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(Dedicated to Professor D. S. Hooda on His 80^{th} *Birth Anniversary Celebrations)*

COMMON FIXED POINT THEOREMS FOR COMMUTATIVE, WEAKLY COMMUTATIVE AND COMPATIBLE MAPPINGS IN MULTIPLICATIVE CONE METRIC SPACE

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Abstract

In this paper, we introduce the notions of commutative, weakly commutative, and compatible mappings in multiplicative cone metric space and prove common fixed point theorems for these mappings with multiplicative normal cone setting. Also, we give an example to show the validity of our results.
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1. Introduction

There exist numerous generalizations of metric space in fixed point theory. One of them is cone metric space, which is introduced by Huang and Zhang [7] in 2007. They analysed convergence and substituted real numbers by ordered Banach space and proved fixed point theorems in this space with normal cone conditions. After that, various authors he proved and extend many fixed point and *CFP* (common fixed point) results to this space with normal and non-normal cone conditions (see, eg., [3, 4, 5, 6, 10, 11, 12, 13, 14, 15, 17, 18]).

Recently in 2017, Ampadu [1] introduced the notion of multiplicative cone metric in which he replaced triangle inequality property in cone metric space by multiplicative triangle inequality property and established a coupled version of higher-order Banach contraction principle with multiplicative normal cone condition, further also in [2], he proved a Hardy-Rogers fixed point theorem in this space uses the c- class multiplicative cone functions.

On the other hand, in 1976 it was the turning point in the theory of the existence of *CFP* for mappings when Jungck [8] introduced the concept of commutative mappings by generalizing the Banach contraction theorem and proved some *CFP* theorems by using these mappings. This opens a new interesting area of research for researchers. Then in the sequel, in 1982, a less restrictive concept was introduced by Sessa [16] called weakly commutativity in order to generalize the commutativity concept. Thereafter, many authors prove and extend a variety of common fixed point theorems by substituting commutativity to weakly commutativity. Further, in 1986, Jungck [9] define a new notion of compatible mappings. These mappings are more general in nature than commutative and weakly commutative mappings that commutative mappings are weakly commutative and weakly commutative mappings are compatible but the converse may not be true. Also, we can notice that commutativity and weakly commutativity are point-based properties of mappings while compatibility is an iteration of sequences-based properties.

In this paper, we first introduce the notions of commutative, weakly commutative, and compatible mappings to multiplicative cone metric space, and then next we prove *CFP* theorems for these mappings. Also, in the last, we show the validity of our proven results by an example.

2. Preliminaries

In 2017, Ampadu [1] gave the perception of multiplicative cone metric space as follows:

Definition 2.1 ([1]). *Let K be a real Banach space. A subset L of K is called a multiplicative cone iff:*

(L_1) *L is closed, nonempty and $L \neq \{1\}$,*
(L_2) *$u^m v^n \in L$, for all $u, v \in L$ and $m, n \geq 0$,*
(L_3) *$u \in L$ and $\frac{1}{u} \in L$ imply $u = 1$ i.e., $L \cap \frac{1}{L} = 1$.*

Definition 2.2 ([1]). *Let $L \subseteq K$ be a multiplicative cone, then partial ordering $\leq$ is defined on L by $u \leq v$ iff $\frac{u}{v} \in L$. Here $u < v$ indicates $u \leq v$ but $u \neq v$ and $u \ll v$ will stand for $\frac{u}{v} \in \text{int}(L)$ (interior of L).*

Definition 2.3 ([1]). Let $L \subseteq K$ *is a multiplicative cone then it is called multiplicative normal if,*

$$\exists\ \Psi > 0 \text{ s.t.}, \forall\ u, v \in K,\ 1 \leq u \leq v \text{ implies that, } \|u\| \leq \|v\|^{\Psi}.$$

The least positive number which satisfies the above condition is called the multiplicative constant of L. Here $\|.\|$ *denotes a multiplicative norm.*

Definition 2.4 ([1]). *Let K be a real Banach space and* $L \subseteq K$ *be a multiplicative cone. Let M be any non-empty set, then if the mapping* $\Upsilon: M \times M \to K$ *satisfies the following:*

(Υ1) $1 < \Upsilon(u, v)$, $\forall\ u, v \in M$ *and* $\Upsilon(u, v) = 1$ *iff* $u \neq v$,

(Υ2) $\Upsilon(u, v) = \Upsilon(v, u)$, $\forall\ u, v \in M$,

(Υ3) $\Upsilon(u, v) \leq \Upsilon(u, w)\ \Upsilon(w, v)\ \forall\ u, w, v \in M$ (*multiplicative triangle inequality*).

Then we say that Υ *is a multiplicative cone metric on M and* (M, Υ) *is a multiplicative cone metric space.*

Example 2.1. Let $K = R^2$, $L = \{(u, v) \in K: u, v \geq 1\} \subseteq R^2$, $M = R$ and mapping $\Upsilon: M \times M \to K$ be such that, $\Upsilon(u, v) = (\omega^{|u-v|},\ \omega^{\sigma|u-v|})$, where $\omega > 1$ and $\sigma \geq 0$ is a constant. Then pair (M, Υ) is a multiplicative cone metric space.

Definition 2.5 ([1]). *Let* (M, Υ) *is multiplicative cone metric space, and* $\{u_n\} \subset M$ *be a sequence, then we say that sequence* $\{u_n\}$ *is*;

(i) *Multiplicative convergent and multiplicative converges to a point* $u \in M$, *if for every* $\mu \in K$ *with* $1 << \mu$, *there is* N *s. t.,* $\forall\ n > N$, $\Upsilon(u_n, u) << \mu$, *i.e.* $\lim_{n\to\infty} u_n = u$.

(ii) *Cauchy sequence, if for any* $\mu \in K$ *with* $1 << \mu$, $\exists\ N$ *s.t.,* $\forall\ n, m > N$, $\Upsilon(u_n, u_m) << \mu$.

Definition 2.6 ([1]). *A multiplicative cone metric space is said to be complete if for every multiplicative Cauchy sequence is multiplicative convergent in M.*

Definition 2.7. *Let* (M, Υ) *be a multiplicative cone metric space, and* $E, F: M \to M$ *are two self-mappings of* (M, Υ). *Then E and F are said to be:*

(i) *Commutative mappings if* $EFu = FEu$, *for all* $u \in M$,

(ii) *Weakly commutative mappings if* $\Upsilon(EFu, FEu) \leq \Upsilon(Eu, Fu)$, *for all* $u \in M$,

(iii) *Compatible mappings if* $\lim_{n\to\infty} \Upsilon(EFu_n, FEu_n) = 1$, *whenever, sequence* $\{u_n\} \subset M$ *be such that* $\lim_{n\to\infty} Eu_n = \lim_{n\to\infty} Fu_n = \lambda$, *for some* $\lambda \in M$.

Remark 2.1. *Commutative mappings are always weakly commutative and weakly commutative mappings are compatible but the converse is not always true, which is clear by the following examples:*

Example 2.2. Let $M = [0, 1]$, $K = R$ and $L = \{u \in K: u \geq 1\}$ be a multiplicative cone in K. Let $\Upsilon: M \times M \to K$ be a multiplicative metric defined as, $\Upsilon(u, v) = \omega^{|u-v|}$, for all $u, v \in M$ and $\omega > 1$, then (M, Υ) is clearly a multiplicative cone metric space. Suppose $E, F: M \to M$ are two self-mappings of (M, Υ) defined by,

$$E(u) = \frac{u}{3-u}, \text{ and } F(u) = \frac{u}{3};\ \forall\ u \in M.$$

Then, we can see that, for any non-zero $u \in M$, we have;

$$EFu = \frac{u}{9-u} < \frac{u}{9-3u} = FEu.$$

That is E and F are not commutative mappings but;

$$\Upsilon(EFu, FEu) = \omega^{\left|\frac{2u^2}{(9-u)(9-3u)}\right|} \leq \omega^{\left|\frac{u^2}{(9-3u)}\right|} = \Upsilon(Eu, Fu) \text{ for any } u \in M.$$

Hence E and F are weakly commutative mappings.

Example 2.3. Let $M = R$, $K = R$, and $L = \{u \in K: u \geq 1\}$ be a multiplicative cone in K. Let $\Upsilon: M \times M \to K$ be a multiplicative metric defined as, $\Upsilon(u, v) = \omega^{|u-v|}$, for all $u, v \in M$ and $\omega > 1$, then (M, Υ) is clearly a multiplicative cone metric space. Suppose $E, F: M \to M$ are two self-mappings of (M, Υ) defined by,

$$E(u) = u^3, \text{ and } F(u) = 2\text{-}u\ \forall\ u \in M.$$

Then, we can see that, for any non-zero $u \in M$, we have;

$EFu = (2\text{-}u)^{3} \neq (2\text{-}u^3) = FEu.$

That is E and F are not commutative mappings and again we have,

$\Upsilon(EFu, FEu) = \omega^{6u^9} > \omega^{u^3} = \Upsilon(Eu, Fu)$ for any $u \in M$.

So, these mappings are also not weakly commutative but

$\lim_{n\to\infty} \Upsilon(Eu_n, Fu_n) = \omega^{|u_n-1||u_n^2+u_n+2|} \to 1$ iff $u_n \to 1$ and

$\lim_{n\to\infty} \Upsilon(EFu_n, FEu_n) = \lim_{n\to\infty} \omega^{6|u_n-1|^2} = 1$ if $u_n \to 1$.

Hence E and F are compatible mappings.

Preposition 2.1. *Let E and F be two compatible self-mappings of a multiplicative cone metric space* (M, Υ). *If* $E\lambda = F\lambda$ *for some* $\lambda \in M$. *Then* $EF\lambda = EE\lambda = FF\lambda = FE\lambda$.

Proof. Let $\{u_n\}$ be a sequence in M defined by $u_n = \lambda$, where $\lambda \in M$ and n = 1, 2, 3... and $E\lambda = F\lambda$. Then we have,

$\lim_{n\to\infty} Eu_n = \lim_{n\to\infty} Fu_n = E\lambda.$

Since E and F are compatible mappings, so we get;

$\Upsilon(EF\lambda, FE\lambda) = \lim_{n\to\infty} \Upsilon(EFu_n, FEu_n) = 1$, therefore, we get $EF\lambda = FF\lambda$. Since $E\lambda = F\lambda$, then finally we get, $EF\lambda = EE\lambda = FF\lambda = FE\lambda$. □

Preposition 2.2. *Let E and F be two compatible self-mappings of a multiplicative cone metric space* (M, Υ) *and* $\lim_{n\to\infty} Eu_n = \lim_{n\to\infty} Fu_n = \lambda$, *for some* $\lambda \in M$. *Then,*

(i) $\lim_{n\to\infty} FEu_n = E\lambda$ *if E is continuous at* λ,
(ii) $\lim_{n\to\infty} EFu_n = F\lambda$ *if F is continuous at* λ,
(iii) $EF\lambda = FE\lambda$ *and* $E\lambda = F\lambda$ *if E and F are continuous at* λ.

Proof. (i) Let E is continuous at λ. Since $\lim_{n\to\infty} Eu_n = \lim_{n\to\infty} Fu_n = \lambda$, for some $\lambda \in M$, so we have $\lim_{n\to\infty} EFu_n = E\lambda$. Also, E and F are compatible mappings therefore, we get;

$\lim_{n\to\infty} \Upsilon(FEu_n, E\lambda) = \lim_{n\to\infty} \Upsilon(FEu_n, EFu_n) \lim_{n\to\infty} \Upsilon(EFu_n, E\lambda) = 1.$

Hence $\lim_{n\to\infty} FEu_n = E\lambda$.

(ii) This can be proven by a similar argument to (i).

(iii) Let E and F be continuous at λ. Since $\lim_{n\to\infty} Fu_n = \lambda$ and E is continuous at λ, then by (i) we have $\lim_{n\to\infty} FEu_n = E\lambda$, also F is continuous at λ. So, $\lim_{n\to\infty} FEu_n = F\lambda$. Thus we get $E\lambda = F\lambda$ and by the uniqueness of the limit and preposition 2.12, we get $EF\lambda = FE\lambda$. □

3. Main Results

We now prove the following *CFP* theorems for commutative, weakly commutative, and compatible mappings which satisfy a contractive condition in the context of multiplicative cone metric space.

Theorem 3.1. *Let* (M, Υ) *be a complete multiplicative cone metric space and L be a multiplicative normal cone with multiplicative constant* Ψ. *Let* $E, F, H, I: M \to M$ *be four self-mappings of* (M, Υ), *which satisfy the following conditions:*

(1) $E(M) \subset I(M)$ *and* $F(M) \subset H(M)$,
(2) $\Upsilon(Eu, Fv) \leq \{\max\{\Upsilon(Hu, Iv), \Upsilon(Hu, Eu), \Upsilon(Iv, Fv), \Upsilon(Eu, Iv), \Upsilon(Hu, Fv)\}\}^{\eta}$, $\forall\, u, v \in M$ *and* $\eta \in (0, \frac{1}{2})$,
(3) *One of the mappings E, F, H, and I is continuous,*
(4) *The pairs* (H, E) *and* (I, F) *are commutative.*

Then mappings E, F, H and I have a unique CFP.

Proof. Since $E(M) \subset I(M)$, consider a point $u_0 \in M$, there exists $u_1 \in M$ such that $Eu_0 = Iu_1 = v_0$. Now for this point u_1, there exist $u_2 \in M$ such that $Fu_1 = Hu_2 = v_1$. This continues to form sequences such that

$$v_{2n} = Eu_{2n} = Iu_{2n+1}, \text{ and } v_{2n+1} = Fu_{2n+1} = Hu_{2n+1}. \quad (3.1)$$

Now, taking $u = u_{2n}$ and $v = u_{2n+1}$ in (2) , we obtain

$$\begin{aligned}\Upsilon(v_{2n}, v_{2n+1}) &= \Upsilon(Eu_{2n}, Fu_{2n+1}) \\ &\leq \{\max\{\Upsilon(Hu_{2n}, Iu_{2n+1}), \Upsilon(Hu_{2n}, Eu_{2n}), \Upsilon(Iu_{2n+1}, Fu_{2n+1}), \Upsilon(Eu_{2n}, Iu_{2n+1}),\end{aligned}$$

$$\begin{aligned}\Upsilon(Hu_{2n}, Fu_{2n+1})\}\}^{\eta} \\ &\leq \{\max\{\Upsilon(v_{2n-1}, v_{2n}), \Upsilon(v_{2n-1}, v_{2n}), \Upsilon(v_{2n}, v_{2n+1}), \Upsilon(v_{2n}, v_{2n}), \Upsilon(v_{2n-1}, v_{2n+1})\}\}^{\eta} \\ &\leq \{\max\{\Upsilon(v_{2n-1}, v_{2n})\, \Upsilon(v_{2n}, v_{2n+1}), \Upsilon(v_{2n-1}, v_{2n})\, \Upsilon(v_{2n}, v_{2n+1}), \Upsilon(v_{2n-1}, v_{2n}) \\ &\quad \Upsilon(v_{2n}, v_{2n+1}), 1, \Upsilon(v_{2n-1}, v_{2n})\Upsilon(v_{2n}, v_{2n+1})\}\}^{\eta} \\ &= \{\Upsilon(v_{2n-1}, v_{2n})\}^{\eta}\ \{\Upsilon(v_{2n}, v_{2n+1})\}^{\eta}.\end{aligned}$$

This implies that

$$\Upsilon(v_{2n}, v_{2n+1}) \leq \{\Upsilon(v_{2n-1}, v_{2n})\}^{\frac{\eta}{1-\eta}},$$

$$\Upsilon(v_{2n}, v_{2n+1}) \leq \{\Upsilon(v_{2n-1}, v_{2n})\}^{h}, \tag{3.2}$$

Here, $h = \frac{\eta}{1-\eta} \in (0, \frac{1}{2})$. Similarly, by using (2) we obtain,

$$\begin{aligned}\Upsilon(v_{2n}, v_{2n+1}) &= \Upsilon(Eu_{2n}, Fu_{2n+1}) \\ &\leq \{\max\{\Upsilon(Hu_{2n}, Iu_{2n+1}), \Upsilon(Hu_{2n}, Eu_{2n}), \Upsilon(Iu_{2n+1}, Fu_{2n+1}), \Upsilon(Eu_{2n}, Iu_{2n+1}), \\ &\quad \Upsilon(Hu_{2n}, Fu_{2n+1})\}\}^{\eta} \\ &\leq \{\max\{\Upsilon(v_{2n-1}, v_{2n}), \Upsilon(v_{2n-1}, v_{2n}), \Upsilon(v_{2n}, v_{2n+1}), \Upsilon(v_{2n}, v_{2n}), \Upsilon(v_{2n-1}, v_{2n+1})\}\}^{\eta} \\ &\leq \{\max\{\Upsilon(v_{2n-1}, v_{2n})\, \Upsilon(v_{2n}, v_{2n+1}), \Upsilon(v_{2n-1}, v_{2n})\, \Upsilon(v_{2n}, v_{2n+1}), \Upsilon(v_{2n-1}, v_{2n}) \\ &\quad \Upsilon(v_{2n}, v_{2n+1}), 1, \Upsilon(v_{2n-1}, v_{2n})\, \Upsilon(v_{2n}, v_{2n+1})\}\}^{\eta} \\ &= \{\Upsilon(v_{2n}, v_{2n+1})\}^{\eta}\ \{\Upsilon(v_{2n+1}, v_{2n+2})\}^{\eta}.\end{aligned}$$

This implies that

$$\Upsilon(v_{2n+1}, v_{2n+2}) \leq \{\Upsilon(v_{2n}, v_{2n+1})\}^{\frac{\eta}{1-\eta}},$$

$$\Upsilon(v_{2n+1}, v_{2n+2}) \leq \{\Upsilon(v_{2n}, v_{2n+1})\}^{h}, \quad h = \frac{\eta}{1-\eta} \in (0, \tfrac{1}{2}). \tag{3.3}$$

So, from (3.2) and (3.3), $\forall\, n \in N$, we get

$$\Upsilon(v_n, v_{n+1}) \leq \Upsilon(v_{n-1}, v_n)^{h} \leq \Upsilon(v_{n-2}, v_{n-1})^{h^2} \leq \ldots \leq \Upsilon(v_0, v_1)^{h^n}.$$

Therefore, by using multiplicative triangle inequality, we obtain $\forall\, n, m \in N$ such that $n < m$,

$$\begin{aligned}\Upsilon(v_n, v_m) &\leq \Upsilon(v_n, v_{n+1})\ \Upsilon(v_{n+1}, v_{n+2}) \ldots \Upsilon(v_{m-1}, v_m) \\ &\leq \Upsilon(v_0, v_1)^{h^n}\ \Upsilon(v_0, v_1)^{h^{n-1}} \ldots \Upsilon(v_0, v_1)^{h^{m-1}} \\ &\leq \Upsilon(v_0, v_1)^{\frac{h^n}{1-h}}.\end{aligned}$$

Now, by using the condition of multiplicative normality of cone, we get

$$\left\|\Upsilon(v_n, v_m)\right\| \leq \left\|\Upsilon(v_0\ \ v_1)\right\|^{\Psi \frac{h^n}{1-h}}.$$

Since $h < 1$ it follows that $\lim_{n,m \to \infty} \Upsilon(v_n, v_m) = 1$.

Hence $\{v_n\}$ is a multiplicative Cauchy sequence in M. Now since M is multiplicative complete so, there is a point $s \in M$ s.t. $\lim_{n\to\infty} v_n = s$. Consequently, we have

$$\lim_{n\to\infty} Eu_{2n} = \lim_{n\to\infty} Iu_{2n+1} = \lim_{n\to\infty} Fu_{2n+1} = \lim_{n\to\infty} Hu_{2n+2} = s, \tag{3.4}$$

because, $\{v_{2n}\} = \{Eu_{2n}\} = \{Iu_{2n+1}\}$, $\{v_{2n+1}\} = \{Fu_{2n+1}\} = \{Hu_{2n+1}\}$ are sub sequences of $\{v_n\}$.

Case (i). Suppose that H is continuous then

$$\lim_{n\to\infty} HEu_{2n} = \lim_{n\to\infty} H^2u_{2n} = Hs.$$

Since, (H, E) is a pair of commutative mappings then, we have

$$\lim_{n\to\infty} EHu_{2n} = \lim_{n\to\infty} HEu_{2n}$$

$$\text{i.e., } \lim_{n\to\infty} EHu_{2n} = Hs. \tag{3.5}$$

On putting $u = u_{2n}$ and $v = u_{2n+1}$ in (2) and using (3.1) and (3.5), we get

$$\Upsilon(EHu_{2n}, Fu_{2n+1}) \leq \{\max\{\Upsilon\left(H^2u_{2n}, Iu_{2n+1}\right), \Upsilon\left(H^2u_{2n}, EHu_{2n}\right), \Upsilon(Iu_{2n+1}, Fu_{2n+1}),$$

$$\Upsilon(EHu_{2n}, Iu_{2n+1}), \Upsilon\left(H^2u_{2n}, Fu_{2n+1}\right)\}\}^{\eta}.$$

Letting $n \to \infty$, we obtain

$$\Upsilon(Hs, s) \leq \{\max\{\Upsilon(Hs, s), \Upsilon(Hs, Hs), \Upsilon(s, s), \Upsilon(Hs, s), \Upsilon(Hs, s)\}\}^{\eta}$$
$$= \{\max\{\Upsilon(Hs, s), 1, 1, \Upsilon(Hs, s), \Upsilon(Hs, s)\}\}^{\eta}$$
$$= \{\Upsilon(Hs, s)\}^{\eta},$$
$$\Upsilon(Hs, s) \leq \{\Upsilon(Hs, s)\}^{\eta}.$$

This implies that, $\Upsilon(Hs, s) = 1$, i.e., $Hs = s$. (3.6)

On putting $u = s$ and $v = u_{2n+1}$ in (2) and using (3.6), we get

$$\Upsilon(Es, Fu_{2n+1}) \leq \{\max\{\Upsilon(Hs, Iu_{2n+1}), \Upsilon(Hs, Es), \Upsilon(Iu_{2n+1}, Fu_{2n+1}), \Upsilon(Es, Iu_{2n+1}), \Upsilon(Hs, Fu_{2n+1})\}\}^{\eta}$$

Letting $n \to \infty$, we obtain

$$\Upsilon(Es, s) \leq \{\max\{\Upsilon(s, s), \Upsilon(s, Es), \Upsilon(s, s), \Upsilon(Es, s), \Upsilon(s, s)\}\}^{\eta}$$
$$= \{\max\{1, \Upsilon(s, Es), 1, \Upsilon(Es, s), 1\}\}^{\eta}$$
$$= \{\Upsilon(Es, s)\}^{\eta},$$
$$\Upsilon(Es, s) \leq \{\Upsilon(Es, s)\}^{\eta}.$$

This implies that, $\Upsilon(Es, s) = 1$, i.e., $Es = s$. (3.7)

From (3.6) and (3.7), we have

$$Hs = Es = s. \qquad (3.8)$$

Now, $s = Es \in E(M) \subset I(M)$, so there exists $\alpha \in M$, such that

$$s = I\alpha. \qquad (3.9)$$

On putting $u = s$ and $v = \alpha$ in (2) and using (3.8)and (3.9), we have

$$\Upsilon(s, F\alpha) = \Upsilon(Es, F\alpha)$$
$$\leq \{\max\{\Upsilon(Hs, I\alpha), \Upsilon(Hs, Es), \Upsilon(I\alpha, F\alpha), \Upsilon(Es, I\alpha), \Upsilon(Hs, F\alpha)\}\}^{\eta}$$
$$= \{\max\{\Upsilon(s, s), \Upsilon(s, s), \Upsilon(s, F\alpha), \Upsilon(s, I\alpha), \Upsilon(s, F\alpha)\}\}^{\eta}$$
$$= \{\Upsilon(s, F\alpha)\}^{\eta},$$
$$\Upsilon(s, F\alpha) \leq \{\Upsilon(s, F\alpha)\}^{\eta}.$$

This implies that, $\Upsilon(s, F\alpha) = 1$, i.e., $F\alpha = s$. (3.10)

Since (I, F) is a pair of commutative mapping and using (3.10), we get

$$Is = IF\alpha = FI\alpha = Fs. \qquad (3.11)$$

On putting $u = s$ and $v = s$ in (2) and using (3.8) and (3.11), we have

$$\Upsilon(s, Fs) = \Upsilon(Es, Fs)$$
$$\leq \{\max\{\Upsilon(Hs, Is), \Upsilon(Hs, Es), \Upsilon(Is, Fs), \Upsilon(Es, Is), \Upsilon(Hs, Fs)\}\}^{\eta}$$
$$= \{\max\{\Upsilon(s, s), \Upsilon(s, s), \Upsilon(s, Fs), \Upsilon(s, s), \Upsilon(s, Fs)\}\}^{\eta}$$
$$= \{\Upsilon(s, Fs)\}^{\eta},$$
$$\Upsilon(s, Fs) \leq \{\Upsilon(s, \mathrm{F}s)\}^{\eta}.$$

This implies that, $\Upsilon(s, Fs) = 1$, i.e., $Fs = s$. (3.12)

Now, from (3.8), (3.9) and (3.12), we have

$$Hs = Es = Is = Fs = s.$$

Hence, s is the *CFP* of mappings H, I, E, and F.

Case (ii). Suppose that I is continuous, then this can be proved similar to case (i).

Case (iii). Suppose that E is continuous then

$$\lim_{n\to\infty} EHu_{2n} = \lim_{n\to\infty} E^2u_{2n} = Es.$$

Since, (H, E) is a pair of commutative mappings then, we have

$$\lim_{n\to\infty} HEu_{2n} = Es. \qquad (3.13)$$

On putting $u = Eu_{2n}$ and $v = u_{2n+1}$ in (2) and using (3.1) and (3.13), we get

$$\Upsilon\left(E^2u_{2n},\ Fu_{2n+1}\right) \le \{\max\{\Upsilon\left(E^2u_{2n},\ Iu_{2n+1}\right), \Upsilon\left(E^2u_{2n},\ E^2u_{2n}\right), \Upsilon\left(Iu_{2n+1},\ Fu_{2n+1}\right), \Upsilon\left(E^2u_{2n},\ Iu_{2n+1}\right), \Upsilon\left(E^2u_{2n},\ Fu_{2n+1}\right)\}\}^{\eta}.$$

Letting $n \to \infty$, we obtain

$$\Upsilon(Es, s) \le \{\max\{\Upsilon(Es, s), \Upsilon(Es, Es), \Upsilon(s, s), \Upsilon(Es, s), \Upsilon(Es, s)\}\}^{\eta}$$
$$= \{\max\{\Upsilon(Es, s), 1, 1, \Upsilon(Es, s), \Upsilon(Es, s)\}\}^{\eta}$$
$$= \{\Upsilon(Es, s)\}^{\eta}.$$
$$\Upsilon(Es, s) \le \{\Upsilon(Es, s)\}^{\eta}.$$

This implies that, $\Upsilon(Es, s) = 1$, i.e., $Es = s$. (3.14)

Now, $s = Es \in E(M) \subset I(M)$, so there exists $\alpha_1 \in M$ such that

$$s = I\alpha_1. \tag{3.15}$$

On putting $u = Eu_{2n}$ and $v = \alpha_1$ in (2), we have

$$\Upsilon\left(E^2u_{2n},\ F\alpha_1\right) \le \{\max\{\Upsilon\left(HEu_{2n},\ I\alpha_1\right), \Upsilon\left(HEu_{2n},\ E^2u_{2n}\right), \Upsilon\left(I\alpha_1,\ F\alpha_1\right), \Upsilon\left(E^2u_{2n},\ I\alpha_1\right), \Upsilon\left(HEu_{2n},\ F\alpha_1\right)\}\}^{\eta}.$$

Letting $n \to \infty$ and using (3.14) and (3.15), we get

$$\Upsilon(Es, F\alpha_1) \le \{\max\{\Upsilon(Es, s), \Upsilon(Es, Es), \Upsilon(s, F\alpha_1), \Upsilon(Es, s), \Upsilon(Es, F\alpha_1)\}\}^{\eta}$$
$$\Upsilon(s, F\alpha_1) \le \{\Upsilon(s, F\alpha_1)\}^{\eta}.$$

This implies that, $\Upsilon(s, F\alpha_1) = 1$ i.e., $F\alpha_1 = s$.

Since, (I, F) is a pair of commutative mappings and by using (3.15), we get

$$Fs = FI\alpha_1 = IF\alpha_1 = Is. \tag{3.16}$$

On putting $u = u_{2n}$ and $v = s$ in (2), and using (3.16), we get

$$\Upsilon(Eu_{2n}, Fs) \le \{\max\{\Upsilon(Hu_{2n}, Is), \Upsilon(Hu_{2n}, Eu_{2n}), \Upsilon(Is, Fs), \Upsilon(Eu_{2n}, Is), \Upsilon(Hu_{2n}, Fs)\}\}^{\eta}$$

Letting $n \to \infty$, we get

$$\Upsilon(s, Fs) \le \{\max\{\Upsilon(s, Fs), \Upsilon(s, s), \Upsilon(Fs, Fs), \Upsilon(s, Fs), \Upsilon(s, Fs)\}\}^{\eta}$$
$$= \{\Upsilon(s, Fs)\}^{\eta},$$
$$\Upsilon(s, Fs) \le \{\Upsilon(s, Fs)\}^{\eta}.$$

This implies that, $\Upsilon(s, Fs) = 1$ i.e., $Fs = s$. (3.17)

Now, $s = Fs \in F(M) \subset H(M)$, so there exists a point $\alpha_2 \in M$ such that

$$s = H\alpha_2. \tag{3.18}$$

On putting $u = \alpha_2$ and $v = s$ in (2), and using (3.17) and (3.18), we get

$$\Upsilon(E\alpha_2, s) = \Upsilon(E\alpha_2, Fs)$$
$$\le \{\max\{\Upsilon(H\alpha_2, Is), \Upsilon(H\alpha_2, E\alpha_2), \Upsilon(Is, Fs), \Upsilon(E\alpha_2, Is), \Upsilon(H\alpha_2, Fs)\}\}^{\eta}$$
$$= \{\Upsilon(E\alpha_2, s)\}^{\eta},$$
$$\Upsilon(E\alpha_2, s) \le \{\Upsilon(E\alpha_2, s)\}^{\eta}.$$

This implies that, $\Upsilon(E\alpha_2, s) = 1$ i.e., $E\alpha_2 = s$. (3.19)

Since, (H, E) is a pair of commutative mappings, therefore from (3.18) and (3.19), we have

$$Hs = HE\alpha_2 = EH\alpha_2 = Es.$$

Hence, $Hs = Es = Is = Fs = s$, (3.20)

i.e., s is a *CFP* of mappings H, I, E, and F.

Case (iv). Suppose that F is continuous, then this can be proved similar to case (iii).

Uniqueness: Let s_1 is another *CFP* of mappings H, I, E, and F, then on putting $u = s_1$ and $v = s$ in (2) and using (3.20), we get

$$\Upsilon(s_1, s) = \Upsilon(Es_1, Fs)$$
$$\le \{\max\{\Upsilon(Hs_1, Is), \Upsilon(Hs_1, Es_1), \Upsilon(Is, Fs), \Upsilon(Es_1, Is), \Upsilon(Hs_1, Fs)\}\}^{\eta}$$

$=\{\max\{\Upsilon(s_1, s), \Upsilon(s_1, s_1), \Upsilon(s, s), \Upsilon(s_1, s), \Upsilon(s_1, s)\}\}^{\eta}$

$=\{\max\{\Upsilon(s_1, s), 1, 1, \Upsilon(s_1, s), \Upsilon(s_1, s)\}\}^{\eta}$

$=\{\Upsilon(s_1, s)\}^{\eta}$,

$\Upsilon(s_1, s) \leq \{\Upsilon(s_1, s)\}^{\eta}$.

This implies that, $\Upsilon(s_1, s) = 1$ i.e., $s_1 = s$.

Hence, mappings H, I, E, and F have a unique *CFP*. □

Theorem 3.2. *Let* (M, Υ) *be a complete multiplicative cone metric space and L be a multiplicative normal cone with multiplicative constant* Ψ. *Let E, F, H, I* : $M \to M$ *be four self-mappings of* (M, Υ), *which satisfy conditions* (1) - (3) *and the following condition:*

(5) *The pairs* (H, E) *and* (I, F) *are weakly commutative.*

Then mappings E, F, H and I have a unique CFP.

Proof. Since $E(M) \subset I(M)$, consider a point $u_0 \in M$, there exists $u_1 \in M$ such that $Eu_0 = Iu_1 = v_0$. Now for this point u_1, there exist $u_2 \in M$ such that $Fu_1 = Hu_2 = v_1$. This continues to form sequences such that;

$$v_{2n} = Eu_{2n} = Iu_{2n+1}, \text{ and } v_{2n+1} = Fu_{2n+1} = Hu_{2n+1}.$$

Then it is clear from the proof of **Theorem 3.1**, that sequence $\{v_n\}$ is a multiplicative Cauchy sequence in M. Now since M is multiplicative complete so, there is a point $s \in M$ s.t. $lim_{n\to\infty} v_n = s$. Consequently, we have

$$\lim_{n\to\infty} Eu_{2n} = \lim_{n\to\infty} Iu_{2n+1} = \lim_{n\to\infty} Fu_{2n+1} = \lim_{n\to\infty} Hu_{2n+2} = s.$$

Because, $\{v_{2n}\} = \{Eu_{2n}\} = \{Iu_{2n+1}\}$, $\{v_{2n+1}\} = \{Fu_{2n+1}\} = \{Hu_{2n+1}\}$ are sub sequences of $\{v_n\}$.

Case (i). First, suppose that H is continuous then, we have

$$\lim_{n\to\infty} HEu_{2n} = \lim_{n\to\infty} H^2u_{2n} = Hs. \quad (3.21)$$

Since, (H, E) is a pair of weakly commutative mappings, so, we have

$$\Upsilon(HEu_{2n}, EHu_{2n}) \leq \Upsilon(Eu_{2n}, Hu_{2n}).$$

Letting $n \to \infty$ and using (3.21), we get

$$\lim_{n\to\infty} \Upsilon(EHu_{2n}, Hs) \leq \Upsilon(s, s) = 1,$$

$$\lim_{n\to\infty} EHu_{2n} = Hs. \quad (3.22)$$

Now, on putting $u = Hu_{2n}$ and $v = u_{2n+1}$ in (2) and using (3.22), we get

$$\Upsilon(EHu_{2n}, Fu_{2n+1}) \leq \{\max\{\Upsilon\left(H^2u_{2n}, Iu_{2n+1}\right), \Upsilon\left(H^2u_{2n}, EHu_{2n}\right), \Upsilon(Iu_{2n+1}, Fu_{2n+1}),$$
$$\Upsilon(EHu_{2n}, Iu_{2n+1}), \Upsilon(EHu_{2n}, Iu_{2n+1}), \Upsilon\left(H^2u_{2n}, Fu_{2n+1}\right)\}\}^{\eta}.$$

Letting $n \to \infty$, we get

$$\Upsilon(Hs, s) \leq \{\max\{\Upsilon(Hs, s), \Upsilon(Hs, Hs), \Upsilon(s, s), \Upsilon(Hs, s), \Upsilon(Hs, s)\}\}^{\eta}$$
$$=\{\max\{\Upsilon(Hs, s), 1, 1, \Upsilon(Hs, s), \Upsilon(Hs, s)\}\}^{\eta},$$
$$\Upsilon(Hs, s) \leq \{\Upsilon(Hs, s)\}^{\eta}.$$

This implies that, $\Upsilon(Hs, s) = 1$, i.e., $Hs = s$. (3.23)

On putting $u = s$ and $v = u_{2n+1}$ in (2) and using (3.23), we get

$$\Upsilon(Es, Fu_{2n+1}) \leq \{\max\{\Upsilon(Hs, Iu_{2n+1}), \Upsilon(Hs, Es), \Upsilon(Iu_{2n+1}, Fu_{2n+1}), \Upsilon(Es, Iu_{2n+1}), \Upsilon(Hs, Fu_{2n+1})\}\}^{\eta}.$$

Letting $n \to \infty$, we obtain

$$\Upsilon(Es, s) \leq \{\max\{\Upsilon(s, s), \Upsilon(s, Es), \Upsilon(s, s), \Upsilon(Es, s), \Upsilon(s, s)\}\}^{\eta}$$
$$=\{\max\{1, \Upsilon(s, Es), 1, \Upsilon(Es, s), 1\}\}^{\eta}$$
$$= \{\Upsilon(Es, s)\}^{\eta},$$
$$\Upsilon(Es, s) \leq \{\Upsilon(Es, s)\}^{\eta}.$$

This implies that, $\Upsilon(Es, s) = 1$, i.e., $Es = s$. (3.24)

From (3.23) and (3.24), we have

$$Hs = Es = s. \tag{3.25}$$

Now, $s = Es \in E(M) \subset I(M)$, so there exists $\beta \in M$, such that

$$s = Es = I\beta. \tag{3.26}$$

On putting $u = s$ and $v = \beta$ in (2) and using (3.25) and (3.26), we have

$$\begin{aligned}\Upsilon(s, F\beta) &= \Upsilon(Es, F\beta),\\ &\leq\{\max\{ \Upsilon(Hs, I\beta), \Upsilon(Hs, Es), \Upsilon(I\beta, F\beta), \Upsilon(Es, I\beta), \Upsilon(Hs, F\beta)\}\}^{\eta}\\ &=\{\max\{ \Upsilon(s, s), \Upsilon(s, s), \Upsilon(s, F\beta), \Upsilon(s, I\beta), \Upsilon(s, F\beta)\}\}^{\eta}\\ &=\{ \Upsilon(s, F\beta)\}^{\eta},\end{aligned}$$

$$\Upsilon(s, F\beta) \leq \{ \Upsilon(s, F\beta)\}^{\eta}.$$

This implies that, $\Upsilon(s, F\beta) = 1$, i.e., $F\beta = s$. (3.27)

Since (I, F) is a pair of weakly commutative mapping and using (3.26) and (3.27), we get

$$\Upsilon(Is, Fs) = \Upsilon(IF\beta, FI\beta) \leq \Upsilon(I\beta, F\beta) = \Upsilon(s, s) = 1.$$

This implies that, $\Upsilon(Is, Fs) = 1$, i.e., $Is = Fs$. (3.28)

On putting $u = s$ and $v = s$ in (2) and using (3.25) and (3.28), we have

$$\begin{aligned}\Upsilon(s, Fs) &= \Upsilon(Es, Fs)\\ &\leq\{\max\{ \Upsilon(Hs, Is), \Upsilon(Hs, Es), \Upsilon(Is, Fs), \Upsilon(Es, Is), \Upsilon(Hs, Fs)\}\}^{\eta}\\ &=\{\max\{ \Upsilon(s, s), \Upsilon(s, s), \Upsilon(s, Fs), \Upsilon(s, s), \Upsilon(s, Fs)\}\}^{\eta}\\ &=\{\Upsilon(s, Fs)\}^{\eta},\end{aligned}$$

$$\Upsilon(s, Fs) \leq \{\Upsilon(s, Fs)\}^{\eta}.$$

This implies that, $\Upsilon(s, Fs) = 1$, i.e., $Fs = s$. (3.29)

Now, from (3.25), (3.28), and (3.29), we have

$$Hs = Es = Is = Fs = s.$$

Hence, s is the *CFP* of mappings H, I, E, and F.

Case (ii). Suppose that I is continuous, then this can be proven similar to case (i).

Case (iii). Suppose that E is continuous then,

$$\lim_{n\to\infty} HEu_{2n} = \lim_{n\to\infty} E^2u_{2n} = Es. \tag{3.30}$$

Since, (H, E) is a pair of weakly commutative mappings then, we have

$$\Upsilon(HEu_{2n}, EHu_{2n}) \leq \Upsilon(Eu_{2n}, Hu_{2n}).$$

Letting $n \to \infty$ and using (3.30), we get

$$\lim_{n\to\infty} \Upsilon(HEu_{2n}, Es) \leq \Upsilon(s,s) = 1.$$

i.e., $\lim_{n\to\infty} HEu_{2n} = Es$. (3.31)

Now, on putting $u = Eu_{2n}$ and $v = u_{2n+1}$ in (2) and using (3.30) and (3.31), we get

$$\begin{aligned}\Upsilon\left(E^2u_{2n}, Fu_{2n+1}\right) \leq\{\max\{&\Upsilon\left(E^2u_{2n}, Iu_{2n+1}\right), \Upsilon\left(E^2u_{2n}, E^2u_{2n}\right), \Upsilon(Iu_{2n+1}, Fu_{2n+1}),\\ &\Upsilon\left(E^2u_{2n}, Iu_{2n+1}\right), \Upsilon\left(E^2u_{2n}, Fu_{2n+1}\right)\}\}^{\eta}.\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}\Upsilon(Es, s) &\leq\{\max\{\Upsilon(Es, s), \Upsilon(Es, Es), \Upsilon(s, s), \Upsilon(Es, s), \Upsilon(Es, s)\}\}^{\eta}\\ &= \{\max\{ \Upsilon(Es, s), 1, 1, \Upsilon(Es, s), \Upsilon(Es, s)\}\}^{\eta}\\ &= \{\Upsilon(Es, s)\}^{\eta},\end{aligned}$$

$$\Upsilon(Es, s) \leq\{\Upsilon(Es, s)\}^{\eta}.$$

This implies that, $\Upsilon(Es, s) = 1$, i.e., $Es = s$. (3.32)

Now, $s = Es \in E(M) \subset I(M)$, so there exists $\beta_1 \in M$ such that

$$s = I\beta_1. \tag{3.33}$$

On putting $u = Eu_{2n}$ and $v = \beta_1$ in (2) and using (3.30), (3.31), (3.32) and (3.33), we get

$$\Upsilon\left(E^2u_{2n},\ F\beta_1\right) \le \{\max\{\Upsilon\left(HEu_{2n},\ I\beta_1\right), \Upsilon\left(HEu_{2n},\ E^2u_{2n}\right), \Upsilon\left(I\beta_1,\ F\beta_1\right), \Upsilon\left(E^2u_{2n},\ I\beta_1\right), \Upsilon\left(HEu_{2n},\ F\beta_1\right)\}\}^{\eta}$$

Letting $n \to \infty$, we get

$$\Upsilon(Es,\ F\beta_1) \le \{\max\{\Upsilon(Es, s), \Upsilon(Es, Es), \Upsilon(s, F\beta_1), \Upsilon(Es, s), \Upsilon(Es, F\beta_1)\}\}^{\eta},$$

$$\Upsilon(s,\ F\beta_1) \le \{\Upsilon(s, F\beta_1)\}^{\eta}.$$

This implies that, $\Upsilon(s, F\beta_1) = 1$ i.e., $F\beta_1 = s$. (3.34)

Since, (I, F) is a pair of weakly commutative mappings and by using (3.33) and (3.34), we get

$$\Upsilon(Fs, Is) = \Upsilon(FI\beta_1, IF\beta_1) \le \Upsilon(F\beta_1, I\beta_1) = \Upsilon(s, s) = 1.$$

This implies that, $\Upsilon(Fs, Is) = 1$ i.e., $Fs = Is$. (3.35)

On putting $u = u_{2n}$ and $v = s$ in (2) and using (3.35), we get

$$\Upsilon(Eu_{2n}, Fs) \le \{\max\{\Upsilon(Hu_{2n}, Is), \Upsilon(Hu_{2n}, Eu_{2n}), \Upsilon(Is, Fs), \Upsilon(Eu_{2n}, Is), \Upsilon(Hu_{2n}, Fs)\}\}^{\eta}.$$

Letting $n \to \infty$, we get

$$\Upsilon(\mathrm{s}, Fs) \le \{\max\{\Upsilon(s, Fs), \Upsilon(s, s), \Upsilon(Fs, Fs), \Upsilon(s, Fs), \Upsilon(s, Fs)\}\}^{\eta}$$

$$= \{\Upsilon(s, Fs)\}^{\eta},$$

$$\Upsilon(s, Fs) \le \{\Upsilon(s, Fs)\}^{\eta}.$$

This implies that, $\Upsilon(s, Fs) = 1$ i.e., $Fs = s$. (3.36)

Now, $s = Fs \in F(M) \subset H(M)$, so there exists a point $\beta_2 \in M$ such that

$s = H\beta_2$. (3.37)

On putting $u = \beta_2$and $v = s$ in (2) and using (3.36) and (3.37), we get

$$\Upsilon(E\beta_2, s) = \Upsilon(E\beta_2, Fs)$$

$$\le \{\max\{\Upsilon(H\beta_2, Is), \Upsilon(H\beta_2, E\beta_2), \Upsilon(Is, Fs), \Upsilon(E\beta_2, Is), \Upsilon(H\beta_2, Fs)\}\}^{\eta}$$

$$= \{\Upsilon(E\beta_2, s)\}^{\eta},$$

$$\Upsilon(E\beta_2, s) \le \{\Upsilon(E\beta_2, s)\}^{\eta}.$$

This implies that, $\Upsilon(E\beta_2, s) = 1$ i.e., $E\beta_2 = s$. (3.38)

Since, (H, E) is a pair of weakly commutative mappings, therefore from (3.37) and (3.38), we get

$$\Upsilon(Hs, Es) = \Upsilon(HE\beta_2, EH\beta_2) \le \Upsilon(H\beta_2, E\beta_2) = \Upsilon(s, s) = 1.$$

This implies that, $\Upsilon(Hs, Es) = 1$ i.e., $Hs = Es$. (3.39)

Hence, $Hs = Es = Is = Fs = s$.

i.e., s is a *CFP* of mappings H, I, E, and F.

Case (iv). Suppose that F is continuous, then this can be proven similar to case (iii).

Uniqueness: It can be easily seen from inequality (2).

Hence mappings H, I, E, and F have a *CFP*. □

Theorem 3.3. *Let* (M, Υ) *be a complete multiplicative cone metric space and L be a multiplicative normal cone with multiplicative constant* Ψ. *Let E, F, H, I*: $M \to M$ *be four self-mappings of* (M, Υ), *which satisfy conditions* (1) - (3) *and the following condition:*

(6) *The pairs* (H, E) *and* (I, F) *are compatible.*

Then mappings E, F, H and I have a unique CFP.

Proof. Since $E(M) \subset I(M)$, consider a point $u_0 \in M$, there exists $u_1 \in M$ such that $Eu_0 = Iu_1 = v_0$. Now for this point u_1, there exist $u_2 \in M$ such that $Fu_1 = Hu_2 = v_1$. This continues to form sequences such that;

$$v_{2n} = Eu_{2n} = Iu_{2n+1}, \text{ and } v_{2n+1} = Fu_{2n+1} = Hu_{2n+1}.$$

Then it is clear from the proof of Theorem 3.1, sequence $\{v_n\}$ is a multiplicative Cauchy sequence in M. Now since M is multiplicative complete so, there is a point $s \in M$ s.t., $\lim_{n\to\infty} v_n = s$. Consequently, we have

$$\lim_{n\to\infty} Eu_{2n} = \lim_{n\to\infty} Iu_{2n+1} = \lim_{n\to\infty} Fu_{2n+1} = \lim_{n\to\infty} Hu_{2n+2} = s.$$

Because, $\{v_{2n}\} = \{Eu_{2n}\} = \{Iu_{2n+1}\}$, $\{v_{2n+1}\} = \{Fu_{2n+1}\} = \{Hu_{2n+1}\}$ are sub sequences of $\{v_n\}$.

Case (i). First, suppose that H is continuous then, we have

$$\lim_{n\to\infty} HEu_{2n} = \lim_{n\to\infty} H^2u_{2n} = Hs. \tag{3.40}$$

Since, (H, E) is a pair of compatible mappings, so it follows from the preposition (2.13), we have

$$\lim_{n\to\infty} EHu_{2n} = Hs. \tag{3.41}$$

Now, on putting $u = Hu_{2n}$ and $v = u_{2n+1}$ in (2) and using (3.41), we get

$$\Upsilon(EHu_{2n},\ Fu_{2n+1}) \le \{\max\{\Upsilon\left(H^2u_{2n},\ Iu_{2n+1}\right), \Upsilon\left(H^2u_{2n},\ EHu_{2n}\right), \Upsilon(Iu_{2n+1},\ Fu_{2n+1}), \Upsilon(EHu_{2n},\ Iu_{2n+1}), \Upsilon\left(H^2u_{2n},\ Fu_{2n+1}\right)\}\}^{\eta}$$

Letting $n \to \infty$, we get

$$\begin{aligned}\Upsilon(Hs, s) &\le \{\max\{\Upsilon(Hs, s), \Upsilon(Hs, Hs), \Upsilon(s, s), \Upsilon(Hs, s), \Upsilon(Hs, s)\}\}^{\eta}\\ &= \{\max\{\Upsilon(Hs, s), 1, 1, \Upsilon(Hs, s), \Upsilon(Hs, s)\}\}^{\eta}\\ &= \{\Upsilon(Hs, s)\}^{\eta},\end{aligned}$$

$$\Upsilon(Hs, s) \le \{\Upsilon(Hs, s)\}^{\eta}.$$

This implies that, $\Upsilon(Hs, s) = 1$, i.e., $Hs = s$. (3.42)

On putting $u = s$ and $v = u_{2n+1}$ in (2) and using (3.42), we get

$$\Upsilon(Es, Fu_{2n+1}) \le \{\max\{\Upsilon(Hs, Iu_{2n+1}), \Upsilon(Hs, Es), \Upsilon(Iu_{2n+1}, Fu_{2n+1}), \Upsilon(Es, Iu_{2n+1}), \Upsilon(Hs, Fu_{2n+1})\}\}^{\eta}.$$

Letting $n \to \infty$, we obtain

$$\begin{aligned}\Upsilon(Es, s) &\le \{\max\{\Upsilon(s, s), \Upsilon(s, Es), \Upsilon(s, s), \Upsilon(Es, s), \Upsilon(s, s)\}\}^{\eta}\\ &= \{\max\{1, \Upsilon(s, Es), 1, \Upsilon(Es, s), 1\}\}^{\eta}\\ &= \{\Upsilon(Es, s)\}^{\eta},\end{aligned}$$

$$\Upsilon(Es, s) \le \{\Upsilon(Es, s)\}^{\eta}.$$

This implies that, $\Upsilon(Es, s) = 1$, i.e., $Es = s$. (3.43)

From (3.42) and (3.43), we have

$$Hs = Es = s. \tag{3.44}$$

Now, $s = Es \in E(M) \subset I(M)$, so there exists $\delta \in M$, such that

$$s = Es = I\delta. \tag{3.45}$$

On putting $u = s$ and $v = \delta$ in (2) and using (3.44) and (3.45), we have

$$\begin{aligned}\Upsilon(s, F\delta) &= \Upsilon(Es, F\delta)\\ &\le \{\max\{\Upsilon(Hs, I\delta), \Upsilon(Hs, Es), \Upsilon(I\delta, F\delta), \Upsilon(Es, I\delta), \Upsilon(Hs, F\delta)\}\}^{\eta}\\ &= \{\max\{\Upsilon(s, s), \Upsilon(s, s), \Upsilon(s, F\delta), \Upsilon(s, I\delta), \Upsilon(s, F\delta)\}\}^{\eta}\\ &= \{\Upsilon(s, F\delta)\}^{\eta},\end{aligned}$$

$$\Upsilon(s, F\delta) \le \{\Upsilon(s, F\delta)\}^{\eta}.$$

This implies that, $\Upsilon(s, F\delta) = 1$, i.e., $F\delta = s$. (3.46)

Since (I, F) is a pair of compatible mappings and using (3.45) and (3.46), we get

$$s = I\delta = F\delta. \tag{3.47}$$

Now, by the preposition (2.12) we get, $IF\delta = FI\delta$, and hence,

$$Is = IF\delta = FI\delta = Fs. \tag{3.48}$$

On putting $u = s$ and $v = s$ in (2) and using (3.44) and (3.48) we have,

$$\begin{aligned}\Upsilon(s, Is) &= \Upsilon(Es, Fs)\\ &\le \{\max\{\Upsilon(Hs, Is), \Upsilon(Hs, Es), \Upsilon(Is, Fs), \Upsilon(Es, Is), \Upsilon(Hs, Fs)\}\}^{\eta}\\ &= \{\max\{\Upsilon(s, Is), \Upsilon(s, s), \Upsilon(Is, Fs), \Upsilon(s, Is), \Upsilon(s, Is)\}\}^{\eta}\\ &= \{\Upsilon(s, Is)\}^{\eta},\end{aligned}$$

$$\Upsilon(s, Is) \le \{\Upsilon(s, Is)\}^{\eta}.$$

This implies that, $\Upsilon(s, Is) = 1$, i.e., $Is = s$. (3.49)

Now, from (3.44), (3.48) and (3.49), we have

$$Hs = Es = Is = Fs = s.$$

Hence, s is the *CFP* of mappings H, I, E, and F.

Case (ii). Suppose that I is continuous, then this can be proven similar to case (i).

Case (iii). Suppose that E is continuous then,

$$\lim_{n\to\infty} EHu_{2n} = \lim_{n\to\infty} E^2u_{2n} = Es. \quad (3.50)$$

Since, (H, E) is a pair of compatible mappings, therefore, from preposition (2.13) it follows

$$\lim_{n\to\infty} HEu_{2n} = Hs. \quad (3.51)$$

On putting $u = Eu_{2n}$ and $v = u_{2n+1}$ in (2) and using (3.50) and (3.51), we get

$$\Upsilon\left(E^2u_{2n},\ Fu_{2n+1}\right) \leq \{\max\{\Upsilon\left(E^2u_{2n},\ Iu_{2n+1}\right), \Upsilon\left(E^2u_{2n},\ E^2u_{2n}\right), \Upsilon(Iu_{2n+1},\ Fu_{2n+1}), \Upsilon\left(E^2u_{2n},\ Iu_{2n+1}\right), \Upsilon(HEu_{2n},\ Fu_{2n+1})\}\}^{\eta}.$$

Letting $n \to \infty$, we obtain

$$\begin{aligned}\Upsilon(Es, s) &\leq \{\max\{\Upsilon(Es, s), \Upsilon(Es, Es), \Upsilon(s, s), \Upsilon(Es, s), \Upsilon(Es, s)\}\}^{\eta}\\ &= \{\max\{\Upsilon(Es, s), 1, 1, \Upsilon(Es, s), \Upsilon(Es, s)\}\}^{\eta}\\ &= \{\Upsilon(Es, s)\}^{\eta},\end{aligned}$$

$$\Upsilon(Es, s) \leq \{\Upsilon(Es, s)\}^{\eta}.$$

This implies that, $\Upsilon(Es, s) = 1$, i.e., $Es = s$. (3.52)

Now, $s = Es \in E(M) \subset I(M)$, so there exists $\delta_1 \in M$ such that

$$s = Es = I\delta_1. \quad (3.53)$$

On putting $u = Eu_{2n}$ and $v = \delta_1$ in (2) and using (3.50), (3.51), (3.52), and (3.53), we get

$$\Upsilon\left(E^2u_{2n},\ F\delta_1\right) \leq \{\max\{\Upsilon(HEu_{2n},\ I\delta_1), \Upsilon\left(HEu_{2n}, E^2u_{2n}\right), \Upsilon(I\delta_1,\ F\delta_1), \Upsilon\left(E^2u_{2n},\ I\delta_1\right), \Upsilon(HEu_{2n},\ I\delta_1)\}\}^{\eta}.$$

Letting $n \to \infty$, we get

$$\Upsilon(s, F\delta_1) \leq \{\max\{\Upsilon(s, s), \Upsilon(s, s), \Upsilon(s, F\delta_1), \Upsilon(s, s), \Upsilon(s, F\delta_1)\}\}^{\eta},$$

$$\Upsilon(s, F\delta_1) \leq \{\Upsilon(s, F\delta_1)\}^{\eta}.$$

This implies that, $\Upsilon(s, F\delta_1) = 1$ i.e., $F\delta_1 = s$. (3.54)

Since, (I, F) is a pair of compatible mappings and by using (3.53) and (3.54), we get $F\delta_1 = I\delta_1 = s$, so by the preposition (2.12), we have, $IF\delta_1 = FI\delta_1$ and hence, we get

$$Is = IF\delta_1 = FI\delta_1 = Fs. \quad (3.55)$$

On putting $u = u_{2n}$ and $v = s$ in (2) and using (3.55), we get

$$\Upsilon(Eu_{2n}, Fs) \leq \{\max\{\Upsilon(Hu_{2n}, Is), \Upsilon(Hu_{2n}, Eu_{2n}), \Upsilon(Is, Fs), \Upsilon(Eu_{2n}, Is), \Upsilon(Hu_{2n}, Fs)\}\}^{\eta}.$$

Letting $n \to \infty$, we get

$$\begin{aligned}\Upsilon(s, Fs) &\leq \{\max\{\Upsilon(s, Fs), \Upsilon(s, s), \Upsilon(Fs, Fs), \Upsilon(s, Fs), \Upsilon(s, Fs)\}\}^{\eta}\\ &= \{\Upsilon(s, Fs)\}^{\eta},\end{aligned}$$

$$\Upsilon(s, Fs) \leq \{\Upsilon(s, Fs)\}^{\eta}.$$

This implies that, $\Upsilon(s, Fs) = 1$ i.e., $Fs = s$. (3.56)

Now, $s = Fs \in F(M) \subset H(M)$, so there exists a point $\delta_2 \in M$ such that

$$s = Fs = H\delta_2. \quad (3.57)$$

On putting $u = \delta_2$ and $v = s$ in (2) and using (3.56) and (3.57), we get

$$\begin{aligned}\Upsilon(E\delta_2, s) &= \Upsilon(E\delta_2, Fs.)\\ &\leq \{\max\{\Upsilon(H\delta_2, Is), \Upsilon(H\delta_2, E\delta_2), \Upsilon(Is, Fs), \Upsilon(E\delta_2, Is), \Upsilon(H\delta_2, Fs)\}\}^{\eta}\\ &= \{\max\{\Upsilon(s, s), \Upsilon(s, E\delta_2), \Upsilon(Fs, Fs), \Upsilon(E\delta_2, s), \Upsilon(s, s)\}\}^{\eta}\\ &= \{\Upsilon(E\delta_2, s)\}^{\eta}.\end{aligned}$$

$\Upsilon(E\delta_2, s) \leq \{\Upsilon(E\delta_2, s)\}^\eta$.

This implies that, $\Upsilon(E\delta_2, s) = 1$ i.e., $E\delta_2 = s$. (3.58)

Since, (H, E) is a pair of compatible mappings in M, therefore, $E\delta_2 = H\delta_2 = s$ and by the preposition (2.12), we have $HE\delta_2 = EH\delta_2$.

Therefore, $Hs = HE\delta_2 = EH\delta_2 = Es$, i.e., $Hs = Es = Is = Fs = s$.

Hence, s is a *CFP* of mappings H, I, E, and F.

Case (iv). Suppose that F is continuous, then this can be proven similar to case (iii).

Uniqueness: It can be easily seen from inequality (2).

Hence mappings H, I, E, and F have a *CFP*. □

Example 3.4. Let $K = R$ and $L = \{u \in K\colon u \geq 1\}$ be a multiplicative cone in K. Let $\Upsilon\colon M \times M \to K$, where $M = R$ is a multiplicative metric defined as:

$$\Upsilon(u, v) = 2^{|u-v|}, \text{ for all } u, v \in M.$$

Then (M, Υ) is clearly a complete multiplicative cone metric space. Also, let the following four self-mappings $E, F, H, I\colon M \to M$ of multiplicative cone metric space (M, Υ) such that,

$$Eu = 2u,\ Fu = u,\ Hu = 4u,\ Iu = 6u,\ \forall\ u \in M.$$

Then, we can easily see that,

(1) Since $E(M) = F(M) = I(M) = H(M) = M$, so $E(M) \subset I(M)$, $F(M) \subset H(M)$.

(2) Let $\eta = \frac{1}{3} \in (0, \frac{1}{2})$, then from the inequality (2) of Theorem 3.1, we obtain

$$2^{|2u-v|} \leq \{\max\{2^{|4u-6v|}, 2^{|4u-2v|}, 2^{|6u-v|}, 2^{|2u-6v|}, 2^{|4u-v|}\}\}^\eta,$$

$$2^{|2u-v|} \leq \{\max\{2^{|4u-6v|\eta}, 2^{|4u-2v|\eta}, 2^{|6u-v|\eta}, 2^{|2u-6v|\eta}, 2^{|4u-v|\eta}\}. \quad (3.59)$$

Since, $v = lnu$ is an increasing mapping, so from (3.59), we get

$$|2u - v| \leq \{\max\{|4u - 6v|\,\eta, |2u|\,\eta, |5v|\,\eta, |2u - 6v|\,\eta, |4u - v|\,\eta\}\}, \forall\ u, v \in M.$$

Hence, mappings $H, I, E,$ and F satisfies the condition (2).

(3) $H, I, E,$ and F all are continuous mappings.

(4) Pair (H, E) and (I, F) are pairs of commutative mappings and according to remark 2.9, they must be weakly commutative and compatible.

Therefore, all the conditions of Theorem 3.1, Theorem 3.2, and Theorem 3.3 are satisfied and $H0 = I0 = E0 = F0 = 0$, i.e., 0 is the unique *CFP* of mappings H, I, E and F.

4. Conclusion

This paper aims to introduce commutative, weakly commutative, and compatible mappings to multiplicative cone metric space and by using these mappings and their properties, develop and generalize the results of common fixed points to multiplicative cone metric space.

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