Discrete Structures and Applications*, Narosa Publishing House, New Delhi, 2008, 109-115.

[7] P. S. Kumar and C. E. V. Madhavan, A New Class of Separators and Planarity of Chordal Graphs, *Lecture Notes in Computer Sciences*, **405** (1989), 30-41.

[8] G. C. Lau, S. Alikhani, S. M. Lee and W. Kocay, On k-Edge-Magic Labelings of Maximal Outerplanar Graphs, *AKCE International Journal of Graphs and Combinatorics*, **12** (2015), 40-46.

[9] S. M. Lee, M. Kitagaki, J. Young and W. Kocay, On Edge-Graceful and Edge-Magic Maximal Outerplanar Graphs, *The Journal of Combinatorial Mathematics and Combinatorial Computing*, **59** (2006), 119-129.

[10] S. Murugesan, D. Jayaraman and J. Shiama, Centered Triangular Sum Labeling of Graphs, *International Journal of Applied Information Systems*, **5**(7) (2013), 1-4.

[11] S. Murugesan, D. Jayaraman and J. Shiama, Second and Third Order Triangular Sum Labeling of Graphs, *International Journal of Mathematical Archive*, **4**(2) (2013), 55-62.

[12] S. Murugesan, D. Jayaraman and J. Shiama, Some Higher Order Triangular Sum Labeling of Graphs, *International Journal of Computer Applications*, **72**(10) (2013), 1-8.

[13] A. Ponmoni, S. Navaneetha Krishnan and A. Nagarajan, Triangular Divisor Cordial Labeling for Some Special Graphs, *International Journal of Development Research*, **5**(8) (2015), 5245-5248.

[14] R. Ponraj, R. Singh, R. Kala and S. Sathish Narayanan, k-Prime Cordial Graphs, *Journal of Applied Mathematics and Informatics*, **34**(3-4) (2016), 227-237.

[15] A. Rosa, On Certain Valuations of the Vertices of a Graph, in: *Theory of Graphs (International Symposium, Rome, July 1966)*, Gordon and Breach, N.Y and Dunod Paris, 1967, 349?355.

[16] K. Ramanjaneyulu, V. Ch. Venkaiah and K. Kothapalli, Cordial Labelings of a Class of Planar Graphs, *AKCE International Journal of Graphs and Combinatorics*, **6**(1) (2009), 171-181.

[17] M. Sundaram, R. Ponraj and S. Somasundram, Prime Cordial Labeling of Graphs, *Journal of the Indian Academy of Mathematics*, **27**(2) (2005), 373-390.

[18] G. Suresh Singh, *Graph theory- A Study of Certain Labeling Problems*, Ph.D. Thesis, Kerala University, Kerala, 1993.