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TOPOLOGICAL SPACES GENERATED BY GRAPH

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Abstract

In this paper we discuss topological spaces generated by simple graphs using adjacency relation and non adjaceny relation on vertices. We establish important results showing relations between complete graph and discrete topological space. We also discuss the topological spaces related to complete graphs, isomorphic graphs and study their properties. Further we discuss the interior and closure operators and their properties. Our motivation is to give an fundamental step toward linkage between topology and graph so as to study different aspects of graphs in terms of topological properties.
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1. Introduction

Graph theory has vast application in various fields like graphics, communication science, computer technology, image processing, data mining, town planning, electric and civil engineering, operation research, medical fields etc. Using graphs we can represent any system involving a binary relation in terms of mathematical model. Topology is a study of spaces invariant under continuous deformations. Topology has applications in various fields like molecular biology, robotics, remote sensing, data analysis, concurrency theory etc.

A graph can be considered as one dimensional topological spaces of a type [4]. Both graph theory and topology claimed to have their origin in famous Eulers seven bridge problem of 1736 . So, the study of interdependence of graph theory and topology can be quite useful. The notions of connected graphs, homeomorphic graphs have the same meaning as in topology, while the notions of homology in topology can be used in solutions of problems of graph theory. Abedal-Hamza Mahadi [1] constructed a topology on undirected graph.Allam[3] discussed a new method to generate topology on graph by using new method of taking neighborhood is determining two fixed vertices on the graph.Zhang and Li[7] discussed topological properties of a pair of relation based generalized approximation operators like interior and closure operators. Allam, Bakeir and Abo-tabl[2] studied the topologies associated with closure operators.

In this paper we define a topology using adjacency relation on a simple graph and study their properties.

2. Preliminaries

Definition 2.1 ([4, 6])**.** *A graph G is a ordered set of vertices $V(G)$ and edges $E(G)$. If $V(G)$ and $E(G)$ are finite then G is said to be finite graph and if either $V(G)$ or $E(G)$ or both are infinite then G is said to be infinite graph.*

Definition 2.2 ([6])**.** *Two vertices a and b are said to be adjucent if they are joined by an edge. Two distinct edges ab and bc are said to be adjacent if they are joined by common vertex b. The vertex a and edges joined to a are said to be incident with each other.*

Definition 2.3 ([4])**.** *A graph G has a loop if G contains an edge joining a vertex to itself. A graph G has multiedges if it has more than one edge joining two vertices. A graph G with multiedges and without loop is called multigraph. A graph G with loop and multiedges is called pseudograph. A graph G is said to be simple if it has no loops and no multiedges.*

Definition 2.4 ([4])**.** *A graph G is said to be connected if every pair of vertices are joined by a path.A graph G is said to be complete graph if every pair of vertices in G are adjacent.*

Definition 2.5 ([4])**.** *Two graphs G and H are isomorphic graphs if there exist one one map between their vertices which preserves adjacency.*

Definition 2.6. *A binary relation R on non empty set X is a subset of $X \times X$.*

Definition 2.7 ([5])**.** *Let X be a non empty ste and P(X) be the power set of X. A collection* $\tau \subseteq P(X)$ *is said to be a topology on X if the following conditions are satisfied,*
1. X and $\phi \in \tau$
2. The intersection of finite members of τ *and an arbitrary union of members of* τ *are in* τ*.*
An ordered pair (X, τ) *is called a topological space and the members of topological spaces are called open sets.*

Definition 2.8 ([5])**.** *A topology* τ *on X is called discrete topology if* $\tau = P(X)$*. A topology* τ *on X is called an indiscrete topology if* $\tau = \{\phi, X\}$*.*

Definition 2.9 ([5])**.** *A collection of subsets of X whose union equals X is called a subbasis S for a topology* τ *on X. The coolection of all unions of finite intersections of members of S is a topology* τ *generated by subbasis S.*

3. Topology Generated by Simple Connected Graph

In this section first we define subbasis and basis for a topology on given simple connected graph by using adjacency relation of vertices.

Definition 3.1. *Let* $G = (V(G), E(G))$ *be the simple connected graph and let* $X = V(G)$*. On set of vertices X we define an adjancy relation R as follows,*

$$(u, v) \in R \text{ if } u \text{ is adjacent to } v \text{ for } u, v \in X$$

Now for $u \in X$*, we define as* $R[u]=\{v \in X / (u, v) \in R\}$*.*
Then the set $S_G = \{R[v]/v \in X\}$ *forms a basis for a topology on X as it is collection of subset of X whose union equals X.Let* β_G *be the finite intersection of members of subbasis* S_G *then clearly* β_G *forms a basis. The collection* τ_G *of all unions of members of* β_G *is a topology on X. We called* τ_G *as a topology generated by a graph G and the ordered pair* (X, τ_G) *as a topological space generated by graph G.*

Example 3.1. Let G_1 be graph with vertex set $V(G_1) = \{a, b, c, d\}$ then

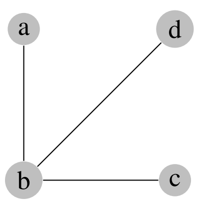

Figure 3.1: G_1

$R[a] = \{b\}, R[b] = \{a, c, d\}, R[c] = \{b\}, R[d] = \{b\},$

$S_{G_1} = \{\{b\}, \{a, c, d\}\},$

$\beta_{G_1} = \{\phi, \{b\}, \{a, c, d\}\},$

$\tau_{G_1} = \{\phi, \{b\}, \{a, c, d\}, X\}.$

Example 3.2. Let G_2 be graph with vertex set $V(G_2) = \{a, b, c, d\}$ then

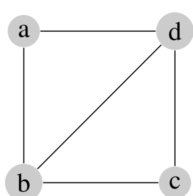

Figure 3.2: G_2

$R[a] = \{b, d\}, R[b] = \{a, c, d\}, R[c] = \{b, d\}, R[d] = \{a, b, c\},$

$S_{G_2} = \{\{b, d\}, \{a, c, d\}, \{a, b, c\}\},$

$\beta_{G_2} = \{\phi, \{b\}, \{d\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, c, d\}\},$

$\tau_{G_1} = \{\phi, \{b\}, \{d\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, c, d\}, X\}.$

Example 3.3. Let G_3 be graph with vertex set $V(G_3) = \{a, b, c, d, e\}$ then

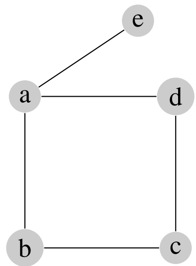

Figure 3.3: G_3

$$R[a] = \{b, d, e\}, R[b] = \{a, c\}, R[c] = \{b, d\}, R[d] = \{a, c\},$$

$$R[e] = \{a\},$$

$$S_{G_3} = \{\{a\}, \{a, c\}, \{b, d\}, \{b, d, e\}\},$$

$$\beta_{G_3} = \{\phi, \{a\}, \{a, c\}, \{b, d\}, \{b, d, e\}\},$$

$$\tau_{G_3} = \{\phi, \{a\}, \{a, c\}, \{b, d\}, \{b, d, e\}, X\}.$$

Theorem 3.1. *If G is complete graph then the topological space generated by G is discrete.*

Proof. Let G be complete graph with n vertices such that $V(G) = \{v_1, v_{2,...,v_n}\}$, denote $V(G) = X$. Let R be adjacency relation on set of vertices X and $R[u] = \{v \in X/(u, v) \in R\}$ for $u \in X$ then by the definition 3.1 , the set $S_G = \{R[v_i]/v_i \in X, i = 1, 2, ..., n\}$ forms a subbasis for a topology on X.

Since G be complete graph with n vertices,therefore on simplifying S_G we get,

$$S_G = \{X - \{v_1\}, X - \{v_2\}, ..., X - \{v_n\}\}. \tag{3.1}$$

Let a basis β_G is the finite intersection of members of subbasis S_G. Clearly from equation 3.1, $\beta_G \subseteq P(X)$, where $P(X)$ is a power set of X and contains all singleton subset of X. Hence the collection of all union of members of β_G is a topology. Therefore $\tau_G = P(X)$. Hence the topological space generated by complete graph is discrete space. □

Example 3.4. Let G_4 be complete graph with vertex set $V(G_4) = \{a, b, c\}$ then

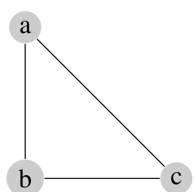

Figure 3.4: G_4

$$S_{G_4} = \{\{a, b\}, \{a, c\}, \{b, c\}\},$$

$$\beta_{G_4} = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\},$$

$$\tau_{G_4} = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}.$$

Therefore $\tau_{G_4} = P(X)$.

Example 3.5. Let G_5 be complete graph with vertex set $V(G_5) = \{a, b, c, d\}$ then

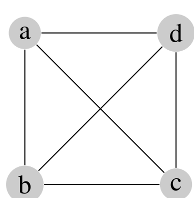

Figure 3.5: G_5

$$S_{G_5} = \{\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\},$$

$$\beta_{G_5} = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, d\}, \{b, d\}, \{c, d\},$$

$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\},$$

$$\tau_{G_5} = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, d\}, \{b, d\}, \{c, d\}.$$

$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}$$

Therefore $\tau_{G_5} = P(X)$.

Remark 3.1. Converse of above theorem need not be true. A topological space generated by a graph G is discrete then G need not be complete graph.

Example 3.6. Let G_6 be a simple connected graph with vertex set $V(G_6) = \{a, b, c, d, e\}$ then clearly G_6 is not a complete graph but a topological space τ_{G_6} generated by G_6 is discrete.

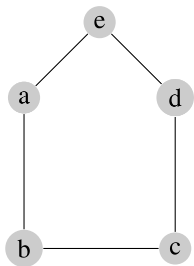

Figure **3.6:** G_6

$$S_{G_6} = \{\{a,d\},\{a,c\},\{b,e\},\{b,d\},\{c,e\}\},$$

$$\beta_{G_6} = \{\phi,\{a\},\{b\},\{c\},\{d\},\{e\},\{a,d\},\{a,c\},\{b,e\},\{b,d\},\{c,e\}\},$$

$$\begin{aligned}\tau_{G_6} = \{&\phi,\{a\},\{b\},\{c\},\{d\},\{e\},\{a,b\},\{a,c\},\{b,c\},\{a,d\},\{b,d\},\{c,d\},\\ &\{a,e\},\{b,e\},\{c,e\},\{d,e\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\},\\ &\{a,c,e\},\{a,d,e\},\{a,b,e\},\{b,c,e\},\{b,d,e\},\{c,d,e\},\{a,b,c,d\},\\ &\{a,b,c,e\},\{a,b,d,e\},\{a,c,d,e\},\{b,c,d,e\},X\} = P(X).\end{aligned}$$

Therefore a topological space τ_{G_6} is discrete space but G_6 is not a complete graph.

Theorem 3.2. *If two simple connected graphs are isomorphic then the topological spaces generated by these graphs are also homeomorphic.*

Proof. Let G and H are two simple connected isomorphic graphs then both of them have same number of vertices and edges which preserves adjacency. Let S_G and S_H are the subbases for the topologies generated by G and H respectively then the sets S_G and S_H are equivalent, since they are constructed by using adjacency relation of vertices. Hence the topological spaces generated by G and H must be homeomorphic. □

Remark 3.2. Converse of above theorem need not be true. If topological spaces generated by two simple connected graphs are homeomorphic then these two graphs may not be isomorphic.

Example 3.7. Let G_7 be a complete graph with vertex set $V(G_7) = \{a,b,c,d,e\}$ them from Theorem 3.1, topological space τ_{G_7} generated by G_7 is discrete, see Figure 3.7 given below. Consider Example 3.6, the topological space τ_{G_6}generated by G_6 is also discrete and vetex set of G_6 is equal to vertex set of G_7.

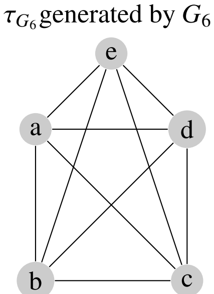

Figure **3.7:** G_7

Hence the topological spaces τ_{G_6} and τ_{G_7} are homeomorphic but the corresponding graphs G_6 and G_7 are not isomprphic because G_6 is not complete graph but G_7 is complete.

Theorem 3.3. *If H is a connected subgraph of a simple connected graph G then $\tau_H \subseteq \tau_G$.*

Proof. Straightforward. □

4. Interior Operator of a Graph

Definition 4.1. *Let G be the simple connected graph with vertex set X. Let R be adjacency relation on set of vertices X and $R[u] = \{v \in X/(u,v) \in R\}$ for $u \in X$. Let A be any subset of X then we define the interior operator of A as $intA = \{v \in A/R[v] \subseteq A\}$.*

Theorem 4.1. *Let G be the simple connected graph with vertex set X. Let R be the adjacency relation on set of vertices X and $R[u] = \{v \in X/(u,v) \in R\}$ for $u \in X$. If A and B are any two subsets of X then*
(a) *$intX = X$ and $int\phi = \phi$,*
(b) *If $A \subseteq B$ then $intA \subseteq intB$,*
(c) *$intA \cap intB = int(A \cup B)$,*
(d) *$intA \cup intB \subseteq int(A \cup B)$.*

Proof. **(a)** Since $intA = \{v \in X/R[v] \subseteq X\}$. Hence if $A = X$ then clearly $intA = X$. Obviously $int\phi = \phi$.
(b) Let $A \subseteq B$. Let $v \in intA$ be any element then $v \in A$ and $R[v] \subseteq A$. Since $A \subseteq B$ inplies $v \in B$ and $R[v] \subseteq B$. Hence by definition of interior operator, $v \in intB$. Therefore $intA \subseteq intB$.
(c) For any element v, $v \in int(A \cap B)$
$\Leftrightarrow v \in (A \cap B)$ such that $R[v] \subseteq (A \cap B)$, since by definition of interior operator.
$\Leftrightarrow v \in A$ such that $R[v] \subseteq A$ and $v \in B$ such that $R[v] \subseteq B$.
$\Leftrightarrow$ $v \in intA$ and $v \in intB$.
Hence $intA \cap intB = int(A \cup B)$.
(d) Let $v \in int(A \cup B)$ be any elements then $v \in intA$ or $v \in intB$.Hence by definition of interior operator, $v \in A$ and $R[v] \subseteq A$ or $v \in B$ and $R[v] \subseteq B$. This implies $v \in A$ or $v \in B$ and $R[v] \subseteq A$ or $R[v] \subseteq B$. Therefore $v \in (A \cup B)$ and $R[v] \subseteq (A \cup B)$.Hence $v \in int(A \cup B)$. Therefore $intA \cup intB \subseteq int(A \cup B)$. □

Example 4.1. Consider Example 3.3, Let $X = \{a, b, c, d, e\}$ and $A = \{a, b, c\}$, $B = \{a, b, e\}$, $C = \{a, b, c, d\}$ and $D = \{a, b, c, e\}$ are all subset of X then $intA = \{b\}$, $intB = \{e\}$, $intC = \{b, c, d\}$ and $intD = \{b, e\}$. Obviously $intX = X$, since $A \subseteq D$ and $intA \subseteq intB$. Hence Theorem 4.1(b) is verified. Now $intA \cap intB$ and since $A \subseteq C = \{a, b, c\}$, therefore $int(A \cap C) = \{b\}$, hence Theorem 4.1(c) is verified.Similarly $(int(B) \cup int) = \{b, c, d, e\}$ and since $(B \cup C) = \{a, b, c, d, e\}$, therefore $int(B \cup C) = \{a, b, c, d, e\}$. Hence Theorem 4.1(d) is verified.

Remark 4.1. $int(A \cup B) \subseteq (intA \cup intB)$ is need not be true in general because from above example $(intB \cup intC) = \{b, c, d, e\}$ and $int(B \cup C) = \{a, b, c, d, e\}$. Therefore $int(A \cup B) \nsubseteq (intA \cup intB)$.

5. Closure Operator of a Graph

Definition 5.1. *Let G be the simple connected graph with vertex set X. Let R be adjacency relation on set of vertices X and $R[u] = \{v \in X/(u, v) \in R\}$ for $u \in X$. Let A be any subset of X then we define the closure operator of A as $clA = A \cup \{v \in X/(R[v] \cap A) \neq \phi\}$.*

Theorem 5.1. *Let G be the simple connected graph with vertex set X. Let R be the adjacency relation on set of vertices X and $R[u] = \{v \in X/(u, v) \in R\}$ for $u \in X$. If A and B are any two subsets of X then*
(a) *$clX = X$ and $cl\phi = \phi$,*
(b) *If $A \subseteq B$ then $clA \subseteq clB$,*
(c) *$(clA \cup clB) = cl(A \cup B)$,*
(d) *$cl(A \cap B) \subseteq (clA \cap clB)$.*

Proof. **(a)** Since $clA = A \cup \{v \in X/(R[v] \cap A) \neq \phi\}$.Hence if $A = X$ then clearly $clX = X$ and obviously $cl\phi = \phi$.
(b) Let $A \subseteq B$.Let $v \in clA$ be any element then $v \in A$ or $R[u] \cap A \neq \phi$.Now since $A \subseteq B$ implies that $v \in B$. Hence $v \in clB$ and $clA \subseteq clB$.
(c) Let $v \in (clA \cup clB)$ be any element then $v \in clA$ or $v \in clB$.Then by definition of closure operator, $v \in A$ or $v \in B$ or $(R[v] \cap A) \neq \phi$ or $(R[v] \cap B) \neq \phi$. This implies $v \in (A \cup B)$ or $R[v] \cap (A \cup B) \neq \phi$. Hence $v \in cl(A \cup B)$. But $v \in (clA \cup clB)$ thus

$$(clA \cup clB) \subseteq cl(A \cup B). \tag{5.1}$$

Now Let $v \in cl(A \cup B)$ be any element.Then by definition of closure operator,$v \in (A \cup B)$ or $R[v] \cap (A \cup B) \neq \phi$.This implies $v \in A$ or $v \in B$ or $(R[v] \cap A) \neq \phi$ or $(R[v] \cap B) \neq \phi$. Hence $v \in (clA \cup clB)$. But $v \in cl(A \cup B)$ thus $v \in cl(A \cup B)$ thus

$$cl(A \cup B) \subseteq (clA \cup clB). \tag{5.2}$$

Therefore from equations 5.1 and 5.2 $(clA \cup clB) = cl(A \cup B)$.
(d) Let $v \in cl(A \cap B)$ be any element. Then by difinition of closure operator, $v \in (A \cap B)$ or $[R[v] \cap (A \cap B)] \neq \phi$. This implies $v \in A$ and $v \in B$ or $R[v] \cap A \neq \phi$ and $R[v] \cap B \neq \phi$.Thus $v \in A$ or $R[v] \cap A \neq \phi$ and $v \in B$ or $R[v] \cap B \neq \phi$. Therefore $v \in cl(A) \cap cl(B)$ and hence $cl(A \cap B) \subseteq (clA \cap clB)$. □

Example 5.1. Consider Example 3.1, let $X = \{a, b, c, d\}$ and $A = \{a, b, c\}$, $B = \{a, b\}$, $C = \{b, d\}$, $D = \{a, c, d\}$, $E = \{a, b, d\}$, $F = \{b, c, d\}$ and $I = \{c, d\}$ are all subset of X then $clA = \{a, b, c, d\}$, $clB = \{a, b, c, d\}$, $clC = \{a, b, c, d\}$, $clD = \{a, b, c, d\}$, $clE = \{a, b, c, d\}$, $clF = \{a, b, c, d\}$ and $clI = \{c, b, d\}$.Obviously $clX = X$. Since $C \subseteq E$ hence $clC \subseteq clE$, Theorem 5.1(b) is verified. Now $clC \cup clD = \{a, b, c, d\}$ and since $C \cup D = \{a, b, c, d\}$ therefore $cl(C \cup D) = \{a, b, c, d\}$, Theorem 5.1(c) is verified. Similarly $clC \cap clI = \{b, c, d\}$ and since $C \cap I = \{\phi\}$, hence $cl(C \cap I) = \{\phi\}$, Theorem .1(d) is verified.

Remark 5.1. $(clA \cap clB) \subseteq cl(A \cap B)$ is need not be true in general because from above example $clC \cap clI = \{b, c, d\}$ and $cl(C \cap I) = \{\phi\}$. Hence $(clC \cap clI) \nsubseteq cl(C \cap I)$.

6. Topology Generated by Simple Graph

Definition 6.1. *Let $G = (V(G), E(G))$ be the simple graph and let $X = V(G)$. Let R be adjacency relation on set of vertices X and $R[u] = \{v \in X/(u, v) \in R\}$ for $u \in X$. Then the set $S_G = X \cup \{R[v]/v \in X\}$ forms a subbasis for a topology on X as it is collection of subset of X whose union equals X.Let β_G be the finite intersection of members of subbasis S_G then clearly β_G forms a basis. The collection τ_G of all unions of members of β_G is a topology on X. We called τ_G as a topology generated by graph G and the ordered pair (X, τ_G) as a topological space generated by graph G.*

Example 6.1. Let G_8 be a graph with vertex set $V(G_8) = \{a, b, c, d\}$ then

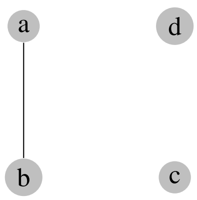

Figure **6.1:** G_8

$$R[a] = \{b\}, R[b] = \{a\}, R[c] = \{\phi\}, R[d] = \{\phi\},$$

$$S_{G_8} = X \cup \{\{a\}, \{b\}\},$$

$$\beta_{G_8} = \{\phi, \{a\}, \{b\}, X\},$$

$$\tau_{G_8} = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}.$$

Theorem 6.1. *G is null graph $\Leftrightarrow$ topology τ_G generated by G is indiscrete.*

Proof. Obvious. □

Example 6.2. Let G_9 be a null graph with vertex set $V(G_9) = \{a, b, c, d, e\}$ then

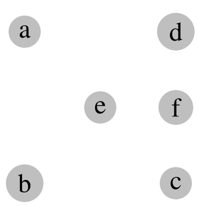

Figure **6.2:** G_9

$R[a] = \{\phi\}, R[b] = \{\phi\}, R[c] = \{\phi\}, R[d] = \{\phi\}, R[e] = \{\phi\}$
and $R[f] = \{\phi\}$,

$$S_{G_9} = X \cup \{\phi\},$$

$$\beta_{G_9} = \{\phi, X\},$$

$$\tau_{G_9} = \{\phi, X\}.$$

Hence τ_{G_9} topology generated by G_9 is indiscrete.

7. Topology generated by simple graph using non adjacency relation

On the same line we define a topology on simple graph by using non adjacency relation of vertices as fallows.

Definition 7.1. *Let $G = (V(G), E(G))$ be the simple graph and let $X = V(G)$. On set of vertices X we define a non adjacency relation R' as $(u, v) \in R'$ if u is not adjacent to v for $u, v \in X$.Now for $u \in X$, we define $R'[u] = \{v \in X/(u, v) \in R'\}$. Then the set $S'_G = X \cup \{R'[u]/v \in X\}$ froms a subbasis for a topology on X, as it is collection of subsets of X whose union equals to X.Let β'_G be the finite intersection of members of subbasis S'_G then clearly β'_G forms a basis. The collection τ'_G of all unions of members of β'_G is a topology on X. We called τ'_G as a topology generated by graph G and the ordered pair (X, τ'_G) as a topological space generated by graph G.*

Example 7.1. Let G_{10} be a graph with vertex set $V(G_{10}) = \{a, b, c, d\}$ then

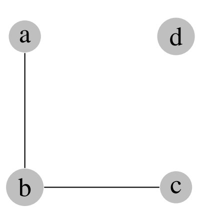

Figure **7.1:** G_{10}

$$R'[a] = \{c, d\}, R'[b] = \{d\}, R'[c] = \{a, d\}, R'[d] = \{a, b, c\},$$

$$S'_{G_{10}} = X \cup \{\{d\}, \{a, d\}, \{c, d\}, \{a, b, c\}\},$$

$$\beta'_{G_{10}} = \{\phi, \{a\}, \{c\}, \{d\}, \{a, d\}, \{c, d\}, \{a, b, c\}, X\},$$

$$\tau'_{G_{10}} = \{\phi, \{a\}, \{c\}, \{d\}, \{a, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, X\}.$$

Theorem 7.1. *If G is complete graph then topological space generated by non adjancency relation on vertices is indiscrete.*

Proof. Let G be a complete graph with vertex set X.Then for each $v \in X$, the non adjancency relation $R'[v] = \phi$. By the definition 7.1, $S'_{_G} = X \cup \{\phi\}$ and $\beta'_G = \{\phi, X\}$.Hence the topology τ'_G related to basis β'_G must be indiscrete. □

Theorem 7.2. *If G is null graph then topological space generated by non adjancency relation on vertices is discrete.*

Proof. Obvious. □

8. Conclusion

In this paper we established a relation between graphs and topological spaces. We have a topology generated on graph by using adjacency relation and non adjacency relation. We studied graphs by using topological aspects.

There is scope for further study on topological aspects of graphs. Using the interior and closure operator as defined in paper topologies can be formed and there interrelationship may give some useful insights.

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APPLICATION OF MATLAB IN REAL DECISION MAKING PROBLEM

By

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Abstract

The aim of this paper is to analyze the maximum age group of women affected by the divorce problem using fuzzy matrix method. Study of this real world problem is based on four types of different matrices, known as initial raw data matrix (*IRDM*), average time dependent data matrix (*ATDM*), refined time dependent data matrix (*RTDM*), and combined effect time dependent data matrix (*CETDM*). For this study the data has been obtained from 110 divorced women in Delhi and *NCR*, India. In order to estimate maximum age group of women influenced by divorce problem, some graphical representations are shown for different values of ?, $0 \leq \alpha \leq 1$ using algebraic applications of fuzzy matrices. Abdul et al. [13] faced different type of problem like if we add one or more attributes row wise or column wise then matrix will become bigger and complexity will increase during calculation. Due to it we use matlab code to solve each part of this problem in this paper and got best results.

2020 Mathematical Sciences Classification: 15B15.

Keywords and Phrases: Fuzzy matrix theory, maximum age group, divorce problem, divorced women.

1. Introduction

Family is one of the essential steps in bulding a socity. The rise in divorce cases and divorce rates is an indicator of the disturbance in that building. In the modern days, as the time passes, divorce cases and divorced womenhave goneup. Divorced women all around the world survive in a hard situation, these women faced many social, economical and financial problems**.** Theory of fuzzy matrixes was firstly proposed by **Vasantha Kandasamy** and Indera [15] to study the transportation problem. They have defined four types of different matrices, initial raw data matrix (*IRDM*), average time dependent data matrix (*ATDM*), refined time dependent data matrix (*RTDM*) and combined effect time dependent data matrix (*CETDM*), respectively. Many other mathematicians and researchers have worked on fuzzy matrices and their applications in the real word situations. Vasantha Khandasamy et al. [16] estimated the maximum age-group of rag pickers using *CETD* matrix method. Kalaichelvi and Kalaivanan [9] worked on Beneficiaries' attitude towards education loan - an analysis through fuzzy matrices in this paper they attempted to analyze the group of beneficiaries having maximum level of favourable attitude towards the education loan using fuzzy matrices. Education, especially the higher education, contributes much for the socio-economic development of our country. Education for all is perceived as the national goal to be achieved and efforts are being taken by launching various schemes by both the Central and State governments. But increasing cost of education caused by liberal privatization of higher education in India resulted in financial hardships to students who are economically poor but academically bright to pursue higher education. In addition to the different scholarships offered by the government, the education loan scheme launched by the commercial banks based on the directions from the government raised the hope of the needy people that their dream of higher education is going to be a true. Education loans are provided by all commercial banks to deserving students for pursuing graduation, post-graduation and professional courses. A maximum of Rs 10 lakhs is sanctioned for studies in India and it is Rs.20 lakhs for studies in abroad. Education loans upto four lakh rupees require neither margin money nor collateral security. In case of higher loan amount, margin money of 5% is insisted for studies in India and it is 15% for studies in abroad. The borrowers need not repay any amount during the moratorium period. The borrowers are expected to repay loan installments only six months after getting job or one year after the course is completed whichever happens earlier. The education loan interest waiver scheme announced by the central government in the union budget 2009-2010 received warm reception with gratitude from the student community, especially the economically poor section. Under this scheme the students whose family income is below four lakh rupees need not pay interest pertaining to the moratorium period. The statistics reveal a steady growth in the number of beneficiaries. Till March,2009 education loan totaling Rs.24,000 crore had been disbursed to 16 lakh students across the country. It is forecasted to touch Rs.50,000 crore within 2015(Union budget 2009-10). Victor Devadoss and Clement Joe Anand [5] investigated dimensions of personality of women using *CETD* matrix and the objective of their work is to find out the peak age of women gets anger in Chennai, for that they have studied the Dimensions of personality of women. It has been classified in to five factors as Openness, Conscientiousness, Extraversion, Agreeableness, and Negative

Emotion. Each Dimension has six facets. Albert William et al. [22] studied the breast cancer problem in women using *RTDM* matrices. Kuppuswami et al. [13] worked on traffic flow problem using *CETD* matrix. Amudhambigai and Sugaipriya [1] workedon demonetization problem using fuzzy matrix theory. Kalaichelvi and Gnanamalar [6] workedon the coffee cultivators in kodai hills. They said the coffee industry of India is the sixth largest producer of coffee in the world. Indian coffee is said to be the finest coffee grown in the shade rather than direct sunlight anywhere in the world. Three southern states of South India (Karnataka, Kerala and Tamil Nadu) account for 98 % of coffee production in India. Well known varieties of coffee grown are the Arabica and Robusta. In the study area Arabica variety is grown. Coffee is a labour-intensive crop and in Kodai Hills coffee is grown under monsoon rainfall conditions. As the coffee contributes significantly for the national economy and the growers face many hardships in coffee cultivation, a research has been conducted to study the problems encountered by them and inferences were drawn using fuzzy matrices. Kokila [11] worked on fuzzy matrix analysis of students information gathering attitude. According to her Fuzzy matrices play an important role in the formulation and analysis of many classes of discrete structural models which are in physical, biological, medical, social and engineering sciences. The objective her work is to study the frequency of subject wise information gathering attitude of college students using fuzzy matrix technique [5,8,14,15].

In the present paper, we have investigated maximum age group of women affected by the divorce problem using fuzzy matrices. The rest of the paper is organized in different sections. In section 2, basic definitions and arithmetic operations are performed on initial raw data matrix. In section 3, ten attributes are considered for collecting the data on the basis of age-group of divorced women. Section 4, defined and divided into different types of fuzzy matrices. In section 5, some graphical representations are shown for distinct values of α, $0 \leq \alpha \leq 1$ using algebraic applications of fuzzy matrices. Finally, results and conclusionsare given in the last section.

2. Motivation and Limitation of Proposed Paper

Abdul et al. [13] faced different type of problem and saw different type of complexity in these problems. Matlab can remove this complexity. This is the first problem which is solve by using Matlab.

- We can increase or decrease attributes row wise/ column wise.
- Calculation of big matrix is easy by using Matlab.
- Converting one matrix into another matrix is so easy.
- We write simple matlab code for each part of the problem. These codes will reduce calculation time and give better results.
- We will represent results graphically by writing matlab codes.
- Matlab code make problems user friendly.

3. Preliminaries

This section begins with fundamental preliminaries based on fuzzy matrix theory.

3.1. Method of obtaining *IRDM* matrix

An initial raw data matrix is obtained by transformation of raw data into the form of matrix. The order of *IRDM* matrix depends on the number of attributes taking for columns and number of class intervals as age groups taken for rows. If there are m attributes and n class intervals, the constructed *IRDM* matrix will be of order $m \times n$.

3.2. Method of obtaining *ATDM* matrix

An average time dependent data matrix is obtained by dividing each entry of the *IRDM* matrix by length of respective class interval. The data values of *ATDM* matrix are uniform in nature.

3.3. Method of obtaining *RTDM* matrix

A number of refined time dependent data matrixesare obtained using simple mean and standard deviation techniques. The data entries represented by *RTDM* matrices are 1, -1 and 0. The refined time dependent data matrices also names as fuzzy matrices. The mathematical formulas used for converting *ATDM* matrix into distinct *RTDM* matrices are as follows:

If $b_{ij} \leq \mu_j - \alpha\sigma_j$ then $e_{ij} = -1$.
If $b_{ij} ? (\mu_j - \alpha\sigma_j, \mu_j + \alpha\sigma_j)$ then $e_{ij} = 0$.
If $b_{ij} \geq \mu_j + \alpha\sigma_j$ then $e_{ij} = 1$.

$$\mu_j = \frac{\sum_{i=1}^{m} x_{ij}}{m} \forall j = 1, 2, \ldots, n,$$

$$\sigma_j = \frac{\sum_{i=1}^{m} (x_{ij} - \mu_j)^2}{m} \forall j = 1, 2, \ldots, n.$$

3.4. Method of obtaining *CETDM* matrix

A combined effect time dependent data matrix is obtained by combining *RTDM* matrices for different values of $\alpha, 0 \leq \alpha \leq 1$. The data represented by *CETDM* matrix gives cumulative effect of all these entries [2, 7].

4. Social attributes and their short descriptions to estimate maximum age group of women affected by divorce system

In this section, we have considered following ten social attributes for constructing a linguistic questionnaire to collect the raw data in the form of initialraw data matrix [6,9,12].

***SA*1**- Husband Extra Marital Affairs
***SA*2**- Husband Unemployment
***SA*3**- Lack of Communication
***SA*4**- Arguing and Abusing
***SA*5**- Quixotic Expectations
***SA*6**- Couple Without Kids
***SA*7**- Lack of Affinity
***SA*8**- Lack of Equality
***SA*9**- Different interests and priorities
***SA*10**- Inability to resolve conflicts

***SA*1- Extra Marital Relations**

Husband extra marital relationsare very hurtful for women and break the loyalty and trust between them. Most of the times husband infidelity may be the main reason for the women divorce cases.

***SA*2- Husband Unemployment**

Money makes people happy and touches almost everything in the world. Clearly, all financial goals are based on employment of husband particularly, when the women are housewives and more spending, causing a breakdown of most marriages into divorces.

***SA*3- Lack of Communication**

As the good communication is the foundation of strong bonding between married coupled. In fact, lack of communication developed some kind of noise, resentment and frustration. Effective communication is very crucial in marriage and it impacts all aspects of a marriage.

***SA*4- Arguing and Abusing**

Constant arguing and abusing (emotional or physical) about same things again and again kill many relationships and come to the end of a marriage.

***SA*5-Quixotic Expectations**

When one or both persons have unrealistic expectations. These quixotic expectations can make a stress on the other person, leaving down feelings and setting up a failure in the pre-assumed expectations.

***SA*6- Couple Without Kids**

If the married couple has no kids, after a long period of married life, it becomes one of the major issues of increasing into divorce cases and divorce rate.

***SA*7- Lack of Affinity**

When one spouse or both persons are not well feeling or connected to each other by emotions and love that can a cause of many divorces.

***SA*8- Lack of Equality**

If any one spouse has more responsibility in the marriage as compare to the other one, it can createresentment and become a main reasonfor ending a marriage life.

***SA*9- Different interests and priorities**

For a successful marriage life, it is essential to have common interests and exploring them together. If the married couple has different priorities and interests, it can also create some unbalancing situations between husband and wife and comes to the end of marriage life.

***SA*10- Inability to resolve conflicts**

Every married couple has some conflicts and disagreements between them, so there must be certain ground rules to resolve these conflicts. Sometimes a third party can help to develop ground rules and teach them to move through conflicts, otherwise these conflicts kill all aspects of a married life.

Based on above assumesten attributes, we have collected data from 110 divorced women in Delhi and NCR, as given below in Table 4.1

Table 4.1: Number of women respondent based on their age group

Age Group	Number of Respondent
18-22	22
23-27	22
28-32	22
33-37	22
38-42	11
43-47	11
	110

The initial raw data matrix of women affected by divorce system is obtained by taking above attributes as the columns and age groups in years 18-22, 23-27, 28-32, 33-37, 38-42, 42-47 as the rows, respectively [7].

Age_Group = {'18-22';'23-27';'28-32';'33-37';'38-42';'43-47'};

$SA1$ = [21; 21; 21; 21; 11; 11];

$SA2$ = [8; 10; 11; 8; 3; 2];

$SA3$ = [9; 16; 19; 13; 7; 4];

$SA4$ = [6; 14; 15; 9; 4; 3];

$SA5$ = [7; 15; 20; 15; 7; 6];

$SA6$ = [8; 16; 17; 12; 4; 2];

$SA7$ = [8; 11; 12; 11; 4; 4];

$SA8$ = [11; 16; 18; 17; 7; 7];

$SA9$ = [16; 17; 18; 18; 6; 6];

$SA10$ = [19; 20; 22; 20; 8; 7];

$IRDM$ = table (Age_Group, $SA1$, $SA2$, $SA3$, $SA4$, $SA5$, $SA6$, $SA7$, $SA8$, $SA9$, $SA10$)

Press enter this will come on screen as shown in Table 4.2.

Table 4.2: Initial raw data matrix of divorced women of the order 6×10

IRDM =

Age_Group	SA1	SA2	SA3	SA4	SA5	SA6	SA7	SA8	SA9	SA10
'18-22'	21	8	9	6	7	8	8	11	16	19
'23-27'	21	10	16	14	15	16	11	16	17	20
'28-32'	21	11	19	15	20	17	12	18	18	22
'33-37'	21	8	13	9	15	12	11	17	18	20
'38-42'	11	3	7	4	7	4	4	7	6	8
'43-47'	11	2	4	3	6	2	4	7	6	7

Transformation of initial raw data matrix into the average time dependent data matrix, by dividing each entry of initial raw data matrix with the length of respective class interval [3].

IRDM{:, 2:end} = IRDM{:, 2:end}*1/5

Press enter this will come on screen.

IRDM =

Age_Group	SA1	SA2	SA3	SA4	SA5	SA6	SA7	SA8	SA9	SA10
'18-22'	4.2	1.6	1.8	1.2	1.4	1.6	1.6	2.2	3.2	3.8
'23-27'	4.2	2	3.2	2.8	3	3.2	2.2	3.2	3.4	4
'28-32'	4.2	2.2	3.8	3	4	3.4	2.4	3.6	3.6	4.4
'33-37'	4.2	1.6	2.6	1.8	3	2.4	2.2	3.4	3.6	4
'38-42'	2.2	0.6	1.4	0.8	1.4	0.8	0.8	1.4	1.2	1.6
'43-47'	2.2	0.4	0.8	0.6	1.2	0.4	0.8	1.4	1.2	1.4

ATDM (: , :) = IRDM (: , :)
Press enter this will come on screen as shown in Table 4.3.

Table 4.3: Average time dependent data matrix of divorced women of the order 6×10

ATDM =

Age_Group	SA1	SA2	SA3	SA4	SA5	SA6	SA7	SA8	SA9	SA10
'18-22'	4.2	1.6	1.8	1.2	1.4	1.6	1.6	2.2	3.2	3.8
'23-27'	4.2	2	3.2	2.8	3	3.2	2.2	3.2	3.4	4
'28-32'	4.2	2.2	3.8	3	4	3.4	2.4	3.6	3.6	4.4
'33-37'	4.2	1.6	2.6	1.8	3	2.4	2.2	3.4	3.6	4
'38-42'	2.2	0.6	1.4	0.8	1.4	0.8	0.8	1.4	1.2	1.6
'43-47'	2.2	0.4	0.8	0.6	1.2	0.4	0.8	1.4	1.2	1.4

Next, we find the mean and standard deviation of every column in the average time dependent data matrix.

a = [4.2, 1.6, 1.8, 1.2, 1.4, 1.6, 1.6, 2.2, 3.2, 3.8; 4.2, 2, 3.2, 2.8, 3, 3.2, 2.2, 3.2, 3.4, 4; 4.2, 2.2, 3.8, 3, 4, 3.4, 2.4, 3.6, 3.6, 4.4; 4.2, 1.6, 2.6, 1.8, 3, 2.4, 2.2, 3.4, 3.6, 4; 2.2, 0.6, 1.4, 0.8, 1.4, 0.8, 0.8, 1.4, 1.2, 1.6; 2.2, 0.4, 0.8, 0.6, 1.2, 0.4, 0.8, 1.4, 1.2, 1.4];

MEAN = mean (a), STDD = std (a)

Press enter this will come on screen as shown in Table 4.4.

Table 4.4: Mean and Standard Deviation of the above average time depedent data matrix

```
MEAN =

    3.5333    1.4000    2.2667    1.7000    2.3333    1.9667    1.6667    2.5333    2.7000    3.2000


STDD =

    1.0328    0.7376    1.1361    1.0178    1.1570    1.2420    0.7230    1.0013    1.1713    1.3327
```

Now we make table of mean for all attributes.

Age_Group = {'MEAN –>'};

SA1 = [3.5333];

SA2 = [1.4000];

SA3 = [2.2667];

SA4 = [1.7000];
SA5 = [2.3333];
SA6 = [1.9667];
SA7 = [1.6667];
SA8 = [2.5333];
SA9 = [2.7000];
SA10 = [3.2000];
Mean_Table = table (Age_Group, SA1, SA2, SA3, SA4, SA5, SA6, SA7, SA8, SA9, SA10)
Press enter this will come on screen as shown in Table 4.5.

Table 4.5: Mean of the above average time depedent data matrix

Mean_Table =

Age_Group	SA1	SA2	SA3	SA4	SA5	SA6	SA7	SA8	SA9	SA10
'MEAN -->'	3.5333	1.4	2.2667	1.7	2.3333	1.9667	1.6667	2.5333	2.7	3.2

Now we make table of standard deviation for all attributes.
Age_Group = {'STAN_DEVIATION'};
SA1 = [1.0328];
SA2 = [0.7376];
SA3 = [1.1361];
SA4 = [1.0178];
SA5 = [1.1570];
SA6 = [1.2420];
SA7 = [0.7230];
SA8 = [1.0013];
SA9 = [1.1713];
SA10 = [1.3327];
STANDERD_DEVIATION_Table = table (Age_Group, SA1, SA2, SA3, SA4, SA5, SA6, SA7, SA8, SA9, SA10)
Press enter this will come on screen as shown in Table 4.6.

Table 4.6: Standard deviation of the above average time depedent data matrix

STANDERD_DEVIATION_Table =

Age_Group	SA1	SA2	SA3	SA4	SA5	SA6	SA7	SA8	SA9	SA10
'STAN_DEVIATION'	1.0328	0.7376	1.1361	1.0178	1.157	1.242	0.723	1.0013	1.1713	1.3327

Converting *ATDM* into distinct *RTDM*
for i = 1:6;
for j=2:11;
Sum = Mean_Table{1,j} + STANDERD_DEVIATION_Table{1,j}*.25;
Diff = Mean_Table{1,j} - STANDERD_DEVIATION_Table{1,j}*.25;
if(le(ATDM{i, j}, Diff))
disp ('-1')
elseif((ATDM{i, j} > Diff) && (ATDM{i,j} < Sum))
disp('0')
elseif(ge(ATDM{i, j},Sum))
disp('1')
end
end

end
Press enter this will come on screen.
A matrix will come on screen of order 60×1.
a1 = [paste here all element of matrix of order 60×1];
$RTDM1$ = reshape(a1,10,6)
Press enter this will come on screen as shown in Table 4.7.

Table 4.7: Converting ATDM into distinct RTDM

RTDM1 =

1	1	1	1	-1	-1
1	1	1	1	-1	-1
-1	1	1	1	-1	-1
-1	1	1	0	-1	-1
-1	1	1	1	-1	-1
-1	1	1	1	-1	-1
0	1	1	1	-1	-1
-1	1	1	1	-1	-1
1	1	1	1	-1	-1
1	1	1	1	-1	-1

Above $RTDM1$ table is not in proper way. To convert it into proper way to use below code.
$RTDM1$ = $RTDM1$'
Press enter this will come on screen as show in Table 4.8.
For $\alpha = .25$;

Table 4.8: Proper distinct RTDM

RTDM1 =

1	1	-1	-1	-1	-1	0	-1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1
1	1	1	0	1	1	1	1	1	1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

Row_Wise_Sum1 = sum(RTDM1, 2)
Press enter this will come on screen.

```
Row_Wise_Sum1 =

    -1
    10
    10
     9
   -10
   -10
```

for i = 1:6;
for j=2:11;
Sum = Mean_Table{1,j} + STANDERD_DEVIATION_Table{1,j}*.5;
Diff = Mean_Table{1,j} - STANDERD_DEVIATION_Table{1,j}*.5;
if(le(ATDM{i, j}, Diff))
disp ('-1')

elseif((ATDM{i, j} > Diff) && (ATDM{i,j} < Sum))
disp('0')
elseif(ge(ATDM{i, j},Sum))
disp('1')
end
end
end
Press enter this will come on screen.
A matrix will come on screen of order 60×1.
a2 = [paste here all element of matrix of order 60×1];
$RTDM2$ = reshape(a2,10,6)
Press enter this will come on screen as shown in Table 4.9.

Table 4.9: Converting ATDM into distinct RTDM

```
RTDM2 =

     1     1     1     1    -1    -1
     0     1     1     0    -1    -1
     0     1     1     0    -1    -1
     0     1     1     0    -1    -1
    -1     1     1     1    -1    -1
     0     1     1     0    -1    -1
     0     1     1     1    -1    -1
     0     1     1     1    -1    -1
     0     1     1     1    -1    -1
     0     1     1     1    -1    -1
```

Above $RTDM$ 2 table is not in proper way. To convert it into proper way to use below code.
RTDM2 = RTDM2'
RTDM for $\alpha = 0.5$
Press enter this will come on screen as show in Table 4.10.

Table 4.10: Proper distinct RTDM

```
RTDM2 =

     1     0     0     0    -1     0     0     0     0     0
     1     1     1     1     1     1     1     1     1     1
     1     1     1     1     1     1     1     1     1     1
     1     0     0     0     1     0     1     1     1     1
    -1    -1    -1    -1    -1    -1    -1    -1    -1    -1
    -1    -1    -1    -1    -1    -1    -1    -1    -1    -1
```

Row_Wise_Sum2 = sum(RTDM2, 2)
Press enter this will come on screen

```
Row_Wise_Sum2 =

     0
    10
    10
     6
   -10
   -10
```

```
for i = 1:6;
for j=2:11;
Sum = Mean_Table{1,j} + STANDERD_DEVIATION_Table{1,j}*.8;
Diff = Mean_Table{1,j} - STANDERD_DEVIATION_Table{1,j}*.8;
if(le(ATDM{i, j}, Diff))
disp ('-1')
elseif((ATDM{i, j} > Diff) && (ATDM{i,j} < Sum))
disp('0')
elseif(ge(ATDM{i, j},Sum))
disp('1')
end
end
end
```

Press enter this will come on screen.

A matrix will come on screen of order 60×1.

a3 = [paste here all element of matrix of order 60×1];

RTDM3 = reshape(a3,10,6)

Press enter this will come on screen as shown in Table 4.11.

Table 4.11: Converting ATDM into distinct RTDM

RTDM3 =

0	0	0	0	−1	−1
0	1	1	0	−1	−1
0	1	1	0	0	−1
0	1	1	0	−1	−1
−1	0	1	0	−1	−1
0	1	1	0	−1	−1
0	0	1	0	−1	−1
0	0	1	1	−1	−1
0	0	0	0	−1	−1
0	0	1	0	−1	−1

Above *RTDM* 3 table is not in proper way. To convert it into proper way to use below code.

RTDM3 = RTDM3'

RTDM for $\alpha = 0.8$

Press enter this will come on screen as show in Table 4.12.

Table 4.12: Proper distinct RTDM

RTDM3 =

0	0	0	0	−1	0	0	0	0	0
0	1	1	1	0	1	0	0	0	0
0	1	1	1	1	1	1	1	0	1
0	0	0	0	0	0	0	1	0	0
−1	−1	0	−1	−1	−1	−1	−1	−1	−1
−1	−1	−1	−1	−1	−1	−1	−1	−1	−1

RTDM3=

Row_Wise_Sum3 = sum(RTDM3, 2)

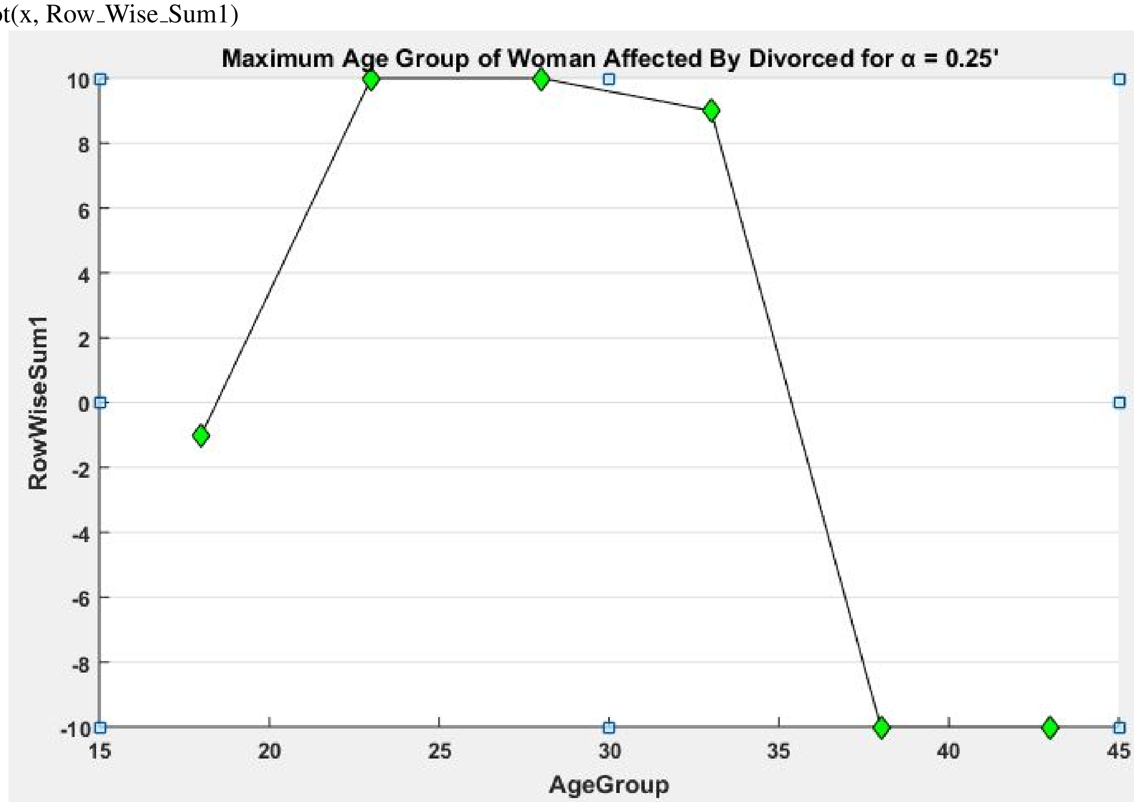
Press enter this will come on screen

```
Row_Wise_Sum3 =

    -1
     4
     8
     1
    -9
   -10
```

5. Plotting graphs for different values of $\alpha, 0 \leq \alpha \leq 1$to depicting the maximum age group of women affected by the divorce system

x =18:5:47;

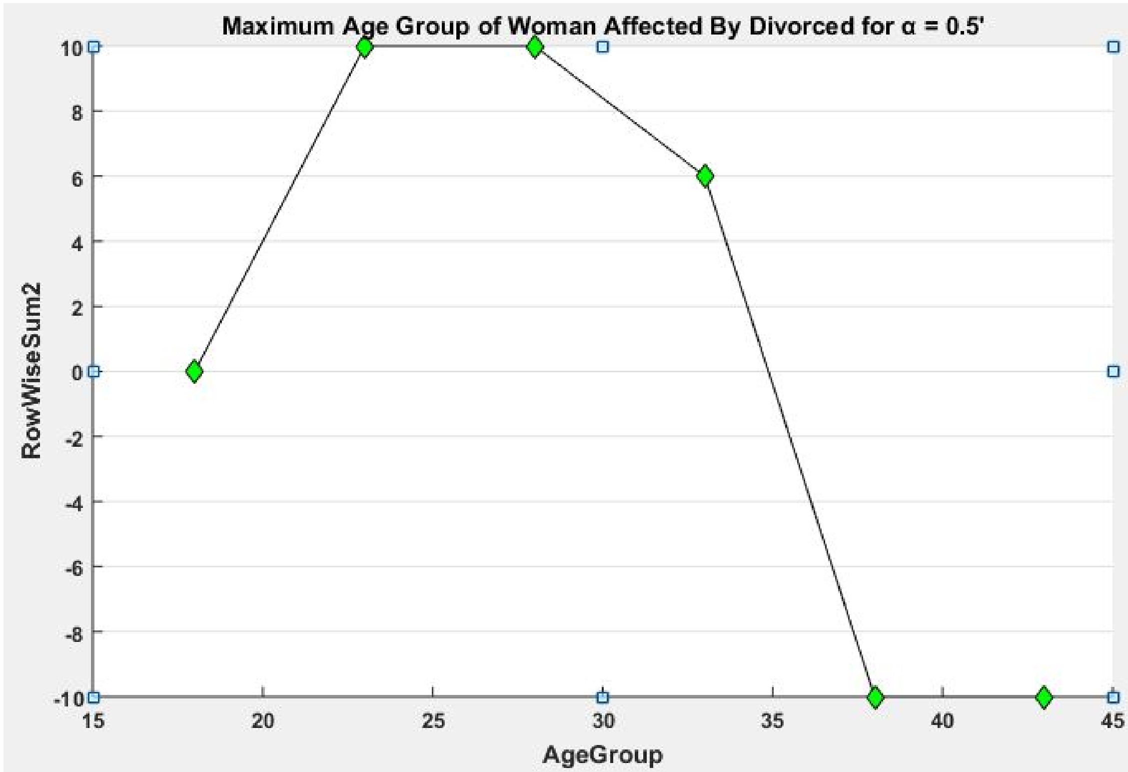
title('Maximum Age Group of Woman Affected By Divorced for α = 0.25'), xlabel('AgeGroup'),ylabel('Row_Wise_Sum1'), plot(x, Row_Wise_Sum1)



***Graph* 5.1: Depicting the maximum age-group of women affected by the divorce for α = 0.2**

x =18:5:47;

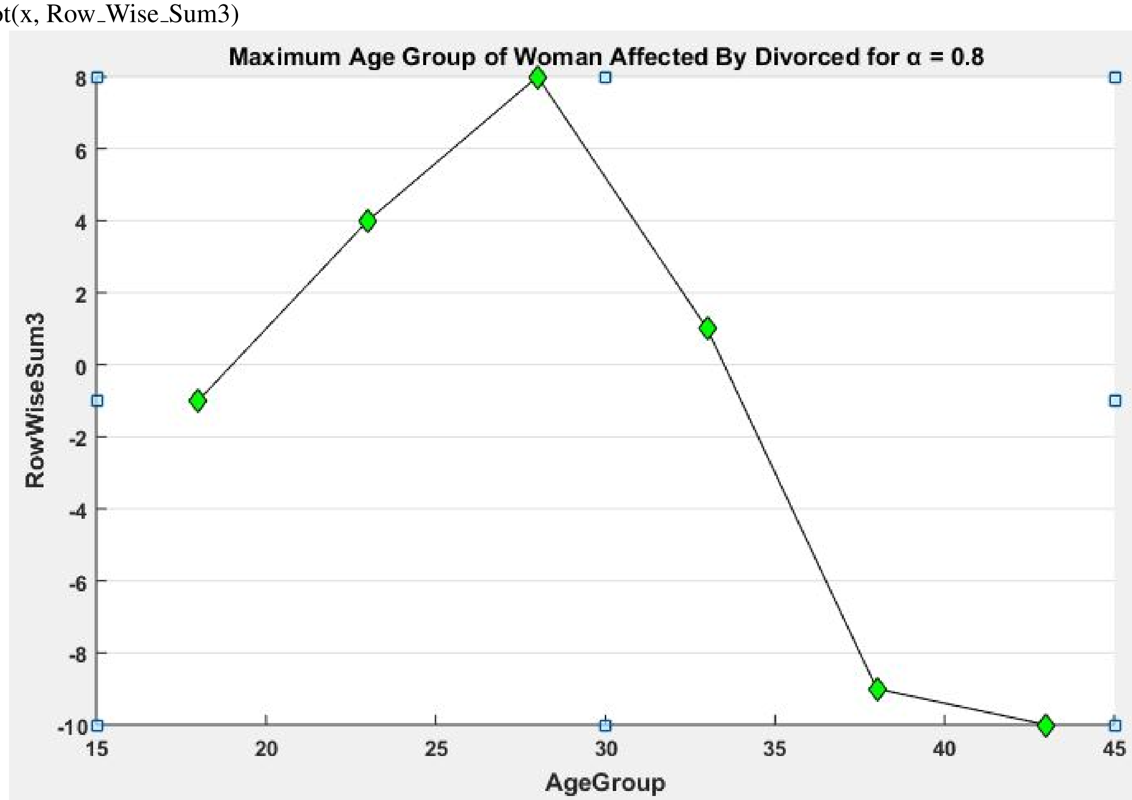
title('Maximum Age Group of Woman Affected By Divorced for α = 0.5'), xlabel('AgeGroup'), ylabel('Row_Wise_Sum2'), plot(x, Row_Wise_Sum2)



***Graph* 5.2: Depicting the maximum age-group of women affected by the divorce for α= 0.5**

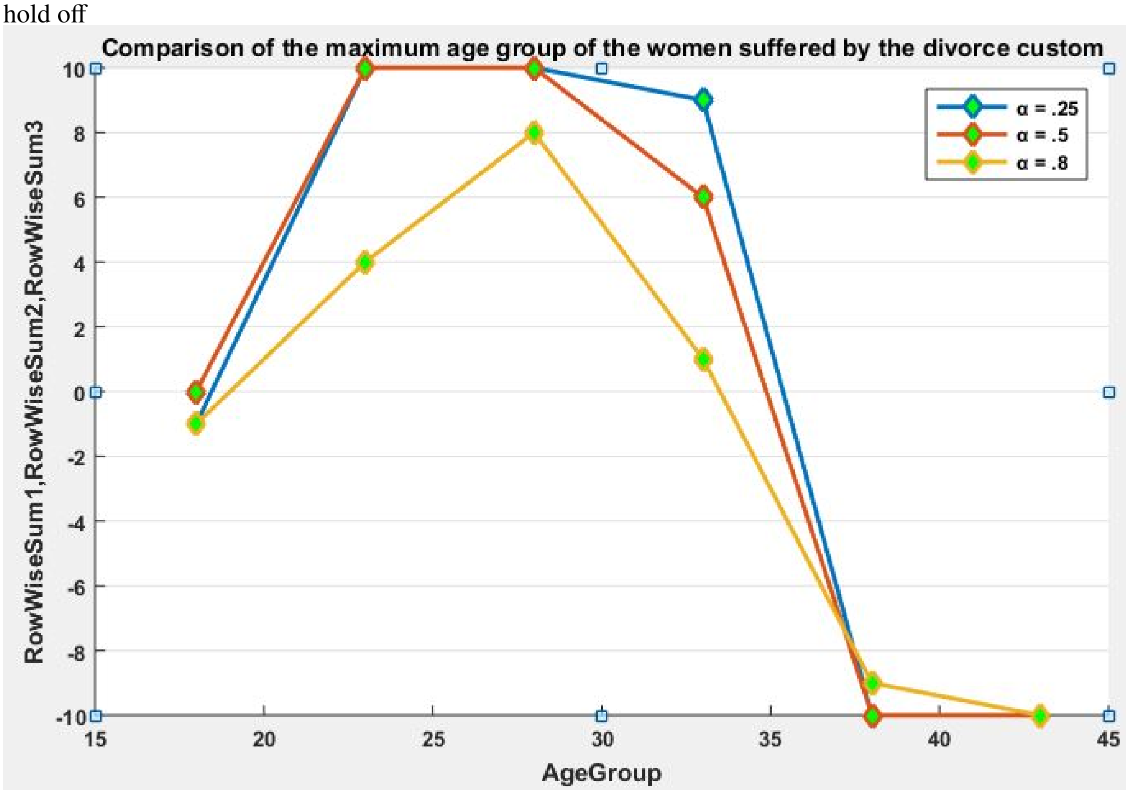
x =18:5:47;

title('Maximum Age Group of Woman Affected By Divorced for α = 0.8'), xlabel('AgeGroup'), ylabel('Row_Wise_Sum3'), plot(x, Row_Wise_Sum3)



Graph **5.3: Depicting the maximum age-group of women affected by the divorce for α= 0.8**

```
x =18:5:47;
plot(x, Row_Wise_Sum1);
hold on
plot(x, Row_Wise_Sum2);
hold on
plot(x, Row_Wise_Sum3);
legend(' ? = .25',' ? = .5',' ? = .8')
hold off
```



Graph **5.4: Graphical comparison of the maximum age-group of women affected by divorce**

6. Combined effect time dependent data matrix and its graphical representation

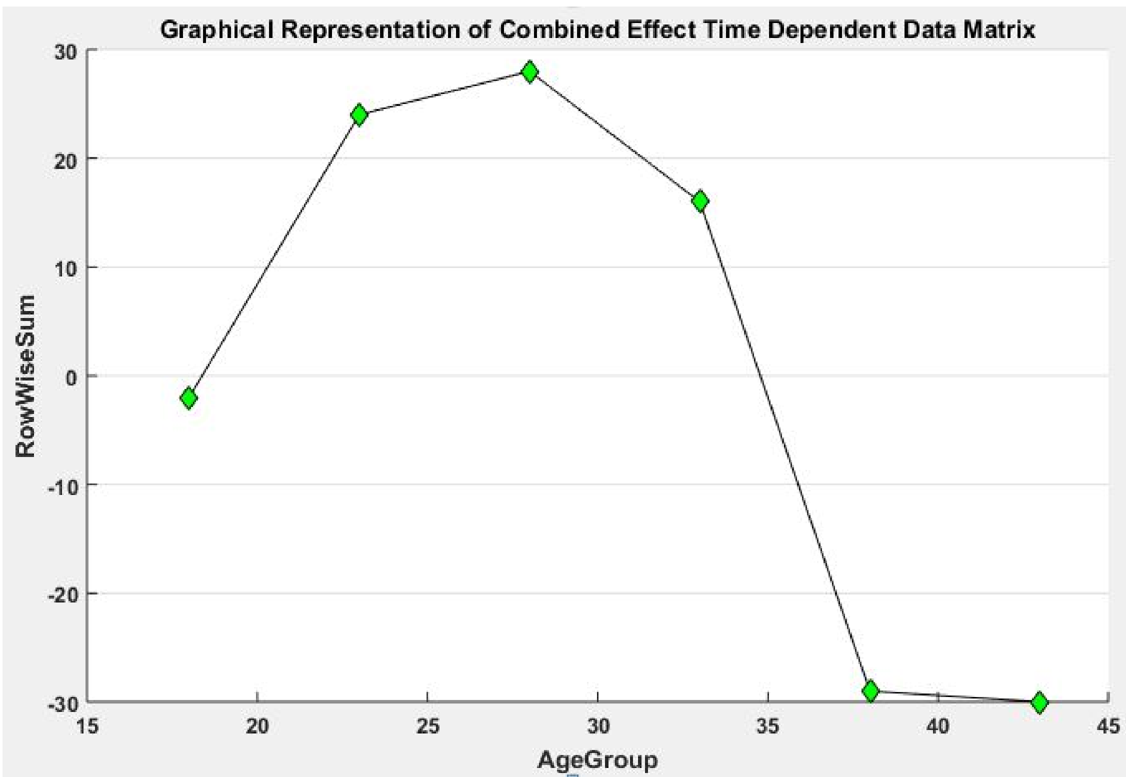
CETDM = RTDM1 + RTDM2 + RTDM3

```
CETDM =

     2     1    -1    -1    -3    -1     0    -1     1     1
     2     3     3     3     2     3     2     2     2     2
     2     3     3     3     3     3     3     3     2     3
     2     1     1     0     2     1     2     3     2     2
    -3    -3    -2    -3    -3    -3    -3    -3    -3    -3
    -3    -3    -3    -3    -3    -3    -3    -3    -3    -3
```

Row_Wise_Sum = Row_Wise_Sum1 + Row_Wise_Sum2 + Row_Wise_Sum3

```
Row_Wise_Sum =

    -2
    24
    28
    16
   -29
   -30
```

x =18:5:47;
title('Graphical Representation of combined effect time dependent data matrix'),xlabel('AgeGroup'), ylabel('Row_Wise_Sum'), plot(x, Row_Wise_Sum)



***Graph* 6.1: Depicting the maximum age-group of women affected by the divorce for CETDM**

7. Results and Conclusions

From the above studies and plots for different values of $\alpha, 0 \leq \alpha \leq 1$, we obtained that the maximum age group of women affected by the divorce problem has been infrequently changed [5,11,14]. The graphical representation is that most of the women affected by divorce problem lie between 24 to 36. The result confirms by combined effect time dependent data matix is also very near to age group of 24 to 36. Also the data obtained for age groups 18 to 22 and 43 to 47 are negative, which simply mean that very few women are affected by divorce problem during this age interval.The main motivation to work on divorce problem is to evaluate determine the maximum age group of women affected by divorce so that family counselors can help to resolve this problem.

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EVALUATION OF RECONNECTION PARAMETERS FOR COLLISIONLESS DISPENSATION IN SOLAR CORONA

By

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Abstract

In the present paper, we have obtained an analytical solution for the thickness of current sheet in the dissipation region of the solar corona, reconnection time and reconnection rate for collisionless magnetic reconnection occuring in the solar corona. For our analysis we have used the length scale of the electron dissipation region based on field reversals. The reconnection electric field and component of the electron pressure tensor used in the Sweet-Parker model have been utilised to finally determine the exact numerical value of the thickness of current sheet, reconnection time and reconnection rate. It is shown that the thickness of the current sheet in the collisionless reconnection model is directly proportional to the electron mass and inversely proportional to the length scale of the electron dissipation region. Also, it is shown that the reconnection time is inversely proportional to the cube of the magnetic field and directly proportional to length scale, whereas reconnection rate is inversely proportional to the length scale of the electron dissipation region.

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Keywords and Phrases: Dissipation region, magnetic reconnection, reconnection rate, solar corona.

1. Introduction

Magnetic reconnection is the fundamental process through which magnetic energy gets transformed into thermal energy and heat of the plasma. Many observational evidences suggest that magnetic reconnection is significantly accountable for the expulsion of solar flares, coronal mass ejection *(CMEs)*, magnetospheric substorms and heating process [2,18,19]. At temperature of few million degrees in the solar corona, the continuous flow of plasma gives rise to magnetic reconnection process and the association of magnetic field lines with tremendous energy released is responsible for heating the corona [11,14,40]. When two pieces of magnetised plasma of opposite polarity come closer, they reconnect and a current sheet is formed also called as dissipation region [31]. This is the region where the magnetic field is considered to be zero, known as neutral region. Magnetic field lines of opposite polarity annihilate and magnetic flux cancelling occurs with a characteristic speed called Alfven speed and the plasma moves outward. As a result, the magnetic composition gets abolished on the inflow sides and a new magnetic arrangement developes on the outflow sides. The magnetic field lines change their direction and give rise to very high electric current density which causes heating of the plasma. An X-point shape is formed where the magnetic field lines meet, this process also takes place in magnetosphere.

A new paradigm for solar coronal heating keeping the plasma marginally collisionless has been proposed by [35]. The plasma collisionality is controlled by the coronal density and thus transition occurs between the slow collisional Sweet-Parker and the fast collisionless reconnection regimes. Continuous repetition of fast reconnection events represents coronal heating. Fast reconnection takes place in a collisionless plasma where resistive *MHD* is not valid. Over a long period, fast collisionless reconnection have been observed in space and solar physics. Many laboratory experiments shows that fast reconnection does indeed take place in the collisionless regime.

For fast collisionless reconnection, there are two physically distinct mechanisms, namely

Hall effect [3,13] and spatially localised anamalous resistivity [28,34]. Thus, the magnetic reconnection is classified into two regimes:- the slow collisional Sweet-Parker reconnection in resistive *MHD* and the fast Petschek-like collisionless reconnection. In collisionless plasma, the transportation is done by magnetic reconnection process[25]. Using localized diffusive effects, magnetic reconnection always gives priority to large scale plasma transport instead of global diffusion [1,26]. Reconnection process results in the generation of waves and turbulence or the onset of reconnection is affected by the preexisting waves and turbulence. Over a wide range of scales and frequencies kinetic processes plays an important role in collisionless plasma because they generate waves in ion, hybrid and electron ranges. In all part of the reconnection region, we can find the presence of these waves such as outflows [15], separatrix regions [37], ion and electron diffusion region [6]. The upper hybrid waves are present near the electron plasma frequency. Deformations in the electron distribution function such as electron beam, ring, shell leads to the generation of these waves. Because of their high frequencies these waves can accelerate at fast rates, and to cause the dissipation of unstable electron distribution, these waves act very quickly in comparison to the timescales over which reconnection evolves.

These waves are often observed near by the source region because these waves have relatively low group velocities as compared to the electron thermal speed. In the separatrix regions, generation of fast electron beams occurs and are often unstable. For weak beams upper hybrid waves are favourable near the plasma frequency. From wind and cluster observations in magnetotail reconnection both upper hybrid and langmuir waves have been reported.

A large number of electrostatic and electromagnetic modes are supported by magnetized beam-plasma system. With the help of linear and non-linear mechanism these modes are exhilarated in situ [7]. They are incorporated at higher altitudes and there is an interchange of energy and momentum between these modes. By the use of axial and spiraling electron and ion beams [30] these large amplitude Trivelpiece-Gould(*TG*) modes or lower hybrid waves were observed in various experiments. The excitation of higher harmonics of a *TG* mode in a low-energy beam-plasma system were observed by [20]. These large amplitude *TG* modes which are also known as lower hybrid waves are used for heating and suppression of micro-instabilities in Q-machine. In magnetized beam-plasma systems, they are also called the *TG* modes, in which these lower hybrid waves are observed as dominant instability.

Large amplitude waves associated with magnetic reconnection are often observed. Because of lower hybrid drift waves, the largest amplitude electric fields are asssociated with magnetopause [24]. During magnetopause reconnection, lower hybrid drift waves are developed in the magnetic separatrix regions due to the sharp density gradient [24]. By introducing electron and ion heating lower hybrid waves performs a vital role in both symmetric and asymmetric reconnection. Within the current sheet near the X-line, electromagnetic lower hybrid waves are developed.

Resistive Magnetohydrodynamics *(MHD)* models in plasma physics give evidences for the support of magnetic reconnection given by ohmic dissipation which generates resistive electric field [8].

For collisionless plasma there are less no. of collisions between ions and electrons, so the nature of dissipation for time and spatial scale illustrating magnetic reconnection is depressed. There is a violation of idealness condition for magnetic reconnection and the dissipative electric field. In MHD, the generalised Ohm's law is given by $J = \sigma(E + V \times B)$. When $\sigma \to \infty$ for ideal MHD, $E = -V \times B$, for this condition the field gets frozen with the fluid, thus no reconnection occurs. The current sheet will not be magnetized unless at the boundaries the flow of magnetization current occurs. Thus magnetization of the current sheet is the basic requirement of reconnection , thus $E + v \times B = 0$, this shows the defficiency of collisions, hence called collisionless plasma. In order to achieve this condition we need to increase resistivity and decrease scale length. This is basically the reason for a very small thickness of the current sheet as calculated in this paper.

In this paper we are using collisionless system, because of the high temperature of Solar Corona [27]. In the diffusion region, the dissipation mechanism occurs which results in the formation of an electric field and due to this electric field the frozen-in-flux condition is violated for the occurence of the reconnection process [29]. Since for collisionless plasma $E + v \times B = 0$, the validity of this condition allows one to visualize transport of magnetic flux through the plasma fluid motion. If this condition does not hold, then the magnetic field line motion is not applicable as it is ill-defined and does not tie to the plasma fluid motion. As a result, magnetic flux transport cannot be visualized to be carried by the plasma fluid motion. This electric field results in the dependence of the dissipation region on the plasma parameters.

In solar corona, the temperature is very high of the order of $(1-2)\times 10^6$ K, and due to large amount of energy that is stored in the form of magnetic field, magnetic reconnection enables the liberation of this enormous energy with solar flares and coronal mass ejections [27]. The evidences for magnetic reconnection in the solar corona have also been provided by the observations from SOHO, TRACE, RHESSI. E Parker and PA Sweet [16], advocated a framework to decipher the phenomena of energy release and heating of solar corona.

In the diffusion region where the current sheet is formed ohmic dissipation takes place due to which magnetic energy gets converted into thermal energy. In the present paper we have used the famous Sweet-Parker model [16] to obtain the expressions for the thicknesss of current sheet, time and rate of magnetic reconnection process taking place in solar corona. We have used the equation for reconnection electric field and component of electron pressure tensor as determined by Hesse [8]. Our results include current sheet thickness, which is in agreement with the results obtained by Spangler [33]. However, in his paper Spangler [33] has used ion inertial length to determine the thickness of the current sheet, plasma density profile based on radio propagation measurements of the corona and magnetic field model of Ingleby et al. [10] . He found the value of thickness of current sheet in terms of R_o which is the heliocentric distance in units of a solar radius. A similar approach has been taken up by Kumar et al. [38] but the results obtained in the current study are quite different. The variation in the results may be attributed to the introduction of the length scale of dissipation region.

2. Current sheet thickness

According to Sweet-Parker model [16], when two oppositely directed magnetic field lines approach each other, reconnection occurs and a current sheet is formed which is called dissipation region. In collisionless magnetic reconnection the dissipation region has two scale structure. The ion diffusion region which has a larger structure has a size of c/w_{pI}, where w_{pI} is the ion plasma frequency. The electron diffusion region which has smaller structure has a size of c/w_{pe}, where w_{pe} is the electron plasma frequency. Energy is released in the form of kinetic energy and thermal energy through this reconnection process. An increase in the number of field lines increases thermal pressure resulting in the flow of plasma from the diffusion region with Alfven speed ($V_{e,A}$) of electron.

$$V_{outflow} = V_{e,A} = \frac{B}{\sqrt{\mu_o m_e n_e}}, \tag{2.1}$$

($V_{outflow}$ is the velocity of the reconnection outflow, B is the magnetic field in the coronal plasma, m_e is the mass of electrons, n_e is the number density of electrons and μ_o is the permeability of free space).

Applying continuity equation,

$$V_{inflow} L = V_{e,A} \delta_{SP}. \tag{2.2}$$

(V_{inflow} is the velocity of the reconnection inflow, L is the length of the diffusion region, δ_{SP} is the width of the diffusion region in Sweet-Parker Model).

The reconnection rate in Sweet-Parker region is defined as,

$$R_{SP} = \frac{V_{inflow}}{V_{e,A}},$$

$$R_{SP} = \frac{1}{\sqrt{S}}.$$

($S = \frac{\mu_o L V_A}{\eta}$ is the Lundquist number and η is the magnetic diffusivity).

Hesse [8], observed the dependence of the structure of the dissipation region on the electron mass. Thus, the trapping of electrons, in the field reversal region, provides a characteristic length scale to determine the size of the dissipation region .

Biskamp and Schindler [4], on the basis of the electron orbits in the field reversals already determined this length scale which is equal to

$$\lambda_x = \left[\frac{2m_e T_e}{e^2(\frac{\partial B_z}{\partial x})^2}\right]^{1/4}. \tag{2.3}$$

Thus, the length scale of the electron dissipation region is proportional to the fourth root of the electron mass [8].

From this result, we can find the reconnection electric field as,

$$E_{rec} = -\frac{1}{n_e e}\frac{\partial P_{xy,e}}{\partial x} \approx \frac{1}{n_e e \lambda_x} P_{xy,e}, \tag{2.4}$$

where $P_{xy,e}$ is the electron pressure tensor, and this can be written as,

$$P_{xy,e} \approx \frac{P_e}{2\Omega_e}\frac{\partial V_{xe}}{\partial x}, \tag{2.5}$$

(Ω_e is the electron cyclotron frequency and P_e denotes the isotropic part of the electron pressure tensor).

Similarly, for z-direction using continuity equation we can write,

$$\frac{\partial V_x}{\partial x} \approx \frac{\partial V_z}{\partial z}.$$

Substituting value of $P_{xy,e}$ from equation (2.5) in equation (2.4) we get,

$$E_{rec} = \frac{1}{n_e e \lambda_x}\frac{P_e}{2\Omega_e}\frac{\partial V_{xe}}{\partial x}, \tag{2.6}$$

Since the magnetic field is guided by the electrons, therefore

$$V_z = V_{e,inflow},$$

$$P_{xy,e} = \frac{P_e}{2\Omega_e}\frac{\partial V_{e,inflow}}{\partial z}, \tag{2.7}$$

Assuming $(\frac{\partial}{\partial z}) \approx (\frac{1}{\Delta z})$,
where Δz is the width of the current sheet.

$$P_{xy,e} = \frac{P_e}{2\Omega e}\frac{V_{e,inflow}}{\Delta z}.$$

Applying Continuity equation (2.2) in Sweet-Parker Model and writing $\delta_{SP} = \Delta z$ we get,

$$V_{e,inflow}L = V_{e,A}\Delta z,$$

therefore,

$$\Delta z = \frac{V_{e,inflow}L}{V_{e,A}}.$$

From equation (2.4),

$$E_{rec} = V_{e,inflow}B_o = \frac{1}{n_e e\lambda_x}P_{xy,e}. \tag{2.8}$$

From equation (2.7),

$$V_{e,inflow}B_o = \frac{1}{n_e, e\lambda_x}\frac{P_e}{2\Omega e}\left(\frac{V_{e,inflow}}{\Delta z}\right),$$

Here, P_e the electron pressure tensor can be written as $P_e = \frac{B_o^2}{2\mu_o}$.

$$V_{e,inflow} = \frac{1}{2n_e e^2\lambda_x}\frac{m_e}{2\mu_o}\left(\frac{V_{e,A}}{L}\right). \tag{2.9}$$

The Lundquist number is given by,

$$S = \frac{\mu_o L V_{e,A}}{\eta},$$

$$S = \frac{\mu_o V_{e_A}\Delta z V_{e,A}}{V_{e,inflow}\eta},$$

$$S = \frac{\mu_o (V_{e,A})^2\Delta z}{V_{e,inflow}\eta},$$

Putting value of $V_{e,inflow}$ from equation (2.9) we get,

$$S = \frac{4(\mu_o)^2 V_{e,A} ne^2\lambda_x\Delta z S\eta}{m_e\eta\mu_o V_{e,A}}.$$

Rearranging the above equation,

$$\Delta z = \frac{m_e}{4\mu_o ne^2\lambda_x}. \tag{2.10}$$

Thus, the width of the current sheet in the collisionless reconnection model is directly proportional to the mass of electron and inversely proportional to the length scale of the electron dissipation region.
The solar corona parameters are; $n = 10^{14} \sim 10^{15} m^{-3}$ and the length scale of electron dissipation region is of the order of c/w_{pe} [23].
where w_{pe} is the electron plasma frequency defined as,

$$w_{pe} = \left(\frac{4\pi n_e e^2}{m_e}\right)^{1/2} \approx 5.64 \times 10^4 (n_e)^{1/2} rad/s. \tag{2.11}$$

Using these values numerically, the width of the current sheet in collisionless magnetic reconnection comes out to be

$$\Delta z = 10^3 m.$$

This result is in agreement with the width of the current sheet determined by Spangler [33]. However, to determine the thickness of current sheet Spangler used ion inertial length, $t_c = \frac{V_A}{\Omega_i}$ where V_A is the Alfven speed and Ω_i is the proton ion cyclotron frequency.

3. Reconnection time

The reconnection rate in Sweet-Parker region is defined as,

$$R_{collissionless} = \frac{v_{e,inflow}}{v_{e,A}},$$

$$R_{collissionless} = \frac{1}{2ne^2\lambda_x}\frac{m_e}{2\mu_o}\left(\frac{1}{L}\right). \tag{3.1}$$

The reconnection time is given by,

$$t_R = \frac{\tau_A}{R_{Collisionless}}, \tag{3.2}$$

Since,

$$\tau_A = \frac{L}{v_{e,A}}.$$

Substituting this value of τ_A in equation (3.2) we get,

$$t_R = \frac{L}{v_{e,A}}\frac{2ne^2\lambda_x}{m_e}.2\mu_o(L),$$

$$t_R = \frac{2ne^2\lambda_x 2\mu_o L^2}{V_{e,A}m_e}. \tag{3.3}$$

The length of diffusion region is given by,

$$L = \frac{S\eta}{\mu_o V_{e,A}}, \tag{3.4}$$

Putting the value of L from equation (3.4) into (3.3) we get,

$$t_R = \frac{2ne^2\lambda_x 2\eta^2 S^2}{V_{e,A}m_e\mu_o V_{e,A}^3},$$

Substituting, $V_{e,A} = \frac{B}{\sqrt{\mu_o m_e n}}$,

$$t_R = \frac{2ne^2\lambda_x 2\eta^2 S^2(\mu_o)^{3/2}(m_e)^{3/2}(n)^{3/2}}{\mu_o m_e B^3},$$

$$t_R = \frac{4(n)^{5/2}e^2\lambda_x\eta^2 S^2(\mu_o)^{1/2}(m_e)^{1/2}}{B^3}. \tag{3.5}$$

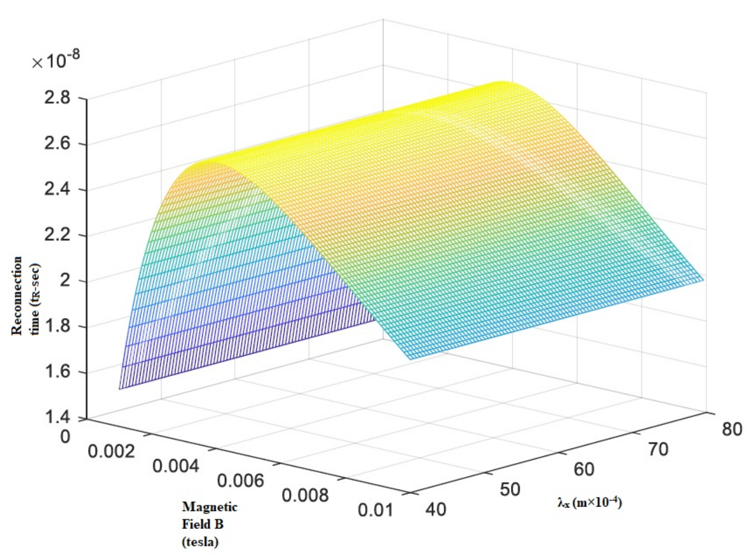

Figure 3.1: 3-D plot showing variation of reconnection time with magnetic field and length scale of electron dissipation region.

Thus, the reconnection time of collisionless magnetic reconnection process is inversely proportional to the cube of magnetic field and directly proportional to the length scale of electron dissipation region λ_x. The inverse dependence of the reconnection time on the cube of magnetic field is in agreement with the findings of Kumar et al. [38]. We have also plotted a graph showing the variation of reconnection time with magnetic field and length scale of electron dissipation region in figure (3.1). From Figure 3.1, it is evident that the range of reconnection time of collisionless magnetic reconnection is of the order of 10^{-8} seconds.

4. Reconnection rate

In order to understand the magnetic reconnection process, it is mandatory to know the rate at which reconnection occurs. The reconnection rate determines the temporal rate of change of magnetic flux defines the reconnection process.

$$R_{collisionless} = \frac{v_{e,inflow}}{v_{e,A}},$$

$$R_{collisionless} = \frac{1}{2ne^2\lambda_x}\frac{m_e}{2_{\mu_o}}\left(\frac{1}{L}\right).$$

The solar coronal parameters are; $n = 10^{14} \sim 10^{15}m^{-3}$, $L = 10^7m$, and the length scale of electron dissipation region is of the order of c/w_{pe} [see,(2.11)].

Using these values numerically, the reconnection rate for collisionless magnetic reconnection comes out to be,

$$R_{collisionless} = 10^{-6}, \tag{4.1}$$

However,

$$R_{collisionless} = \frac{1}{\sqrt{S}}. \tag{4.2}$$

Substituting the numerical value of reconnection rate from equation (4.1) in (4.2), we have obtained the value of Lundquist number $S = 10^{12}$. For solar corona, the pre-determined value of Lundquist number is of the order of $10^{12} \sim 10^{14}$ [22].

The numerical value of Lundquist number obtained in this paper is in agreement with the value obtained by Shibata and Magara [32].

5. Conclusions

In this paper, we have obtained analytical solutions for the thickness of current sheet, reconnection time and reconnection rate using Sweet Parker model. The thickness of the current sheet for collisionless magnetic reconnection is found to be 10^3m. The reconnection time for collisionless magnetic reconnection process is found to be inversely proportional to the cube of the magnetic field and directly proportional to the length scale of electron dissipation region. We have also predicted the reconnection rate which is numerically equal to 10^{-6}. Using this value we have obtained $S = 10^{12}$, which is in agreement with the pre-determined value of Lundquist number.

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ON THE SEMI-DIFFERENTIALS OF SOME COMPLETE ELLIPTIC INTEGRALS AND THEIR DIFFERENCES

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Abstract

In this article we aim at obtaining the semi-differentials of Complete Elliptic integrals of different kinds and thier differences in terms of algebraic functions by using series manipulation technique and Pfaff-Kümmer linear transformation.

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Keywords and Phrases: Hypergeometric functions; Complete Elliptic integrals; Pfaff-Kümmer linear transformation.

1. Introduction

The Fractional Calculus is a generalization of classical calculus concerned with operations of integration and differentiation of non-integer (fractional) order. The concept of fractional operators has been introduced almost simultaneously with the development of the classical ones. The concept of differentiation (and integration) to a non-integer order has appeared surprisingly early in the history of the Calculus. It is mentioned in a letter dated September 30, 1695, from G.W. Leibniz to G.A. L'Hôpital, and in another letter dated May 28, 1697, from Leibniz to J. Wallis. This question consequently attracted the interest of many wellknown mathematicians, including Euler, Liouville, Laplace, Riemann, Grnwald, Letnikov and many others.

In 1731, L. Euler extended the derivative formula in general form [3, p.80, Eq.(2.37), [12], p.285, Eq.(5)]:

$$D_x^{\alpha}\left\{x^{\beta}\right\} = \frac{d^{\alpha}}{dx^{\alpha}}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}, \tag{1.1}$$

where α is not restricted to integer values and α may be an arbitrary complex number and $\Gamma(1+\beta)$, $\Gamma(1+\beta-\alpha)$ are well defined. When α is positive real number, then above formula stands for fractional differentiation and when α is negative real number, then above formula represents fractional integration.

Fractional Calculus adds another dimension to understand or describe basic nature in a better way. For past three centuries this subject was with mathematicians and only in last few years, this is pulled to several (applied) fields of engineering science and economics. Next decade will see several applications based on this three hundred years (old) new subject, which can be thought of as superset of fractional differintegral calculus, the conventional integer order calculus being a part of it. Differintegration is operator doing differentiation and sometimes integrations in a general sense.

The classical Pochhammer symbol $(\alpha)_p$ $(\alpha, p \in \mathbb{C})$ is defined by([10, p.22, Eq.(1), p.32, Q.N.(8) and Q.N.(9), see also [12] p.23, Eq.(22) and Eq.(23)]).

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$ is accomplished by introducing any arbitrary number of numerator and denominator parameters [12, p.42, Eq.(1)].

Each of the following results will be needed in our present study:

Some complete Elliptic integrals [4, p.321, Eq.(25)]

$$\mathbf{B}(x) = \int_0^{\frac{\pi}{2}} \frac{\cos^2\theta}{\sqrt{(1-x^2\ \sin^2\theta)}}\,d\theta = \frac{\pi}{4}\,{}_2F_1\left[\begin{array}{c}\frac{1}{2},\ \frac{1}{2};\\ \\ 2;\end{array} x^2\right];\ \ |x|<1, \tag{1.2}$$

$$\mathbf{C}(x) = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta\cos^2\theta}{\left(\sqrt{(1-x^2\ \sin^2\theta)}\right)^3}\,d\theta = \frac{\pi}{16}\,{}_2F_1\left[\begin{array}{c}\frac{3}{2},\ \frac{3}{2};\\ \\ 3;\end{array} x^2\right];\ \ |x|<1, \tag{1.3}$$

$$\mathbf{D}(x)=\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta}{\sqrt{(1-x^2\ \sin^2\theta)}}\,d\theta=\frac{\pi}{4}\,{}_2F_1\left[\begin{array}{c}\frac{1}{2},\ \frac{3}{2};\\ \\ 2;\end{array}\ x^2\right];\ \ |x|<1. \tag{1.4}$$

Complete Elliptic integral of second kind [4, p.317, Eq.(2)]

$$\mathbf{E}(x)=\int_0^{\frac{\pi}{2}}\sqrt{(1-x^2\ \sin^2\theta)}\,d\theta=\frac{\pi}{2}\,{}_2F_1\left[\begin{array}{c}\frac{1}{2},\ -\frac{1}{2};\\ \\ 1;\end{array}\ x^2\right];\ \ |x|<1. \tag{1.5}$$

Complete Elliptic integral of first kind [4, p.317, Eq.(1)]

$$\mathbf{K}(x)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{(1-x^2\ \sin^2\theta)}}=\frac{\pi}{2}\,{}_2F_1\left[\begin{array}{c}\frac{1}{2},\ \frac{1}{2};\\ \\ 1;\end{array}\ x^2\right];\ \ |x|<1. \tag{1.6}$$

Pfaff-Kümmer linear transformation[10, p.60, Eq.(4),[11], p.67, Eq.(19), [12], p.33, Eq.(19)]:

$${}_2F_1\left[\begin{array}{c}\alpha,\ \beta;\\ \\ \gamma;\end{array}\ z\right]=(1-z)^{-\alpha}\,{}_2F_1\left[\begin{array}{c}\alpha,\ \gamma-\beta;\\ \\ \gamma;\end{array}\ \tfrac{-z}{1-z}\right];\ \ |z|<1, \tag{1.7}$$

where $|\arg\ (1-z)|<\pi$ and $\gamma\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

See ref. [10, p.70, Q.N.(10)]

$${}_2F_1\left[\begin{array}{c}\alpha,\ \alpha-\frac{1}{2};\\ \\ 2\alpha;\end{array}\ z\right]=\left(\frac{2}{1+\sqrt{(1-z)}}\right)^{2\alpha-1}, \tag{1.8}$$

where $|z|<1$ and $2\alpha\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

$${}_2F_1\left[\begin{array}{c}\alpha,\ \alpha+\frac{1}{2};\\ \\ 2\alpha;\end{array}\ z\right]=\frac{1}{\sqrt{(1-z)}}\left(\frac{2}{1+\sqrt{(1-z)}}\right)^{2\alpha-1}, \tag{1.9}$$

where $|z|<1$ and $2\alpha\in\mathbb{C}\backslash\mathbb{Z}_0^-$.

Motivated by the work collected in beautiful monographs of Abramowitz *et al.*[1], Andrews[2, 3], Gradshteyn *et al.*[5], Magnus *et al.*[6], Prudnikov *et al.*[7] and the papers of Qureshi *et al.*[8, 9], we aim at obtaining semi-differentials of complete Elliptic integrals. In section 2, semi-differentials of complete Elliptic integrals of different kinds in terms of certain algebraic expressions are mentioned. In section 3, their proofs are given by using series manipulation technique and Pfaff-Kümmer linear transformation.

2. Some results involving semi-differentiation

In this section, we obtain the semi-differentiation of complete Elliptic integrals of different kinds and their differences in terms of algebraic expressions.

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{B}(\sqrt{x})\right\}=\frac{\sqrt{\pi}}{2\sqrt{x}\left(1+\sqrt{(1-x)}\right)}\ . \tag{2.1}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{C}(\sqrt{x})\right\}=\frac{\sqrt{\pi}}{4x^{\frac{5}{2}}}\left(\frac{6-4x}{\sqrt{(1-x)}}-6+x\right). \tag{2.2}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{D}(\sqrt{x})\right\}=\frac{1}{2}\sqrt{\frac{\pi}{x(1-x)}}\ \frac{1}{\left(1+\sqrt{(1-x)}\right)}. \tag{2.3}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{E}(\sqrt{x})\right\}=\frac{\sqrt{\pi}}{2}\sqrt{\left(\frac{1-x}{x}\right)}. \tag{2.4}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{K}(\sqrt{x})\right\}=\frac{1}{2}\sqrt{\left(\frac{\pi}{x(1-x)}\right)}. \tag{2.5}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\frac{1}{\sqrt{(1+x)}}\mathbf{B}\left(\sqrt{\left(\frac{x}{1+x}\right)}\right)\right\}=\sqrt{\frac{\pi}{x(1+x)}}\ \frac{1}{2\left(1+\sqrt{(1+x)}\right)}. \tag{2.6}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\frac{1}{(1+x)^{\frac{3}{2}}}\mathbf{C}\left(\sqrt{\left(\frac{x}{1+x}\right)}\right)\right\} = \frac{\sqrt{\pi}}{4x^{\frac{5}{2}}}\left(\frac{2(3+2x)}{\sqrt{(1+x)}} - 6 - x\right). \tag{2.7}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\frac{1}{\sqrt{(1+x)}}\mathbf{D}\left(\sqrt{\left(\frac{x}{1+x}\right)}\right)\right\} = \frac{1}{2}\sqrt{\frac{\pi}{x}}\left(\frac{1}{1+\sqrt{(1+x)}}\right). \tag{2.8}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\sqrt{(1+x)}\,\mathbf{E}\left(\sqrt{\left(\frac{x}{1+x}\right)}\right)\right\} = \frac{\sqrt{\pi}}{2}\sqrt{\left(\frac{1+x}{x}\right)}. \tag{2.9}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\frac{1}{\sqrt{(1+x)}}\mathbf{K}\left(\sqrt{\left(\frac{x}{1+x}\right)}\right)\right\} = \frac{1}{2}\sqrt{\frac{\pi}{x(1+x)}}. \tag{2.10}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{B}(\sqrt{x}) - 4\mathbf{C}(\sqrt{x})\right\} = \frac{-\sqrt{\pi}}{2x^{\frac{5}{2}}}\left(\frac{20-17x+x^2}{\sqrt{(1-x)}} - 20 + 7x\right). \tag{2.11}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{B}(\sqrt{x}) - \mathbf{D}(\sqrt{x})\right\} = \frac{-\sqrt{\pi x}}{2\sqrt{(1-x)}\left(1+\sqrt{(1-x)}\right)^2}. \tag{2.12}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{B}(\sqrt{x}) - \frac{1}{2}\mathbf{E}(\sqrt{x})\right\} = \frac{\sqrt{\pi}\left\{2-(2+x)\sqrt{(1-x)}\right\}}{4x^{\frac{3}{2}}}. \tag{2.13}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{B}(\sqrt{x}) - \frac{1}{2}\mathbf{K}(\sqrt{x})\right\} = \frac{-\sqrt{\pi x}}{4\sqrt{(1-x)}\left(1+\sqrt{(1-x)}\right)^2}. \tag{2.14}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{C}(\sqrt{x}) - \frac{1}{4}\mathbf{D}(\sqrt{x})\right\} = \frac{3\sqrt{\pi x}}{8\sqrt{(1-x)}\left(1+\sqrt{(1-x)}\right)^3}. \tag{2.15}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{C}(\sqrt{x}) - \frac{1}{8}\mathbf{K}(\sqrt{x})\right\} = \frac{1}{16}\sqrt{\left(\frac{\pi x}{1-x}\right)}\left\{\frac{(8\sqrt{(1-x)} - x + 8)}{\left(1+\sqrt{(1-x)}\right)^4}\right\}. \tag{2.16}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{C}(\sqrt{x}) - \frac{1}{8}\mathbf{E}(\sqrt{x})\right\} = \frac{\sqrt{(\pi x)}}{8}\,{}_4F_3\left[\begin{matrix}\frac{1}{2},\ 2,\ \frac{73+\sqrt{145}}{32},\ \frac{73-\sqrt{145}}{32}; \\ 4,\ \frac{41+\sqrt{145}}{32},\ \frac{41-\sqrt{145}}{32};\end{matrix}\quad x\right]. \tag{2.17}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{D}(\sqrt{x}) - \frac{1}{2}\mathbf{E}(\sqrt{x})\right\} = \frac{\sqrt{\pi}}{4x^{\frac{3}{2}}}\left\{\frac{x^2 - x + 2\left(1-\sqrt{(1-x)}\right)}{\sqrt{(1-x)}}\right\}. \tag{2.18}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{D}(\sqrt{x}) - \frac{1}{2}\mathbf{K}(\sqrt{x})\right\} = \frac{\sqrt{\pi x}}{4\sqrt{(1-x)}\left(1+\sqrt{(1-x)}\right)^2}. \tag{2.19}$$

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{K}(\sqrt{x}) - \mathbf{E}(\sqrt{x})\right\} = \frac{1}{2}\sqrt{\left(\frac{\pi x}{1-x}\right)}. \tag{2.20}$$

3. Demonstration of the semi-differentials

Proof of (2.1).

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{B}(\sqrt{x})\right\} = \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\frac{\pi}{4}\,{}_2F_1\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2}; \\ 2;\end{matrix}\quad x\right]\right\}$$

$$= \frac{\pi}{4}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(2)_n\ n!}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}x^n$$

$$= \frac{\pi}{4} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n \Gamma(n+1)}{(2)_n \; \Gamma\left(n+\frac{1}{2}\right)} \; \frac{x^{n-\frac{1}{2}}}{n!}$$

$$= \frac{1}{4} \sqrt{\left(\frac{\pi}{x}\right)} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n (1)_n \, x^n}{(2)_n \; n!}$$

$$= \frac{1}{4} \sqrt{\left(\frac{\pi}{x}\right)} \, {}_2F_1 \left[\begin{matrix} 1, \; \frac{1}{2}; \\ \\ 2; \end{matrix} \; x \right]. \tag{3.1}$$

Using equation (1.8) in equation (3.1) and after further simplification, we arrive at the result (2.1).

Proof of (2.2).

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \left\{ \mathbf{C}(\sqrt{x}) \right\} = \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \left\{ \frac{\pi}{16} \, {}_2F_1 \left[\begin{matrix} \frac{3}{2}, \; \frac{3}{2}; \\ \\ 3; \end{matrix} \; x \right] \right\}$$

$$= \frac{\pi}{16} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{(3)_n \; n!} \; \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \; x^n$$

$$= \frac{\pi}{16} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n}{(3)_n} \; \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)} \; \frac{x^{n-\frac{1}{2}}}{n!}$$

$$= \frac{1}{16} \sqrt{\left(\frac{\pi}{x}\right)} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n \left(\frac{3}{2}\right)_n (1)_n}{(3)_n \left(\frac{1}{2}\right)_n} \; \frac{x^n}{n!}$$

$$= \frac{1}{16} \sqrt{\left(\frac{\pi}{x}\right)} \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n (1)_n \; x^n}{(3)_n \; n!} \; (1+2n)$$

$$= \frac{1}{16} \sqrt{\left(\frac{\pi}{x}\right)} \left\{ \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n (1)_n \; x^n}{(3)_n \; n!} + 2 \sum_{n=1}^{\infty} \frac{\left(\frac{3}{2}\right)_n (1)_n \; x^n}{(3)_n \; (n-1)!} \right\}$$

$$= \frac{1}{16} \sqrt{\left(\frac{\pi}{x}\right)} \left\{ \sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\right)_n (1)_n \; x^n}{(3)_n \; n!} + x \sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}\right)_n (2)_n \; x^n}{(4)_n \; n!} \right\}$$

$$= \frac{1}{16} \sqrt{\left(\frac{\pi}{x}\right)} \left\{ {}_2F_1 \left[\begin{matrix} \frac{3}{2}, \; 1; \\ \\ 3; \end{matrix} \; x \right] + x \, {}_2F_1 \left[\begin{matrix} \frac{5}{2}, \; 2; \\ \\ 4; \end{matrix} \; x \right] \right\}. \tag{3.2}$$

Using equations (1.8) and (1.9) in equation (3.2), we get

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} \left\{ \mathbf{C}(\sqrt{x}) \right\} = \frac{1}{16} \sqrt{\left(\frac{\pi}{x}\right)} \left\{ \left(\frac{2}{1+\sqrt{1-x}} \right)^2 + \frac{x}{\sqrt{1-x}} \left(\frac{2}{1+\sqrt{1-x}} \right)^3 \right\}$$

$$= \frac{\sqrt{\pi}}{4x^{\frac{5}{2}}} \left\{ \left(1-\sqrt{1-x}\right)^2 + \frac{2}{\sqrt{1-x}} \left(1-\sqrt{1-x}\right)^3 \right\}$$

$$= \frac{\sqrt{\pi}}{4x^{\frac{5}{2}}} \left\{ \frac{-6\sqrt{1-x} - 4x + 6 + x\sqrt{1-x}}{\sqrt{1-x}} \right\}. \tag{3.3}$$

On simplifying further, we arrive at the result (2.2).

Proof of (2.3) *to* (2.5).

The proof of the results (2.3) to (2.5) are obtained by following the same steps as in the proof of the results (2.1) and (2.2). So we omit the details.

Proof of (2.6).

In the proof of the result (2.6), we proceed same as above and make use of Pfaff-Kümmer's transformation (1.7) and the equation (1.9). So we omit the details here.

Proof of (2.7).

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\frac{1}{(1+x)^{\frac{3}{2}}}\,\mathbf{C}(\sqrt{x})\right\}=\frac{\pi}{16}\,\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\frac{1}{(1+x)^{\frac{3}{2}}}\,{}_2F_1\left[\begin{array}{c}\frac{3}{2},\ \frac{3}{2};\\ 3;\end{array}\ \frac{x}{1+x}\right]\right\}. \tag{3.4}$$

Using Pfaff-Kümmer's transformation (1.7) in equation (3.4), we get

$$\begin{aligned}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\frac{1}{(1+x)^{\frac{3}{2}}}\,\mathbf{C}(\sqrt{x})\right\}&=\frac{\pi}{16}\,\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{{}_2F_1\left[\begin{array}{c}\frac{3}{2},\ \frac{3}{2};\\ 3;\end{array}\ -x\right]\right\}\\
&=\frac{\pi}{16}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n(-1)^n}{(3)_n\ n!}\ \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\ x^n\\
&=\frac{\pi}{16}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n(-1)^n}{(3)_n}\ \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\ \frac{x^{n-\frac{1}{2}}}{n!}\\
&=\frac{1}{16}\sqrt{\left(\frac{\pi}{x}\right)}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n(1)_n(-1)^n}{(3)_n\left(\frac{1}{2}\right)_n}\ \frac{x^n}{n!}\\
&=\frac{1}{16}\sqrt{\left(\frac{\pi}{x}\right)}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n(1)_n\ (-1)^n\ x^n}{(3)_n\ n!}\ (1+2n)\\
&=\frac{1}{16}\sqrt{\left(\frac{\pi}{x}\right)}\left\{\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n(1)_n\ (-1)^n\ x^n}{(3)_n\ n!}+2\sum_{n=1}^{\infty}\frac{\left(\frac{3}{2}\right)_n(1)_n\ (-1)^n\ x^n}{(3)_n\ (n-1)!}\right\}\\
&=\frac{1}{16}\sqrt{\left(\frac{\pi}{x}\right)}\left\{\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n(1)_n\ (-1)^n\ x^n}{(3)_n\ n!}-x\sum_{n=0}^{\infty}\frac{\left(\frac{5}{2}\right)_n(2)_n\ (-1)^n\ x^n}{(4)_n\ n!}\right\}\\
&=\frac{1}{16}\sqrt{\left(\frac{\pi}{x}\right)}\left\{{}_2F_1\left[\begin{array}{c}\frac{3}{2},\ 1;\\ 3;\end{array}\ -x\right]-x\,{}_2F_1\left[\begin{array}{c}\frac{5}{2},\ 2;\\ 4;\end{array}\ -x\right]\right\}.\end{aligned} \tag{3.5}$$

Using equations (1.8) and (1.9) in equation (3.5), we get

$$\begin{aligned}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\frac{1}{(1+x)^{\frac{3}{2}}}\,\mathbf{C}(\sqrt{x})\right\}&=\frac{1}{16}\sqrt{\left(\frac{\pi}{x}\right)}\left\{\left(\frac{2}{1+\sqrt{1+x}}\right)^2-\frac{x}{\sqrt{1+x}}\left(\frac{2}{1+\sqrt{1+x}}\right)^3\right\}\\
&=\frac{\sqrt{\pi}}{4x^{\frac{5}{2}}}\left\{\left(1-\sqrt{1+x}\right)^2+\frac{2}{\sqrt{1+x}}\left(1-\sqrt{1+x}\right)^3\right\}\\
&=\frac{\sqrt{\pi}}{4x^{\frac{5}{2}}}\left\{\frac{-6\sqrt{1+x}+4x+6-x\sqrt{1+x}}{\sqrt{1+x}}\right\}.\end{aligned} \tag{3.6}$$

On simplifying further, we arrive at the result (2.7).

Proof of (2.8) *to* (2.10).

By following the same steps as in the proof of the result (2.7) and making use of Pfaff-Kümmer's transformation (1.7), we arrive at the results (2.8) to (2.10). So we omit the details here.

Proof of (2.11).

$$\begin{aligned}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{B}(\sqrt{x})-4\mathbf{C}(\sqrt{x})\right\}&=\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\frac{\pi}{4}\,{}_2F_1\left[\begin{array}{c}\frac{1}{2},\ \frac{1}{2};\\ 2;\end{array}\ x\right]-\frac{\pi}{4}\,{}_2F_1\left[\begin{array}{c}\frac{3}{2},\ \frac{3}{2};\\ 3;\end{array}\ x\right]\right\}\\
&=\frac{\pi}{4}\left\{\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(2)_n\ n!}\ \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\ x^n-\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n}{(3)_n\ n!}\ \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\ x^n\right\}\\
&=\frac{\pi}{4}\left\{\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(2)_n}\ \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\ \frac{x^{n-\frac{1}{2}}}{n!}-\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n}{(3)_n}\ \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\ \frac{x^{n-\frac{1}{2}}}{n!}\right\}\end{aligned}$$

$$= \frac{1}{4}\sqrt{\left(\frac{\pi}{x}\right)}\sum_{n=0}^{\infty}\frac{(1)_n\, x^n}{n!}\left\{\frac{\left(\frac{1}{2}\right)_n}{(2)_n} - \frac{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n}{(3)_n\left(\frac{1}{2}\right)_n}\right\}$$

$$= \frac{1}{4}\sqrt{\left(\frac{\pi}{x}\right)}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n(1)_n\ x^n}{(2)_n\ n!}\left\{1 - \frac{2(1+2n)^2}{(2+n)}\right\}$$

$$= -\frac{7}{8}\sqrt{\left(\frac{\pi}{x}\right)}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{15}{8}\right)_n(1)_n\ x^n}{\left(\frac{7}{8}\right)_n(3)_n\ (n-1)!}. \tag{3.7}$$

Replacing n by $n+1$ in equation (3.7), we get

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{B}(\sqrt{x}) - 4\mathbf{C}(\sqrt{x})\right\} = \frac{-5\sqrt{\pi x}}{16}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n(2)_n\left(\frac{23}{8}\right)_n\ x^n}{(4)_n\left(\frac{15}{8}\right)_n\ n!}$$

$$= \frac{-5\sqrt{\pi x}}{16}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n(2)_n\ x^n}{(4)_n\ n!}\left(1+\frac{8n}{15}\right)$$

$$= \frac{-5\sqrt{\pi x}}{16}\left\{\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n(2)_n\ x^n}{(4)_n\ n!} + \frac{8}{15}\sum_{n=1}^{\infty}\frac{\left(\frac{3}{2}\right)_n(2)_n\ x^n}{(4)_n\ (n-1)!}\right\}$$

$$= \frac{-5\sqrt{\pi x}}{16}\left\{\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n(2)_n\ x^n}{(4)_n\ n!} + \frac{2x}{5}\sum_{n=0}^{\infty}\frac{\left(\frac{5}{2}\right)_n(3)_n\ x^n}{(5)_n\ n!}\right\}$$

$$= \frac{-5\sqrt{\pi x}}{16}\left\{{}_2F_1\left[\begin{matrix}\frac{3}{2},\ 2;\\ 4;\end{matrix}\ x\right] + \frac{2x}{5}\ {}_2F_1\left[\begin{matrix}\frac{5}{2},\ 3;\\ 5;\end{matrix}\ x\right]\right\}. \tag{3.8}$$

Using equations (1.8) and (1.9) in equation (3.8), we get

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{B}(\sqrt{x}) - 4\mathbf{C}(\sqrt{x})\right\} = \frac{-5\sqrt{\pi x}}{16}\left\{\left(\frac{2}{1+\sqrt{1-x}}\right)^3 + \frac{2x}{5\sqrt{1-x}}\left(\frac{2}{1+\sqrt{1-x}}\right)^4\right\}$$

$$= \frac{-5\sqrt{\pi}}{2x^{\frac{5}{2}}}\left\{\left(1-\sqrt{1-x}\right)^3 + \frac{4}{5\sqrt{1-x}}\left(1-\sqrt{1-x}\right)^4\right\}$$

$$= \frac{-\sqrt{\pi}}{2x^{\frac{5}{2}}}(1-\sqrt{1-x})^3\left\{\frac{4-\sqrt{1-x}}{\sqrt{1-x}}\right\}. \tag{3.9}$$

On further simplification, we arrive at the result (2.11).

Proof of (2.12).

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{B}(\sqrt{x}) - \mathbf{D}(\sqrt{x})\right\} = \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\frac{\pi}{4}\ {}_2F_1\left[\begin{matrix}\frac{1}{2},\ \frac{1}{2};\\ 2;\end{matrix}\ x\right] - \frac{\pi}{4}\ {}_2F_1\left[\begin{matrix}\frac{1}{2},\ \frac{3}{2};\\ 2;\end{matrix}\ x\right]\right\}$$

$$= \frac{\pi}{4}\left\{\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(2)_n\ n!}\ \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\ x^n - \sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{3}{2}\right)_n}{(2)_n\ n!}\ \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\ x^n\right\}$$

$$= \frac{\pi}{4}\left\{\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(2)_n}\ \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\ \frac{x^{n-\frac{1}{2}}}{n!} - \sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{3}{2}\right)_n}{(2)_n}\ \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\ \frac{x^{n-\frac{1}{2}}}{n!}\right\}$$

$$= \frac{1}{4}\sqrt{\left(\frac{\pi}{x}\right)}\sum_{n=0}^{\infty}\frac{(1)_n\, x^n}{(2)_n\ n!}\left\{\left(\frac{1}{2}\right)_n - \left(\frac{3}{2}\right)_n\right\}$$

$$= -\frac{1}{2}\sqrt{\left(\frac{\pi}{x}\right)}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n(1)_n\ x^n}{(2)_n\ (n-1)!}. \tag{3.10}$$

Replacing n by $n+1$ in equation (3.10), we get

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{B}(\sqrt{x}) - \mathbf{D}(\sqrt{x})\right\} = \frac{-\sqrt{\pi x}}{8}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n(2)_n\ x^n}{(3)_n\ n!}$$

$$= \frac{-\sqrt{\pi x}}{8}\ {}_2F_1\left[\begin{array}{c} \frac{3}{2},\ 2; \\ \\ 3; \end{array} x\right]. \tag{3.11}$$

Using equation (1.9) in equation (3.11) and after further simplification, we get the result (2.12).

Proof of (2.13) *and* (2.14).

For the proof of the results (2.13) and (2.14), we follow the same steps as in the proof of the results (2.11) and (2.12) and make use of the equation (1.8). So we omit the details here.

Proof of (2.15).

Similarly, the proof of the result (2.15) is obtained by following the same steps as in the proof of the results (2.11) and (2.12) and making use of the equation (1.9). So we omit the details here.

Proof of (2.16).

$$\begin{aligned}
\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{C}(\sqrt{x}) - \frac{1}{8}\mathbf{K}(\sqrt{x})\right\} &= \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\frac{\pi}{16}\ {}_2F_1\left[\begin{array}{c} \frac{3}{2},\ \frac{3}{2}; \\ \\ 3; \end{array} x\right] - \frac{\pi}{16}\ {}_2F_1\left[\begin{array}{c} \frac{1}{2},\ \frac{1}{2}; \\ \\ 1; \end{array} x\right]\right\} \\
&= \frac{\pi}{16}\left\{\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n}{(3)_n\ n!}\ \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\ x^n - \sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(1)_n\ n!}\ \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\ x^n\right\} \\
&= \frac{\pi}{16}\left\{\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n}{(3)_n}\ \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\ \frac{x^{n-\frac{1}{2}}}{n!} - \sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(1)_n}\ \frac{\Gamma(n+1)}{\Gamma\left(n+\frac{1}{2}\right)}\ \frac{x^{n-\frac{1}{2}}}{n!}\right\} \\
&= \frac{1}{16}\sqrt{\left(\frac{\pi}{x}\right)}\sum_{n=0}^{\infty}\frac{x^n}{n!}\left\{\frac{\left(\frac{3}{2}\right)_n\left(\frac{3}{2}\right)_n(1)_n}{\left(\frac{1}{2}\right)_n(3)_n} - \left(\frac{1}{2}\right)_n\right\} \\
&= \frac{1}{16}\sqrt{\left(\frac{\pi}{x}\right)}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\ x^n}{n!}\left\{\frac{2(1+2n)(1+2n)}{(1+n)(2+n)} - 1\right\} \\
&= \frac{5}{32}\sqrt{\left(\frac{\pi}{x}\right)}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{12}{7}\right)_n(1)_n\ x^n}{\left(\frac{5}{7}\right)_n(3)_n\ (n-1)!}.
\end{aligned} \tag{3.12}$$

Replacing n by $n+1$ in equation (3.12), we get

$$\begin{aligned}
\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{C}(\sqrt{x}) - \frac{1}{8}\mathbf{K}(\sqrt{x})\right\} &= \frac{\sqrt{\pi x}}{16}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n\left(\frac{19}{7}\right)_n(2)_n\ x^n}{\left(\frac{12}{7}\right)_n(4)_n\ n!} \\
&= \frac{\sqrt{\pi x}}{16}\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n(2)_n\ x^n}{(4)_n\ n!}\left(1 + \frac{7n}{12}\right) \\
&= \frac{\sqrt{\pi x}}{16}\left\{\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n(2)_n\ x^n}{(4)_n\ n!} + \frac{7}{12}\sum_{n=1}^{\infty}\frac{\left(\frac{3}{2}\right)_n(2)_n\ x^n}{(4)_n\ (n-1)!}\right\} \\
&= \frac{\sqrt{\pi x}}{16}\left\{\sum_{n=0}^{\infty}\frac{\left(\frac{3}{2}\right)_n(2)_n\ x^n}{(4)_n\ n!} + \frac{7x}{16}\sum_{n=0}^{\infty}\frac{\left(\frac{5}{2}\right)_n(3)_n\ x^n}{(5)_n\ n!}\right\} \\
&= \frac{\sqrt{\pi x}}{16}\left\{{}_2F_1\left[\begin{array}{c} \frac{3}{2},\ 2; \\ \\ 4; \end{array} x\right] + \frac{7x}{16}\ {}_2F_1\left[\begin{array}{c} \frac{5}{2},\ 3; \\ \\ 5; \end{array} x\right]\right\}.
\end{aligned} \tag{3.13}$$

Using equations (1.8) and (1.9) in equation (3.13), we get

$$\begin{aligned}
\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left\{\mathbf{C}(\sqrt{x}) - \frac{1}{8}\mathbf{K}(\sqrt{x})\right\} &= \frac{\sqrt{\pi x}}{16}\left\{\left(\frac{2}{1+\sqrt{1-x}}\right)^3 + \frac{7x}{16\sqrt{1-x}}\left(\frac{2}{1+\sqrt{1-x}}\right)^4\right\} \\
&= \frac{\sqrt{\pi}}{2x^{\frac{5}{2}}}\left\{\left(1-\sqrt{1-x}\right)^3 + \frac{7}{8\sqrt{1-x}}\left(1-\sqrt{1-x}\right)^4\right\}.
\end{aligned} \tag{3.14}$$

On further simplification, we arrive at the result (2.16).

Proof of (2.17) *to* (2.20).
By following the same steps as in the proof of the results (2.11) to (2.16), we arrive at the results (2.17) to (2.20). So we omit the details here.

4. Concluding remarks and observations

In this paper, we have obtained the semi-differentials of Complete Elliptic integrals of different kinds in terms of algebraic functions by using series manipulation technique and Pfaff-Kümmer linear transformation. We conclude this paper with the remark that the results deduced above are expected to lead some potential applications in several fields of Applied Mathematics, Statistics and Engineering Sciences.

Conflicts of interests. The authors declare that there are no conflicts of interest.

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UNSTEADY MHD FLOW OVER A PERMEABLE STRETCHING SURFACE WITH SUCTION/INJECTION AND HEAT SOURCE/SINK

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Abstract

The inspiration driving this article is to explore the unsteady *MHD* flow across stretching surface along with suction/injection and heat source/sink inserted into a porous medium. Governing *PDEs* are altered into non-linear *ODEs*, which are numerically solved by utilizing *MATLAB* built-in routine Bvp4c. Impact of physical parameters affecting velocity and energy are investigated using portrayal and tabular expressions. The results displayed that by improving the heat source and permeability parameter, temperature profiles increase while antipodal effect is observed in velocity profiles. The effect of various parameters like magnetic, suction/injection, Eckert and Prandtl number are also examined through graphs. The achieved results thus affirm that an exquisite agreement is acquired with those available in the open literature.
2020 Mathematical Sciences Classification: 76D05, 76D10, 76W05
Keywords and Phrases: *MHD*, Permeable stretching surface, Porous medium, Heat source/sink.

1. Introduction

These days applications on stretchable surfaces are extensively engaged in the field of mechanical as well as industrial engineering. Scientists and researcher are continuously looking to create better systems and increase their performance by improving heat transfer from them. Structure and utilization of the stretchable surfaces are well known and quickest tactics to accomplish this goal. Stretchable surfaces have a various of characteristics, like fins and heat sinks are often utilized in various applications as an example aviation industry, microelectronics, cooling electrical devices, heat exchangers, gas turbines , etc. The behaviour of boundary layer flow across continuous solid surfaces was first addressed by Sakiadis [23]. Crane [4] investigated the flow through a stretching sheet that was kept at a consistent length. Temperature distribution across a stretching sheet with homogeneous heat flow was described by Dutta et al. [8]. Heat transmission on a continuously stretching plate with suction or blowing was investigated by Chen and Char [5]. The simultaneous heat and mass transport in a continuous liquid film on a lateral stretching sheet are examined by Andersson et al. [1]. Andersson's approach is extended by Elbashbeshy and Bazid [10] across an unsteady stretching surface.

Flow through porous media occurs into various engineering situations and has noteworthy as well as engineering applications, as illustration, flow across ion-exchange beds and packed beds, production of energy from geothermal areas, Filtration of minerals from liquids. Ali [2] looked at the heat transmission properties of a stretched surface using suction/injection. Gupta and Gupta [11] investigated the effects of suction or blowing on a stretched sheet. Heat transmission through an unstable stretched permeable surface with a fixed wall temperature was explored by Ishak et al. [12]. Cortell [10] investigated the influence of viscous dissipation and heat radiation on fluid flow through a non-linearly stretched permeable surface.

Flow of *MHD* across a stretching permeable surface has important applications in industrial processes, modern metallurgical and metal-working process such as hot rolling, glass blowing, paper production, plastic films, metal spinning, plasma studies, wire coating, nuclear reactors, geothermal energy extraction, electromagnetic propulsion and polymer extrusion, etc. Choudhary et al. [7] used suction/injection to examine unsteady *MHD* flow and heat transfer across a stretched permeable surface. Butt et al. [3] investigated the impact of magnetic fields on entropy production in viscous flow through a stretching cylinder contained in a porous material using numerical simulations. Radiation effects on *MHD* flow near the stretching sheet's stagnation point were explored by Jat and Chaudhary [14]. In the presence of a heat source/sink, Mukhopadhyay and Layek [18] investigated the effects of altering fluid viscosity on flow through a heated stretched sheet inserted in a porous media. Several research analyses are performed regarding some relevant work on *MHD* flow for different geometry as provided Refs [9,13,16,17,19,21,22,25].Using semi-analytical methods, Jabeen et al. [15] investigated the Magnetohydrodynamic fluid flow across a nonlinear stretching sheet in a porous material.

Based on the aforementioned review of the literature, it is observed that few works are available on the fluid flow across stretchable surface with suction/injection and heat source/sink embedded in a porous medium. Our main goal in this article is to numerical investigate the flow of unsteady *MHD* across permeable stretching surface with the effect of heat source/sink and suction/injection. The numerical solution of suggested issue is attained by supplicating bvp4c from *MATLAB* and appearances of all involving parameters are discussed.

2. Mathematical formulation

Let us consider the 2-D unsteady flow of a viscous incompressible fluid across a permeable stretching surface adjacent to a porous medium. The flow is limited in the semi infinite region $y > 0$ and the sheet is coincides with the plane $y=0$. The x-axis is taken along the sheet (Figure 2.1) and a homogeneous magnetic field H_0^2 is applied along the y-axis. The flow is generated by linear stretching of the sheet and there is no free stream velocity within the boundary layer.

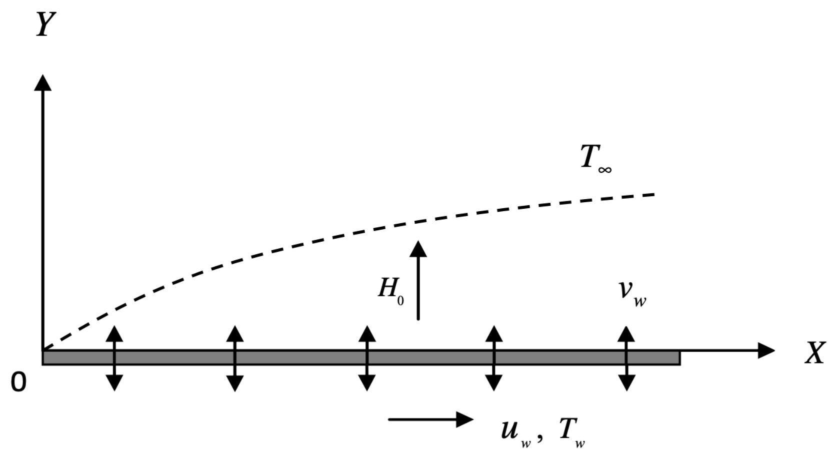

Figure 2.1: Physical model of the problem.

Under usual boundary-layer approximations the continuity, momentum and energy equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \left(\frac{\sigma_e \mu_e^2 {H_0}^2}{\rho} + \frac{\nu}{k_p}\right)u, \tag{2.2}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{Q\,(T - T_\infty)}{\rho C_p}, \tag{2.3}$$

and the boundary conditions are given by:

$$u = u_w(x,t), \quad v = v_w(x,t), \quad T = T_w(x,t) \qquad at \quad y = 0, \tag{2.4}$$
$$u \to 0, \quad T \to T_\infty \qquad at \ \ y \to \infty,$$

where u and v are velocity component in the x and y directions, respectively, ρ is the fluid density, ν is the kinematic viscosity, C_p is the specific heat at constant pressure, T is fluid temperature, σ_e is the electrical conductivity, μ_e is the magnetic permeability, α is the thermal diffusivity, μ is the coefficient of viscosity, k_p is the permeability of the porous medium and Q is the heat generation/absorption coefficient.

By introducing the following quantities Ishak et al. [12]

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial y}, \qquad \eta = y\sqrt{\frac{u_w}{\nu x}}, \qquad \psi\,(x, y, t) = \sqrt{\nu x u_w} f\,(\eta) \tag{2.5}$$
$$and \qquad T = T_\infty + \frac{b}{a} u_w \theta\,(\eta),$$

where $\psi(x, y, t)$ is the stream function, $f(\eta)$ is the dimensionless stream function, η is the similarity variable, and $\theta(\eta)$ is the dimensionless temperature. Equation (2.2) and (2.3) thus reduce to the following non-dimensional form

$$f''' - f'^2 + f'' f - (M + k_1)\,f' - A\left(f' + \frac{\eta}{2} f''\right) = 0, \tag{2.6}$$

$$\frac{1}{Pr}\theta'' + f\theta' - A\left(\theta + \frac{\eta}{2}\theta'\right) - (f' - \lambda)\,\theta + Ecf''^2 = 0. \tag{2.7}$$

Boundary conditions (2.4) reduce as:

$$\eta = 0: \qquad f(\eta) = f_0, \qquad f'(\eta) = 1, \qquad \theta(\eta) = 1$$
$$\eta \to \infty: \qquad f'(\eta) \to 0, \qquad \theta(\eta) \to 0 \tag{2.8}$$

where primes denote differentiation with respect to η . $A = c/a$ is the unsteadiness parameter, $M = (\sigma_e \mu_e^2 H_0^2 \nu Re_x)/(\rho u_w^2)$ is the magnetic field parameter, $Re_x = (u_w x)/\nu$ is the local Reynolds number, $Pr = \nu/\alpha$ is the Prandtl number, $Ec = u_w^2/C_p(T_w - T_\infty)$ is the Eckert number, $f_0 = -(v_w/u_w)\sqrt{Re_x}$ the suction/injection parameter, $k_1 = \nu x/k_p u_w$ is the permeability parameter and $\lambda = Qx/\rho C_p u_w$ is heat source/sink parameter.

The physical quantities skin friction coefficient C_f and local Nusselt number Nu_x are defined as

$$C_f = \frac{2\mu\left(\frac{\partial u}{\partial y}\right)\Big|_{y=0}}{\rho {u_w}^2} = \frac{2f''(0)}{\sqrt{Re_x}}, \tag{2.9}$$

$$Nu_x = -\frac{x}{(T_w - T_\infty)}\frac{\partial T}{\partial y}\Big|_{y=0} = -\theta'(0)\sqrt{Re_x}\,, \tag{2.10}$$

where $Re_x = (u_w x)/\nu$ local Reynolds number.

3. Results and Discussion

The system of nonlinear ordinary equations (2.6) and (2.7) along with the boundary conditions (2.8) are solved numerically using the Bvp4c in *MATLAB*. The effects of the governing parameters, namely magnetic field parameter M, Prandtl number Pr, suction/injection parameter f_0, permeability parameter k_1, heat source/sink parameter λ, unsteady parameter A and Eckert number Ec on the flow, and temperature profiles are examined. The physical parameters are involved by subsequent manageable assortments: $0.1 \le M \le 4$, $0.7 \le Pr \le 5$, $-1 \le f_0 \le 1$, $0.2 \le K_1 \le 0.8$, $0.1 \le A \le 3$, $-0.2 \le \lambda \le 0.2$, $0.1 \le Ec \le 2$. A comparative investigation is made in order to authorize the current numerical results with predetermined outcomes Table 3.2. Based on the findings, the calculated results show that this is an outstanding and significant study.

Figures 3.1-3.2 represent the prominences of f_0 on velocity and temperature profiles. By enhancing the f_0 parameter, velocity as well as temperature profile get cut down. Figures 3.3-3.4 exhibit consequences of unsteady parameter A on f' and θ profiles. An increase in unsteady parameter A reduces f' and θ profiles.

Figures 3.5-3.6 show the influence of M on f' and θ profiles. As values of M are enhanced, the f' profiles decrease whereas temperature profiles θ is improved. The antagonistic force identified as Lorentz force comes into play when magnetic field is incumbent on the field of flow with which the boundary layer thickness for momentum diminishes under the effect of strong magnetic domain.

Figures 3.7-3.8 represent the prominences of permeability parameter k_1 on f' and θ profiles. By improving the k_1 values velocity profiles reduce rapidly whereas θ profile increase. Figure 3.9 exhibits consequences of Eckert number Ec on temperature profiles. It is notice that temperature profiles grow as Ec values are increased.

Figure 3.10 show the influence of Pr on temperature profiles. It is observed that temperature profiles are improved as Pr values are increased. The proportion of the amount of momentum diffusivity with that of thermal diffusivity is termed as Prandtl number. In fact, fluids with lower Prandtl number possess higher heat conductivities than those with higher Prandtl number therefore rate of thermal diffusion from surface considered is faster in the fluids carrying lower Prandtl number. Figure 3.11 represents the prominences of heat source/sink parameter λ on θ profiles. Rising the value of λ, enhancement in θ profiles is noticed.

Table 3.2 represents the investigations on the internal assessment with those previously issued consequences for listed researchers namely Ishak et al. [12] and Choudhary et al. [7]. Under various limiting cases the compression with previously available issued outcomes is made and excellent agreement is achieved which validates the presented investigations with prescribed accuracy.

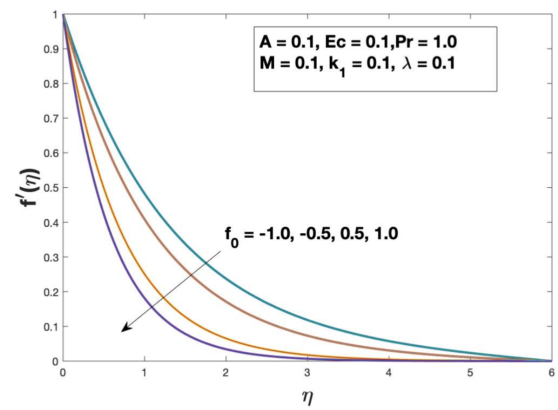

Figure 3.1: Influence of Suction/injection on velocity profiles.

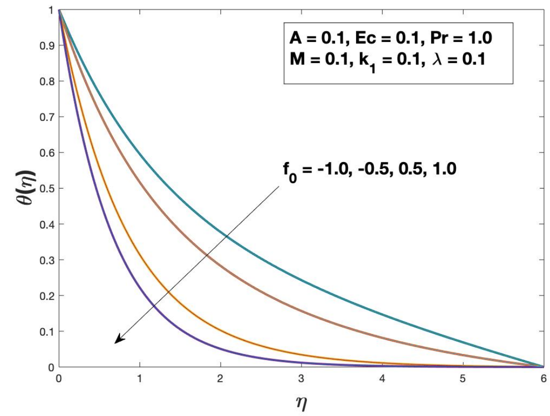

Figure 3.2: Influence of Suction/injection on temperature profiles.

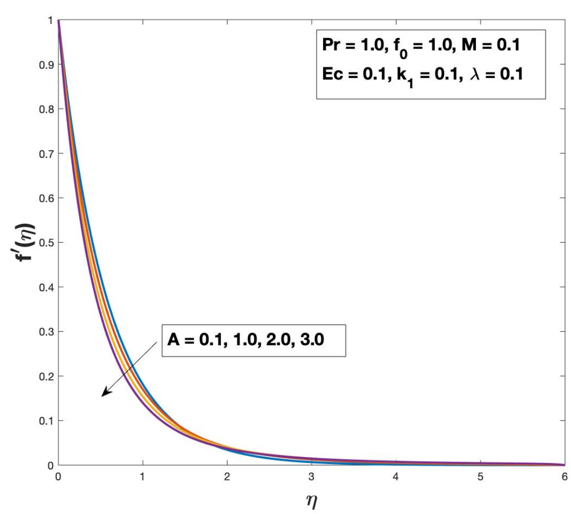

Figure 3.3: Influence of A on velocity profiles.

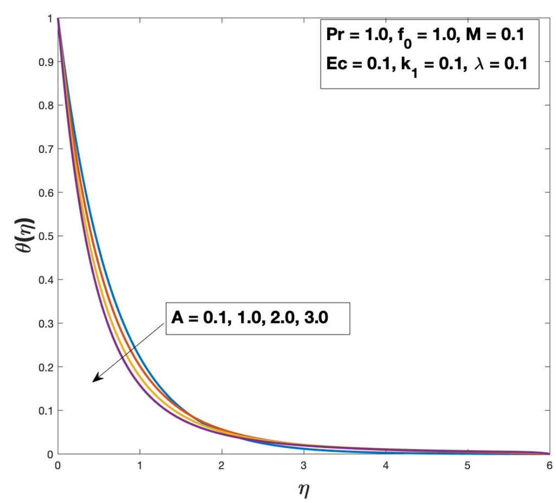

Figure 3.4: Influence of A on temperature profiles.

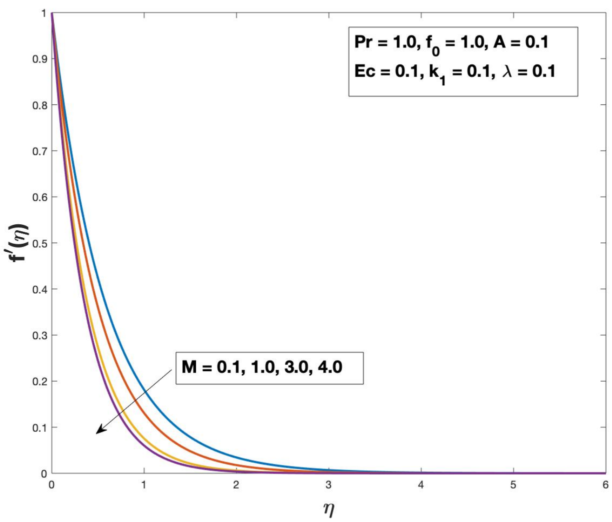

Figure 3.5: Influence of *M* on velocity profiles.

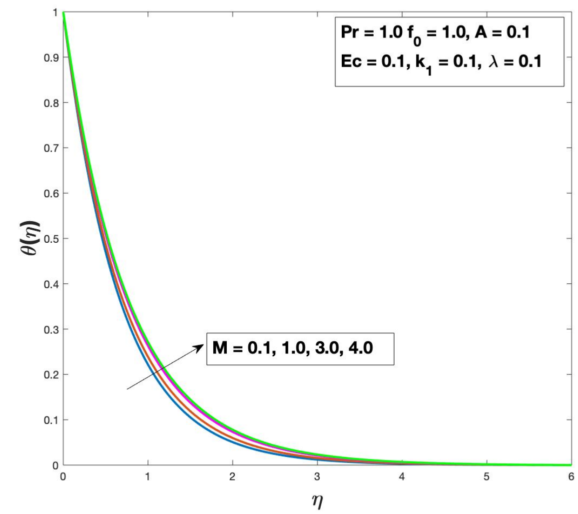

Figure 3.6: Influence of *M* on temperature profiles.

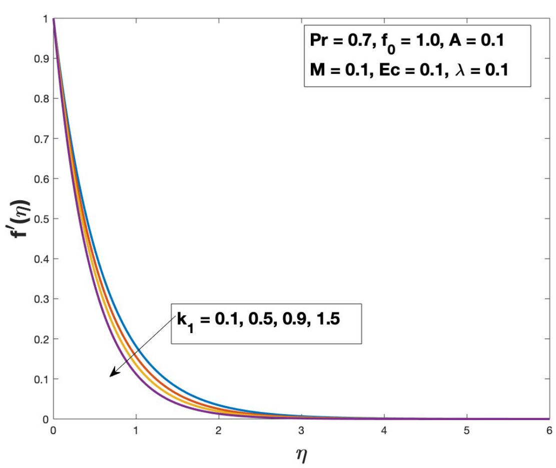

Figure 3.7: Influence of Permeability parameter on velocity profiles.

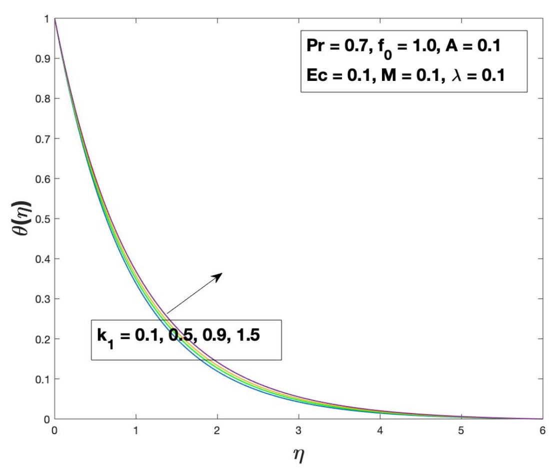

Figure 3.8: Influence of Permeability parameter on temperature profiles.

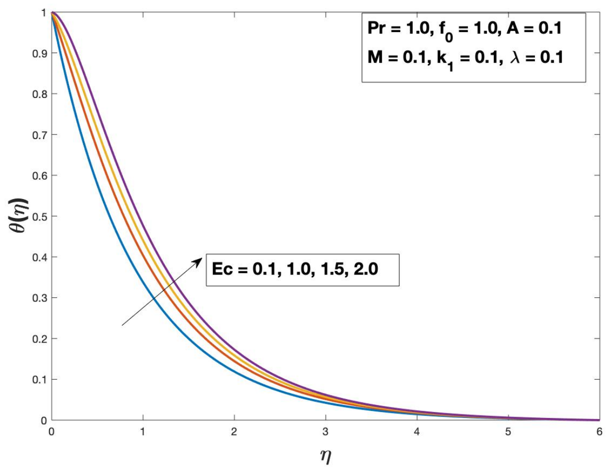

Figure 3.9: Influence of *Ec* on temperature profiles.

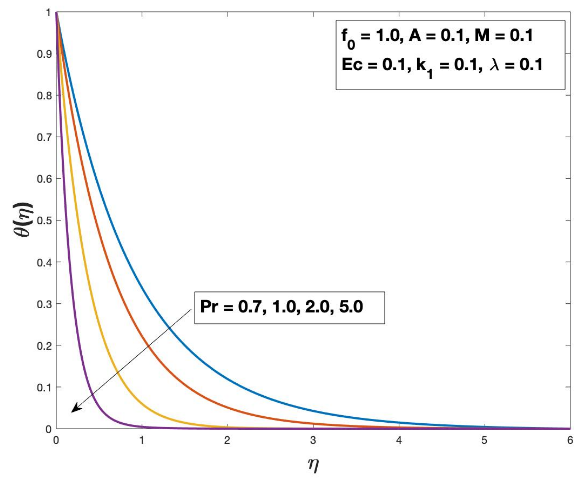

Figure 3.10: Influence of *Pr* on temperature profiles.

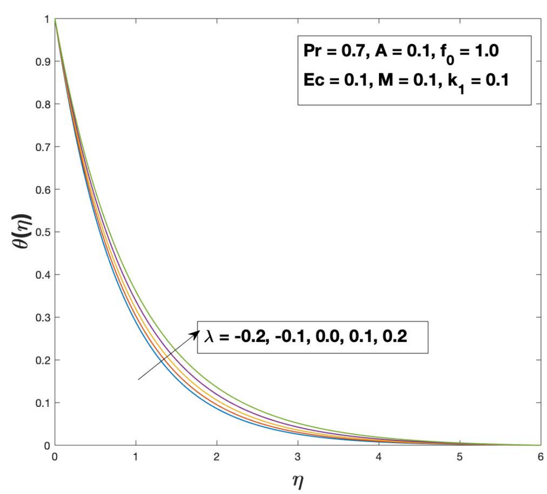

Figure 3.11: Influence of Heat source/sink parameter on Temperature profiles.

Table **3.1** Comparison of $-f''(0)$ and $-\theta'(0)$ for varying parameters: f_0 ,A, M, Pr, Ec, k_1, λ

f_0	A	M	Pr	Ec	k_1	λ	present results $-f''(0)$	present result $-\theta'(0)$
-1.0	0.1	0.1	0.1	1.0	0.1	0.1	0.73811	0.55720
-0.5							0.90749	0.69902
0.5							1.40230	1.17331
1.0							1.72921	1.50172
-0.5	1.0	0.0	1.0	0.0	0.0	0.0		0.80951
0.0								1.32052
0.5								2.22236
1.0	1.0	0.1	1.0	0.1	0.1	0.1	1.94957	1.74740
	2.0						2.17661	1.99300
	3.0						2.38244	2.20858
1.0	0.1	1.0	1.0	0.1	0.1	0.1	2.05394	1.43771
		3.0					2.60189	1.34110
		4.0					2.82782	1.30500
1.0	0.1	0.1	0.7	0.1	0.1	0.1	1.72915	1.10878
			2.0					2.68577
			5.0					5.83928
1.0	0.1	0.1	1.0	1.0	0.1	0.1	1.72915	0.80698
				1.5				0.42100
				2.0				0.03503
1.0	0.1	0.1	1.0	0.1	0.2	0.1	1.76943	1.49350
					0.4		1.84630	1.47802
					0.6		1.91895	1.46368
1.0	0.1	0.1	1.0	0.1	0.1	-0.2	1.72915	1.65513
						0.0		1.55612
						0.2		1.44297

Table **3.2** Comparison of $-f''(0)$ and $-\theta'(0)$ for varying parameters: f_0, A, M, Pr and Ec

f_0	A	M	Pr	Ec	Ishak et al. [9] $-\theta'(0)$	Choudhary et al. [11] $-f''(0)$	Choudhary et al. [11] $-\theta'(0)$	present results $-f''(0)$	present result $-\theta'(0)$
-1.0	0.1	0.1	0.1	1.0		0.71244	0.71121	0.71243	0.71120
-0.5						0.82241	0.81408	0.82231	0.81407
0.5						1.31617	1.30514	1.31615	1.30512
1.0						1.64870	1.63647	1.64868	1.63645
-0.5	1.0	0.0	1.0	0.0	0.8095		0.80957		0.80951
0.0					1.3205		1.32064		1.32052
0.5					2.2224		2.22255		2.22236

4. Conclusion

In this study numerical solutions for unsteady *MHD* flow across a stretching permeable surface with suction/ injection and heat source/sink embedded in a porous medium is analysied. The influence of distinct parameters on the behaviour of fluid flow including temperature profiles are abridged as given below:

- Velocity profiles decrease as the parameter f_0 increases.
- The velocity profiles get cut down with increasing values of M while the temperature profiles rise.
- Temperature profiles increase as Eckert number improved.
- Temperature profiles increase by increasing permeability parameter and source/sink parameter.

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ON A LOWER BOUND INEQUALITY FOR THE DERIVATIVE OF A POLYNOMIAL

By

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Abstract

Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| < K, K > 0$ while s-fold zeros are located at origin. In this paper, by motivation with a result of Aziz[A Refinement of an Inequality of S. Bernstein, *Journal of Mathematical Analysis and Applications*, **144** (1989), 226-235.], we propose some new estimates of the lower bound of $|P'(z)|$ in terms of $\max |P(z)|$ on $|z| = 1$.

2020 Mathematical Sciences Classification: 30D15, 30A10.

Keywords and Phrases: Polynomials; Inequalities; maximum modulus; Zeros.

1. Introduction and statement of results

Let a polynomial $P(z)$ of degree n has all its zeros in $|z| \le 1$.Then it was shown by Turan[12], that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.1}$$

The result is sharp and equality in (1.1) holds if all the zeros of $P(z)$ lie on $|z| = 1$.

Aziz[1] proved that if $P(z)$ is a polynomial of degree n which has s-fold zeros at the origin, $0 \le s \le n$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \left(2 \max_{|z|=1} |P(z)| - \left(1 - \frac{s}{n}\right)(M_1^* + M_2^*) \right), \tag{1.2}$$

where

$$M_1^* = \max_{1\le k\le (n-s)} |P(e^{2k\pi i/(n-s)})| \tag{1.3}$$

and

$$M_2^* = \max_{1\le k\le (n-s)} |P(e^{(1+2k)\pi i/(n-s)})|. \tag{1.4}$$

In literature (see [2,3,4,8,9,11]), there are several results concerning the estimation of upper bound of maximum modulus of polynomial and its derivative. For deep understanding of the subject matter, we have studied other research work (see [5,6,7,10]). Here, we propose to obtain some estimates for the lower bound of maximum modulus of the derivative of a polynomial.

2. Main results

In this paper, we shall present certain new inequalities to the polynomials $P(z)$ of degree n having all its zeros inside a disk of prescribed radius of which s-fold zeros lying at origin.

Theorem 2.1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le 1$ and s-fold zeros at the origin, then*

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \left(2 \max_{|z|=1} |P(z)| - \left(1 - \frac{s}{n}\right)(M_\alpha^{*2} + M_{\alpha+\pi}^{*2})^{1/2} \right), \tag{2.1}$$

where

$$M_\alpha^* = \max_{1\le k\le (n-s)} |P(e^{i(\alpha+2k\pi)/(n-s)})| \tag{2.2}$$

and $M^*_{\alpha+\pi}$ is obtained from (2.2) *by replacing α with $\alpha + \pi$.*

On taking $s = 0$ in Theorem 2.1, we obtain the following

Corollary 2.1. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \le 1$, then*

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \left(2 \max_{|z|=1} |P(z)| - (M_\alpha^2 + M_{\alpha+\pi}^2)^{1/2} \right), \tag{2.3}$$

where M_α is obtained from (2.2) *for $s = 0$.*

Further, we are able to generalise our Theorem 2.1. More precisely, we prove the

Theorem 2.2. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K, K \leq 1$ and s-fold zeros at the origin, then*

$$\max_{|z|=1}|P'(z)| \geq n\max_{|z|=1}|P(z)| - \frac{(n-s)K}{\sqrt{2(1+K^2)}}(M_\alpha^{*2}+M_{\alpha+\pi}^{*2})^{1/2}, \tag{2.4}$$

where M_α^* is defined by (2.2).

Remark 2.1. For $K = 1$, Theorem 2.2 reduces to Theorem 2.1

On taking $s = 0$ in Theorem 2.2, we obtain the following

Corollary 2.2. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K, K \leq 1$, then*

$$\max_{|z|=1}|P'(z)| \geq n\max_{|z|=1}|P(z)| - \frac{nK}{\sqrt{2(1+K^2)}}(M_\alpha^{2}+M_{\alpha+\pi}^{2})^{1/2}, \tag{2.5}$$

where M_α is obtained from (2.2) *for $s = 0$.*

While seeking the corresponding complimentary result of (2.4) for polynomial having its zeros in $|z| \leq K, K \geq 1$, we prove following

Theorem 2.3. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K, K \geq 1$ and s-fold zeros at the origin, then*

$$\max_{|z|=1}|P'(z)| \geq n\max_{|z|=1}|P(z)| - \frac{(n-s)K^{(n-s)}}{\sqrt{2(1+K^{2(n-s)})}}(M_\alpha^{*2}+M_{\alpha+\pi}^{*2})^{1/2}, \tag{2.6}$$

where M_α^* is defined by (2.2).

If we take $s = 0$ in Theorem 2.3, we get the following

Corollary 2.3. *If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq K, K \geq 1$, then*

$$\max_{|z|=1}|P'(z)| \geq n\max_{|z|=1}|P(z)| - \frac{nK^n}{\sqrt{2(1+K^{2n})}}(M_\alpha^{2}+M_{\alpha+\pi}^{2})^{1/2}, \tag{2.7}$$

where M_α is obtained from (2.2) *for $s = 0$.*

3. Lemmas

For the proofs of main results,we need the following Lemmas

Lemma 3.1. *If $P(z)$ is a polynomial of degree n having no zeros in $|z| < K, K \geq 1$, then for every real α,*

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{\sqrt{2(1+K^2)}}(M_\alpha^{2}+M_{\alpha+\pi}^{2})^{1/2}, \tag{3.1}$$

where

$$M_\alpha = \max_{1\leq k\leq n}|P(e^{i(\alpha+2k\pi)/n})| \tag{3.2}$$

and $M_{\alpha+\pi}$ is obtained from (3.2) *by replacing α with $\alpha + \pi$.*

Lemma 3.2. *If $P(z)$ is a polynomial of degree n having no zeros in $|z| < K, K \leq 1$, then for every real α,*

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{\sqrt{2(1+K^{2n})}}(M_\alpha^{2}+M_{\alpha+\pi}^{2})^{1/2}, \tag{3.3}$$

provided $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$ where $Q(z) = z^n\overline{P(1/\overline{z})}$ and M_α is defined by (3.2).

Lemma 3.1 and Lemma 3.2 are special cases of Rather and Shah's results ([9], Theorem 3 and Theorem 5, for $m = 0$).

4. Proof of the main results

As Theorem 2.1 is a special case of Theorem 2.2 (for $K = 1$), so we only need here to prove Theorem 2.2.

Proof of Theorem 2.2. Let $P(z) = z^s H(z)$, where $H(z)$ is a polynomial of degree $(n-s)$ having all its zeros in $|z| \leq K, K \leq 1$ and $H(0) \neq 0$. If $G(z) = z^{n-s}\overline{H(1/\overline{z})}$, then $G(z)$ has no zeros in $|z| < 1/K, 1/K \geq 1$.

If $Q(z) = z^n\overline{P(1/\overline{z})}$, then $|Q(z)| = |P(z)|$ for $|z| = 1$. Clearly, $Q(z)$ is a polynomial of degree $(n-s)$ and

$$\begin{aligned} Q(z) =& z^n\overline{(1/\overline{z})^s H(1/\overline{z})} \\ =& z^{(n-s)}\overline{H(1/\overline{z})} \\ =& G(z). \end{aligned}$$

It is immediate for $|z| = 1$,

$$|Q(z)| = |G(z)| = |H(z)|.$$

Moreover,

$$zQ'(z) = nz^n\overline{P(1/\overline{z})} - z^{(n-1)}\overline{P'(1/\overline{z})},$$

from which it follows that for $|z| = 1$,

$$|Q'(z)| = |z^{(n-1)}\overline{Q'(1/\overline{z})}| = |nP(z) - zP'(z)|.$$

We conclude that

$$|P'(z)| + |Q'(z)| \geq n|P(z)|.$$

Choosing θ such that $|P(e^{i\theta})| = \max_{|z|=1} |P(z)|$, we get

$$|P'(e^{i\theta})| + |Q'(e^{i\theta})| \geq n \max_{|z|=1} |P(z)|,$$

which implies that

$$\max_{|z|=1} |P'(z)| + \max_{|z|=1} |Q'(z)| \geq n \max_{|z|=1} |P(z)|. \tag{4.1}$$

As $|Q(z)| = |P(z)|$ for $|z| = 1$, then

$$\max_{1\leq k\leq(n-s)} |Q(e^{i(\alpha+2k\pi)/(n-s)})| = \max_{1\leq k\leq(n-s)} |P(e^{i(\alpha+2k\pi)/(n-s)})| = M^*_\alpha,$$

and

$$\max_{1\leq k\leq(n-s)} |Q(e^{i(\alpha+(1+2k)\pi)/(n-s)})| = \max_{1\leq k\leq(n-s)} |P(e^{i(\alpha+(1+2k)\pi)/(n-s)})| = M^*_{\alpha+\pi},$$

Since $G(z)$ has no zeros in $|z| < 1/K, 1/K \geq 1$, applying Lemma 3.1 to $G(z)$, we get,

$$\max_{|z|=1} |G'(z)| \leq \frac{(n-s)}{\sqrt{2(1+(1/K)^2)}}(M^{*2}_\alpha + M^{*2}_{\alpha+\pi})^{1/2}.$$

As $Q(z) = G(z)$, so we have $Q'(z) = G'(z)$ and $|Q'(z)| = |G'(z)|$ for $|z| = 1$. So,

$$\max_{|z|=1} |Q'(z)| \leq \frac{(n-s)}{\sqrt{2(1+(1/K)^2)}}(M^{*2}_\alpha + M^{*2}_{\alpha+\pi})^{1/2}. \tag{4.2}$$

Therefore, from (4.1) and (4.2), we get

$$\max_{|z|=1} |P'(z)| + \frac{(n-s)K}{\sqrt{2(1+K^2)}}(M^{*2}_\alpha + M^{*2}_{\alpha+\pi})^{1/2} \geq n \max_{|z|=1} |P(z)|.$$

which follows the required result.

Proof of Theorem 2.3. Let $P(z) = z^s H(z)$,where the polynomial $H(z)$ has all its $(n-s)$ zeros in $|z| \leq K, K \geq 1$ and $H(0) \neq 0$. If $G(z) = z^{(n-s)}\overline{H(1/\overline{z})}$, then $G(z)$ has no zeros in $|z| < 1/K, 1/K \leq 1$. If $Q(z) = z^n\overline{P(1/\overline{z})}$, then $|Q(z)| = |P(z)|$ for $|z| = 1$. Clearly, $Q(z)$ is a polynomial of degree $(n-s)$ and

$$\begin{aligned} Q(z) =& z^n\overline{(1/\overline{z})^s H(1/\overline{z})} \\ =& z^{(n-s)}\overline{H(1/\overline{z})} \\ =& G(z). \end{aligned}$$

It is immediate that

$$|Q(z)| = |G(z)| = |H(z)| = |P(z)| \quad \text{for} \quad |z| = 1,$$

then

$$\max_{1\le k\le (n-s)} |Q(e^{i(\alpha+2k\pi)/(n-s)})| = \max_{1\le k\le (n-s)} |P(e^{i(\alpha+2k\pi)/(n-s)})| = M_\alpha^*,$$

$$\max_{1\le k\le (n-s)} |Q(e^{i(\alpha+(1+2k)\pi)/(n-s)})| = \max_{1\le k\le (n-s)} |P(e^{i(\alpha+(1+2k)\pi)/(n-s)})| = M_{\alpha+\pi}^*,$$

As polynomial $G(z)$ of degree $(n-s)$ has no zeros in $|z| < 1/K, 1/K \le 1$, therefore applying Lemma 3.2 to $G(z)$, we obtain

$$\max_{|z|=1} |G'(z)| \le \frac{(n-s)}{\sqrt{2(1+(1/K)^{2(n-s)})}} (M_\alpha^{*2} + M_{\alpha+\pi}^{*2})^{1/2}, \tag{4.3}$$

when $|G'(z)|$ and $|H'(z)|$ attain maximum at the same point on $|z| = 1$. If we choose β such that

$$\max_{|z|=1} |G'(z)| = |G'(e^{i\beta})| \quad \text{where} \quad 0 \le \beta < 2\pi,$$

then

$$\max_{|z|=1} |H'(z)| = |H'(e^{i\beta})|.$$

As $Q(z) = G(z)$, so we have $Q'(z) = G'(z)$ and $|Q'(z)| = |G'(z)|$ for $|z| = 1$. Therefore,

$$|Q'(e^{i\beta})| = |G'(e^{i\beta})|. \tag{4.4}$$

Also,we have

$$zQ'(z) = nz^n\overline{P(1/\overline{z})} - z^{n-1}\overline{P'(1/\overline{z})},$$

from which it follows that for $|z| = 1$,

$$|Q'(z)| = |z^{n-1}\overline{Q'(1/\overline{z})}| = |nP(z) - zP'(z)|.$$

We conclude that

$$|P'(z)| + |Q'(z)| \ge n|P(z)|.$$

Choosing θ such that $|P(e^{i\theta})| = \max_{|z|=1} |P(z)|$, we get

$$|P'(e^{i\theta})| + |Q'(e^{i\theta})| \ge n \max_{|z|=1} |P(z)|,$$

which implies that

$$\max_{|z|=1} |P'(z)| + \max_{|z|=1} |Q'(z)| \ge n \max_{|z|=1} |P(z)|. \tag{4.5}$$

Making an appeal to (4.3), (4.4) and (4.5), we derive

$$\max_{|z|=1} |P'(z)| + \frac{(n-s)}{\sqrt{2(1+(1/K)^{2(n-s)})}} (M_\alpha^{*2} + M_{\alpha+\pi}^{*2})^{1/2} \ge n \max_{|z|=1} |P(z)|,$$

from which the result follows.

5. Conclusion

In this paper, our results are generalization of previously known results and open avenues to find new estimates of lower bounds for the maximum modulus of polynomial and its derivatives as well as polar derivatives.

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MODELLING AND ANALYSIS OF THE VECTOR BORNE DISEASES WITH FREE LIVING PATHOGEN GROWING IN THE ENVIRONMENT

By

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Abstract

In various infectious diseases infection is spread by vectors such as infected mosquito, black flies, ticks etc. In this paper a non-linear mathematical model is proposed and analysed for vector-borne infectious diseases like: Yellow Fever, Dengue Fever, Malaria etc. that are caused through direct transmission or through biting of an infectious vector by considering the effect of environment on the pathogen. It is further assumed that pathogen population increases with increase in discharge by human population in the environment, thereby increasing vector population. This model is analysed by using Sylvester's criterion and by Lyapunov's direct method. It is found that if growth of pathogen population caused by conductive human related activity increases, the spread of infectious disease increases.

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1. Introduction

Vector-transmitted diseases are transmitted through vectors that are basically the biological agents that carry infectious agents in their guts and are released in the blood when these vectors bite human population. The vectors carry the disease without getting themselves infected. Environment plays a significant role in the spread of vector-borne infectious diseases. A conducive environment is provided for the survival [3] and even growth [17] of pathogens in the environment by various kinds of household and other wastes, discharged into the environment. The environment-to-host disease transmission is now more evident as contaminated contact surface: household wastes like food, water etc., and soil may transmit infection to susceptible hosts [4, 6, 24]. Thus, the contaminated environment, due to human activities, is liable for the fast spread of vector-borne diseases.

Mathematical models have been used to study the various aspects of vector-borne infectious diseases. In early 19th century, the first model proposed by Ross [23] and modified by Macdonald [20], has influenced the mathematical theory of vector-host infectious diseases. Since then, mathematical models have been used to evaluate various specific infectious diseases [1, 2, 8, 18, 28]; and models related to vector-borne diseases have been analyzed by considering various aspects and effects [11, 22, 16, 15]. However, very little attention has been paid to the study of vector-borne diseases with Pathogen growing in the environment. Human migration has a significant role in the transmission and spread of vector-borne diseases. Cosner et al. [10] described vector-borne diseases on different physical locations, socio-behavioural and socio-economical classes. The effect of immigration of human individuals on the host-vector disease has been addressed by Tumwiine et al. [25]. The transmission model of dengue fever, by considering temperature-dependent parameters with vectorhost transmission, has been used to discuss: the variation of pre-adult mosquito maturation, adult mosquito death rate, oviposition rate and virus incubation rate in the mosquito [9]. Mosquito dispersal is one of the fundamental aspects that influence the persistence and revival of many vector-borne diseases. Lutambi et al. investigated effect of heterogeneous dispersal of mosquito resources (breeding sites) on the spatial spread dynamics and persistence of mosquito populations [19].

The role of vector biting and its effect, on the spread of disease and vector mortality, provide further insight on the disease transmission in age-structured model [21]. An epidemic model of vector-borne disease possessing partial immunity to reinfection by considering time delay in vector population could destabilize the system and lead to Hopf bifurcation [27]. Britton and Traore described the spread and time to extinction of vector-borne disease in a community; where host and vector die, and new hosts and vectors are borne by considering the stochastic model [5]. Disease transmission between a host and vector and host population has been analyzed by the diffusive age-structured epidemic model using the reaction-diffusion equations [13]. Waikhom et al. investigated effect of temperature on transmission dynamics of host-pathogen systems and studied the relationship between climate and thermal adaptability in pathogens [26]. Ghosh et al. studied the environmental effect on the direct (infected to susceptible) and indirect (through carriers) transmission of carrier-dependent infectious diseases [14]. However, in these studies, the effect of the environment on the pathogen population has not been considered either directly or indirectly.

2. Mathematical Model

In this section an *ODE* model for the spread of vector transmitted disease in host is presented. We assume that the total host population at time t; $N_1(t)$ is partitioned as Susceptible class $S(t)$, Infectious class $I(t)$ and Recovered class $R(t)$. Furthermore, the host population is recruited at a constant rate b_1 and dies at natural death rate μ_1. We further assume that:

a) The vertical transmission in the host population is negligible, so as all the newly recruited individuals are susceptibles.
b) The recovered individuals acquire permanent immunity such and cannot again become susceptible.
c) The Susceptible host can become infected either through direct transmission (contact with the infected person: possibly through blood transfusion) or through biting of an infectious vector.

We consider λ_1 as the rate of direct transmission for new infections so that the simple mass action term $\lambda_1 SI$ gives the incidence of new infection. Similarly, the new infections spread by the pathogen-carrier vectors is given by the mass action term $\beta_1 SV$ where β_1 is the biting rate of a vector (pathogen-carrier) on the susceptible host. The dynamics of disease in host population can be formulated by using following differential equations.

Host Population Dynamics:

$$\frac{dS}{dt} = b_1 - \lambda_1 SI - \beta_1 SV - \mu_1 S, \tag{2.1}$$

$$\frac{dI}{dt} = \lambda_1 SI + \beta_1 SV - (\alpha + \mu_1) I, \tag{2.2}$$

$$\frac{dR}{dt} = \alpha I - \mu_1 R, \tag{2.3}$$

where V denotes the number of pathogen-carrier vectors at time t. The number of the susceptible (pathogen-free) vectors at time t is denoted by M, so that the total Vector Population $N_2(t)$ is given by $N_2(t) = M + V$. In addition, we suppose that the vector population is recruited at a rate b_2 and the natural death rate is denoted by μ_2 . We consider all the new-born vectors as susceptible despite the fact that pathogen of several vector-borne diseases (Yellow fever, Lyme disease) can be transmitted from female parent (female) to offspring in the vector population. The susceptible vectors become infected after biting the infected host at a rate λ_2 such that $\lambda_2 MI$ represents the incidence of newly infected vectors. The susceptible vectors after becoming infected will remain infected throughout the life and will carry pathogen for whole life.

The free-living pathogen in the environment are denoted by compartment P. The pathogen shed by the Infectious individuals are capable of growth and survival in the environment. Further $E(t)$ is the cumulative density of environmental factors and the growth rate of environmental factors which depends on the human action (household emission, waterlogging, etc.) is denoted by Q_0 and the depletion rate coefficient of environmental factors be θ_1. The growth rate coefficient of environmental factors due to human and vector population density related factors is denoted by θ_2.

Vector Population Dynamics:

$$\frac{dM}{dt} = b_2 - \lambda_2 MI - \mu_2 M, \tag{2.4}$$

$$\frac{dV}{dt} = \lambda_2 MI - \mu_2 V, \tag{2.5}$$

$$\frac{dP}{dt} = \eta IP + \theta P\left(1 - \frac{P}{c(E)}\right) - \gamma P, \tag{2.6}$$

$$\frac{dE}{dt} = Q_0 - \theta_1 E + \theta_2 N, \tag{2.7}$$

where η represents the shedding rate of pathogen from infected hosts, γ gives the decay rate of pathogen in the environment, θ represents the growth rate of pathogen and $c(E)$ is the carrying capacity of the environment. We assume some of the parameters of the model as $b_i > 0, \mu_i > 0$ for $i = 1, 2$ $\alpha > 0, \theta > 0$ and the initial conditions for vector and host population as

$$S(0) = S_0, I(0) = I_0, R(0) = R_0, V(0) = V_0, M(0) = M_0, P(0) = P_0, E(0) = E_0.$$

Since R does not appear in other equations (2.1) – (2.7) and we have

$N(t) = N_1(t) + N_2(t)$.

Also $\lim_{t\to\infty} N_1(t) = \frac{b_1}{\mu_1}$ and $\lim_{t\to\infty} N_2(t) = \frac{b_2}{\mu_2}$.

So we assume that

$N_1(t) = \frac{b_1}{\mu_1}$ and $N_2(t) = \frac{b_2}{\mu_2}$; $S(t) + I(t) + R(t) = N_1(t)$ and $M(t) + V(t) = N_2(t)$.
Therefore, $M(t) = N_2(t) - V(t) = \left(\frac{b_2}{\mu_2} - V(t)\right)$.
Thus, the model can be reduced as:

$$\frac{dS}{dt} = b_1 - \lambda_1 S I - \beta_1 S V - \mu_1 S, \tag{2.8}$$

$$\frac{dI}{dt} = \lambda_1 S I + \beta_1 S V - (\alpha + \mu_1) I, \tag{2.9}$$

$$\frac{dV}{dt} = \lambda_2 \left(\frac{b_2}{\mu_2} - V(t)\right) I - \mu_2 V, \tag{2.10}$$

$$\frac{dP}{dt} = \eta I P + \theta P \left(1 - \frac{P}{c(E)}\right) - \gamma P, \tag{2.11}$$

$$\frac{dE}{dt} = Q_0 - \theta_1 E + \theta_2 N. \tag{2.12}$$

The set Ω attracts all the solutions (initiating in the positive orthant) of the reduced system. We need the solution to be non-negative for the biological reasons and the reduced system is studied in the closed set

$$\Omega = \{(S, I, V, P, E) \in \mathbb{R}_+^5, 0 \leqslant S + I \leqslant \frac{b_1}{\mu_1}; 0 \leqslant V \leqslant \frac{b_2}{\mu_2}; S, I, V \geqslant 0; 0 \leqslant P \leqslant m; 0 \leqslant E \leqslant E_m\},$$

due to mathematical properties of the solution,
where

$$P \leqslant \frac{c(E)}{\theta}\left(\frac{\eta b_1 + \mu_1(\theta - \gamma)}{\mu_1}\right) = m, \qquad E_m = \frac{Q_0 + \theta_2 N}{\theta_1}.$$

2.1. Existence of equilibrium and Reproduction number:

The reduced model (2.8) – (2.12) has following three non-negative equilibria:

i) **Disease-Free equilibrium:** $E_0 = \left(S^0, I^0, V^0, P^0, E^0\right) = \left(S^0, 0, 0, P^0, E^0\right)$
where

$$S^0 = \frac{b_1}{\mu_1}, \quad P^0 = (\theta - \gamma)\frac{c(E)}{\theta}, \quad E^0 = \frac{Q_0 + \theta_2 N}{\theta_1},$$

which exists if $\eta I + \theta > \gamma$.

ii) **Pathogen- free equilibrium:** $E_1 = \left(S^1, I^1, V^1, P^1, E^1\right) = \left(S^1, 0, 0, 0, E^1\right)$
where

$$S^1 = \frac{b_1}{\mu_1}, \quad E^1 = \frac{Q_0 + \theta_2 N}{\theta_1}.$$

We will discuss in detail the existence of endemic equilibrium E_2. Before that we will obtain the expression of basic reproduction number. The basic reproduction number denoted by $\mathcal{R}_0$ and defined as the average number of secondary infections produced when a single infectious host is introduced into a totally susceptible population. We use the next generation matrix method described in Diekmann et al.[12] to define the basic reproductive number $\mathcal{R}_0$. The associated linearized matrices of system (2.8) – (2.12), at Disease-free Equilibrium E_0, for the computations of $\mathcal{R}_0$ are given by

$$F = \begin{pmatrix} \alpha_1 S^0 & \beta_1 S^0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad V = \begin{pmatrix} \alpha + \mu_1 & 0 \\ -\frac{\lambda_2 b_2}{\mu_2} & \mu_2 \end{pmatrix}.$$

The basic reproduction number is the spectral radius of $\left(FV^{-1}\right)$, which is

$$\mathcal{R}_0 = \frac{b_1}{\mu_1}\left(\frac{\lambda_2}{(\alpha + \mu_1)}\frac{\beta_1}{\mu_2}\frac{b_2}{\mu_2} + \frac{\lambda_1}{(\alpha + \mu_1)}\right).$$

iii) **Endemic equilibrium:** $E_2 = (S^*, I^*, V^*, P^*, E^*)$
where

$$S^* = \frac{b_1 - (\alpha + \mu_1) I^*}{\mu_1}, V^* = \frac{\lambda_2 b_2}{\mu_2(\lambda_2 I^* + \mu_2)} I^*, P^* = \left(\frac{\eta I^* + \theta - \gamma}{\theta}\right) c(E), E^* = \frac{Q_0 + \theta N}{\theta_1}.$$

Substituting these values in equation (2.9) we get

$$F(I) = a_0 I^{*2} + a_1 I^* + a_2, \tag{2.1.1}$$

where

$$\begin{aligned}
a_0 =& \mu_2\lambda_1\lambda_2(\alpha+\mu_1),\\
a_1 =& \left[\mu_1\mu_2\lambda_2(\alpha+\mu_1)+\lambda_1\mu_2^2(\alpha+\mu_1)+\beta_1\lambda_2 b_2(\alpha+\mu_1)-b_1\mu_2\lambda_1\lambda_2\right],\\
a_2 =& \left[\mu_1\mu_2^2(\alpha+\mu_1)-b_1\left(\lambda_1\mu_2^2+\beta_1\lambda_2 b_2\right)\right]=\mu_1\mu_2^2(\alpha+\mu_1)(1-\mathcal{R}_0).
\end{aligned}$$

Since, it is obvious from the expression of a_0, a_1, a_2 that a_0 is always positive. Using Descartess rule of signs, equation (2.1.1) has a positive root whenever $\mathcal{R}_0 > 1$. Further, equation (2.1.1) may have more than one positive root in case $\mathcal{R}_0 < 1$ and $a_1 > 0$. To be precise, this will result in complicated system, i.e. the system (2.8) − (2.12) may undergo backward bifurcation if $\mathcal{R}_0 < 1$. In the next section, a detailed analysis is carried out to investigate the existence of backward bifurcation.

3. Existence of backward Bifurcation

The model (2.8) − (2.12) may undergo backward bifurcation if $\mathcal{R}_0 < 1$, and backward bifurcation requires extra effort to eradicate the disease, as it violates the basic requirement of disease eradication. To discuss whether or not the system (2.8) − (2.12) exhibits backward bifurcation; we calculate the bifurcation coefficients using centre manifold theory [7]. The Jacobian matrix around the disease-free equilibrium of the model (2.8) − (2.12)) is given by:

$$J_0=\begin{pmatrix}
-\mu_1 & -\lambda_1 S^0 & -\beta_1 S^0 & 0 & 0\\
0 & \lambda_1 S^0-(\alpha+\mu_1) & \beta_1 S^0 & 0 & 0\\
0 & \frac{\lambda_2 b_2}{\mu_2} & -\mu_2 & 0 & 0\\
0 & \eta P & 0 & \frac{\theta-\gamma-2\theta P^0}{c(E)} & \frac{\theta P+\theta P^2 c(E)}{(c(E))^2}\\
0 & 0 & 0 & 0 & -\theta_1
\end{pmatrix}.$$

Now, for $\mathcal{R}_0 = 1$,

we select $\lambda_2^* = \frac{\mu_2^2}{b_1 b_2\beta_1}(\mu_1(\alpha+\mu_1)-\lambda_1 b_1)$ as the bifurcation parameter.

The right eigen vectors of J_0 at $\lambda_2=\lambda_2^*$ are given by $\omega=[\omega_1,\omega_2,\omega_3,\omega_4,\omega_5]^T$,

where

$$\omega_1=\left[\frac{\lambda_1 b_1}{\mu_1}+\frac{1}{\mu_1}(\mu_1(\alpha+\mu_1)-\lambda_1 b_1)\right],\qquad \omega_2=1,$$

$$\omega_3=\frac{1}{b_1\beta_1}(\mu_1(\alpha+\mu_1)-\lambda_1 b_1),\qquad \omega_4=\frac{\theta}{\eta c},\qquad \omega_5=0.$$

Similarly, the left eigenvalues are given by $v=(v_1,v_2,v_3,v_4,v_5)$,

where

$$v_1=0, v_2=1, v_3=\frac{\beta_1 b_1}{\mu_1\mu_2}, v_4=0, v_5=0.$$

The expression for the coefficients a and b given in Castillo-Chavez and Song [7] are as:

$$a=\sum_{k,j,i=1}^{n} v_k w_i w_j\frac{\partial^2 f_k}{\partial x_i\partial x_j}(E_0,\lambda_2^*),$$

$$b=\sum_{k,i=1}^{n} v_k w_i\frac{\partial^2 f_k}{\partial x_i\partial\psi}(E_0,\lambda_2^*).$$

Using the above expressions, the coefficients a and b for the proposed system (2.8) − (2.12) are computed as:

$$\begin{aligned}
a &= v_2\left[2\omega_1\omega_2\frac{\partial^2 f_2}{\partial S\partial I}(E_0,\lambda_2^*)+2\omega_2\omega_3\frac{\partial^2 f_2}{\partial I\partial V}(E_0,\lambda_2^*)\right]+v_3\left[2\omega_2\omega_3\frac{\partial^2 f_2}{\partial I\partial V}(E_0,\lambda_2^*)\right]\\
&= -\left[\frac{\lambda_1^2 b_1}{\mu_1^2}+\frac{2}{b_1\mu_1}\left(\frac{\beta_1 b_2+\mu_1\mu_2}{\mu_1\beta_1 b_2}\right)(\mu_1(\alpha+\mu_1)-\lambda_1 b_1)^2\right].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
b &= v_3\left(\omega_2\frac{\partial^2 f_2}{\partial I\partial\lambda_2}(E_0,\lambda_2^*)\right)\\
&= \frac{\beta_1 b_1 b_2}{\mu_1\mu_2^2}.
\end{aligned}$$

It is clear from the expressions of a and b that $a < 0$ and $b > 0$. This eliminates the possibility of backward bifurcation. Thus, equation (2.1.1) does not possess a positive root for $\mathcal{R}_0 < 1$.

4. Stability of Disease-free Equilibrium

4.1. Local Stability of Disease-free Equilibrium

Theorem 4.1. *The disease-free equilibrium point E_0 is locally asymptotically stable if $\mathcal{R}_0 < 1$, $\mu_1(\mu_2+\alpha+\mu_1) > \lambda_1 b_1$ and $\frac{2\theta P}{c(E)} + \gamma > \theta$ otherwise unstable.*

Proof. The Jacobian of linearised system around $E_0 = \left(S^0, 0, 0, P^0, E^0\right)$ is:

$$J_0 = \begin{pmatrix} -\mu_1 & -\lambda_1 S^0 & -\beta_1 S^0 & 0 & 0 \\ 0 & \lambda_1 S^0 - (\alpha+\mu_1) & \beta_1 S^0 & 0 & 0 \\ 0 & \frac{\lambda_2 b_2}{\mu_2} & -\mu_2 & 0 & 0 \\ 0 & \eta P & 0 & \frac{\theta-\gamma-2\theta P^0}{c(E)} & \frac{\theta P+\theta P^2 c(E)}{(c(E))^2} \\ 0 & 0 & 0 & 0 & -\theta_1 \end{pmatrix}.$$

The corresponding three eigenvalue values are $-\theta_1, -\left(\frac{2\theta P}{c(E)} + \gamma - \theta\right), -\mu_1$ and the eigenvalue values are negative provided $\frac{2\theta P}{c(E)} + \gamma > \theta$.

The other two eigenvalue values can be obtained from $\left(\lambda_1 S^0 - (\alpha+\mu_1) - \lambda\right)(\mu_2 - \lambda) - \frac{\beta_1\lambda_2 b_2 S^0}{\mu_2} = 0$. By simple algebraic calculations, we have

$$\begin{gathered} \mu_1\lambda^2 + (\mu_1(\mu_2+\alpha+\mu_1) - \lambda_1 b_1)\lambda + \mu_1\mu_2(\alpha+\mu_1)(1-\mathcal{R}_0) = 0 \\ A_1\lambda^2 + A_2\lambda + A_3 = 0, \end{gathered}$$

where $A_1 = \mu_1$, $A_2 = (\mu_1(\mu_2+\alpha+\mu_1) - \lambda_1 b_1)$ and $A_3 = \mu_1\mu_2(\alpha+\mu_1)(1-\mathcal{R}_0)$.

Using Routh-Hurwitz criterion, a second-degree polynomial with all the coefficients positive will obviously have negative roots.

Now, $A_2 > 0$ if $\mu_1(\mu_1+\mu_2+\alpha) > \lambda_1 b_1$ and $A_3 > 0 \iff \mathcal{R}_0 < 1$.

Hence the theorem. □

4.2. Non-linear stability of Disease-free Equilibrium

Theorem 4.2. *The disease-free equilibrium $E_0 = \left(S^0, 0, 0, P^0, E^0\right)$ is non-linearly asymptotically stable in the region Ω provided the following conditions are satisfied:*

i) $\mu_1 > \frac{1}{2}\frac{\beta_1}{\mu_1}(\lambda_1+\beta_1)$,
ii) $2\mu_1(\alpha+\mu_1) > b_1(3\lambda_1+\beta_1)$,
iii) $\mu_2 > \beta_1\frac{b_1}{\mu_1} + \lambda_2\left(\frac{b_2}{\mu_2} + \frac{b_1}{\mu_1}\right)$,
iv) $\frac{2}{c(E)}\theta m + \gamma > \eta m + \theta$.

Proof. See 8 Appendix I

5. Stability of Pathogen free Equilibrium

5.1. Linear Stability of Pathogen free Equilibrium

The Jacobian of the linearized system around $E_1 = \left(S^1, 0, 0, 0, E^1\right)$ is

$$J_{E_1} = \begin{pmatrix} -\mu_1 & -\lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 S - (\alpha+\mu_1) & \beta_1 S & 0 & 0 \\ 0 & \frac{\lambda_2 b_2}{\mu_2} & -\mu_2 & 0 & 0 \\ 0 & 0 & 0 & \theta-\gamma & 0 \\ 0 & 0 & 0 & 0 & \theta_1 \end{pmatrix}.$$

Clearly, the corresponding three eigenvalues $-\theta_1, -(\gamma-\theta), -\mu_1$ are negative provided $\gamma > \theta$ and the remaining two eigenvalues can be obtained from

$$\lambda^2 + \left(\mu_2 + \alpha + \mu_1 - \lambda_1\frac{b_1}{\mu_1}\right)\lambda + \mu^2(\alpha+\mu_1)(1-\mathcal{R}_0) = 0.$$

Now, by using Routh-Hurwitz criterion, the above second degree equation will possess the negative roots if all the coefficients are positive i.e.

$$\mu_1(\mu_2+\alpha+\mu_1) > \lambda_1 b_1$$

and

$$\mu^2(\alpha+\mu_1)(1-\mathcal{R}_0) > 0 \iff \mathcal{R}_0 < 1.$$

Thus, we state the following theorem to establish the stability of pathogen-free equilibrium.

Theorem 5.1. *The pathogen-free equilibrium E_1 is locally asymptotically stable if $\mathcal{R}_0 < 1$, $\mu_1(\mu_2 + \alpha + \mu_1) > \lambda_1 b_1$ and $\gamma > \theta$ otherwise unstable.*

5.2. Non-linear stability of Pathogen-free Equilibrium

Theorem 5.2. *The disease-free equilibrium $E_1 = \left(\bar{S}, 0, 0, 0, \bar{E}\right)$ is non-linearly asymptotically stable in the region Ω provided the following conditions are satisfied:*

i) $2\mu_1(\alpha + \mu_1) > b_1(2\lambda_1 + \beta_1)$,
ii) $\mu_1\mu_2^2 > (\lambda_2\beta_2\mu_1 + b_1\mu_2)$,
iii) $2\mu_1^2 > b_1(\lambda_1 + \beta_1)$,
iv) $m > 0$.

Proof. See 9, Appendix II.

6. Stability Analysis of Endemic Equilibrium

6.1. Linear stability analysis of Endemic Equilibrium

By using the bifurcation coefficients a and b obtained in the 'Existence of Backward Bifurcation' section and using the application of the theorem 4.1 given in [7] provide the conditions for the local stability of endemic equilibrium. The theorem can be stated as:

Theorem 6.1. *The endemic equilibrium E_2 is locally asymptotically stable for $\mathcal{R}_0 > 1$.*

6.2. Non-linear stability of Endemic Equilibrium

Theorem 6.2. *The disease-free equilibrium $E_2 = (S^*, V^*, I^*, P^*, E^*)$ is non-linearly asymptotically stable in the region Ω provided the following conditions are satisfied:*

i) $2\mu_1(\alpha + \mu_1) > b_1(2\lambda_1 + \beta_1)$,
ii) $\mu_1\mu_2^2 > (\lambda_2\beta_2\mu_1 + b_1\mu_2)$,
iii) $2\mu_1^2 > b_1(\lambda_1 + \beta_1)$,
iv) $m > 0$.

Proof. See 10, Appendix III.

7. Numerical Analysis

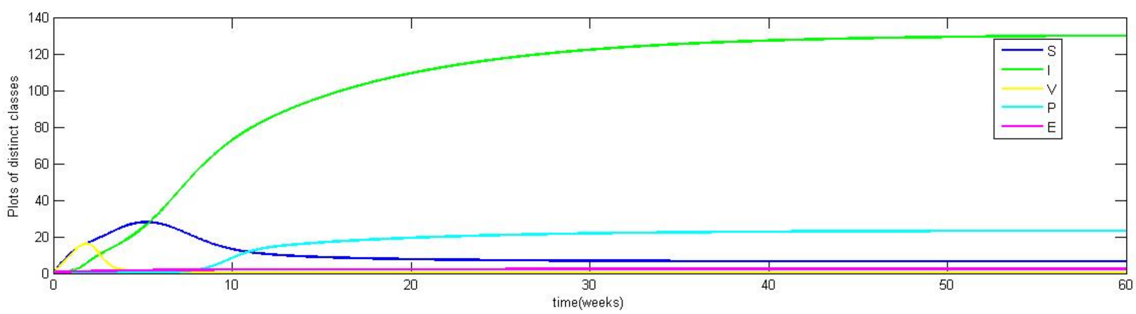

Figure 7.1: Time versus all population and Pathogens

In this section numerical simulation is presented to study the dynamic behaviour of the system (2.8) − (2.12) and to explain the applicability of the results discussed above. The parameter values considered for simulation are as: $b_1 = 12, \lambda_1 = 0.0001, \beta_1 = 0.062, \mu_1 = 0.04, \alpha = 0.05, \lambda_2 = 0.00094, \mu_2 = 0.5, \eta = 0.005, \theta = 0.08, c(E) = 0.001, \theta_2 = 0.005, \gamma = 0.02, \theta_1 = 0.2, Q = 0.45, b_2 = 5$. The equilibrium values for endemic equilibrium are computed as:
$E_2 = (342.42, 14.482, 0.038035, 0.0016551, 1.3577)$. The eigenvalues corresponding to variational matrix of endemic equilibrium are :

$$-0.0417574653, -0.1324089419, -0.1526889240, -0.2, -2.3765070091$$

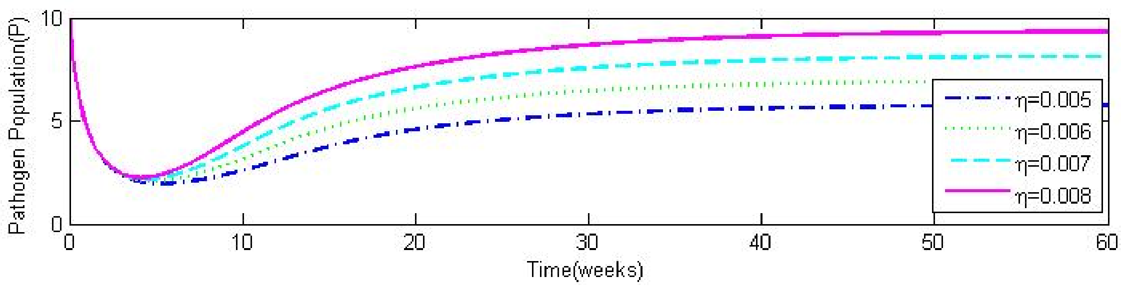

Figure 7.2: Variation of pathogen population with time for distinct values of η

Since all the eigenvalues corresponding to E_2 are found to be negative, for the above set of parameter values, therefore, the endemic equilibrium E_2 is locally asymptotically stable. Figure 7.1 depicts plot between time versus all population and pathogen and shows that all the trajectories lead to the endemic equilibrium.

Figure 7.2 shows the variation of pathogen with time for distinct values of η, the rate of release of pathogen from infective host population. It is seen that as the rate of shedding of pathogen from infected host increases, the pathogen population in the environment increases. Figure 7.3 shows the variation of infected (host) population with time for different values of β_1, the biting rate of vector (pathogen carrier) on the susceptible host. It is noted that infective population also increases with increase in the value of β_1. This indicates that to keep the spread of infected host population under control, the vector population present in the environment needs to be curbed by way of some suitable strategies like reducing areas where vectors can easily breed.

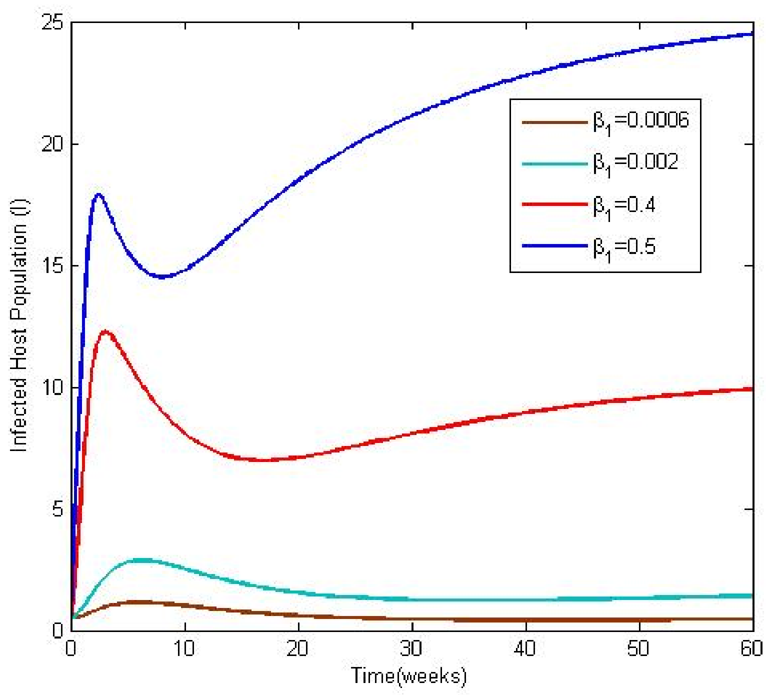

Figure 7.3: Variation of infected (host) population with time for distinct values of β_1

The effect of carrying capacity on pathogen population with time is shown in Figure 7.4. It is observed that as the carrying capacity of environment increases, the pathogen population also increases. Therefore, in order to control the pathogen population, the feasibility of environment for the breading and growth of pathogen population should be minimised. Figure 7.5 shows the variation of cumulative density of environment with the values of Q_0, the growth rate of environmental factors depending on the human action.

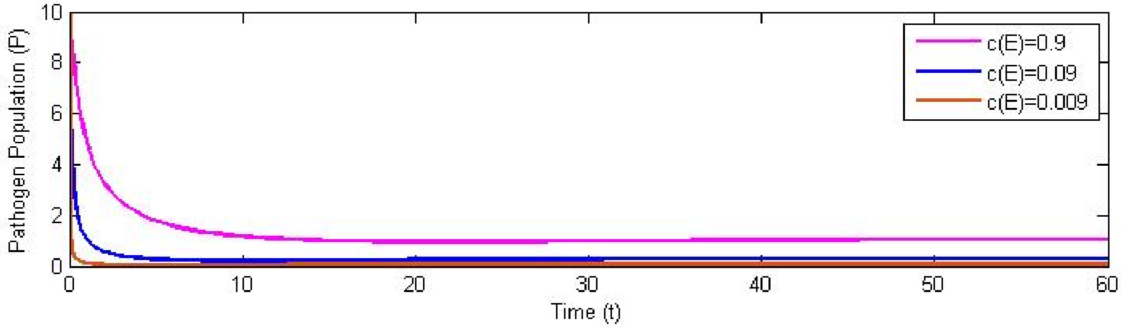

Figure 7.4: Variation of pathogen population with time for distinct values of *c(E)*

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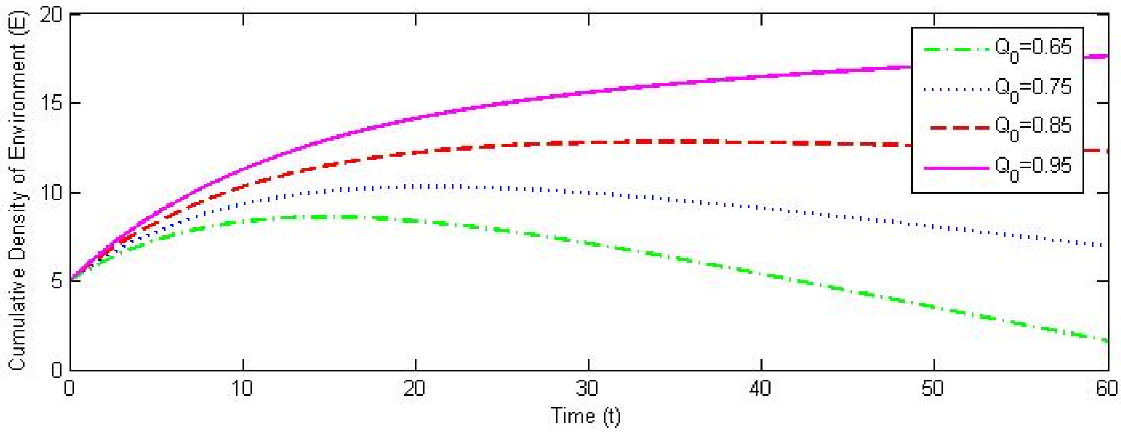

Figure 7.5: Variation of cumulative density of environment with time for distinct values of Q_0

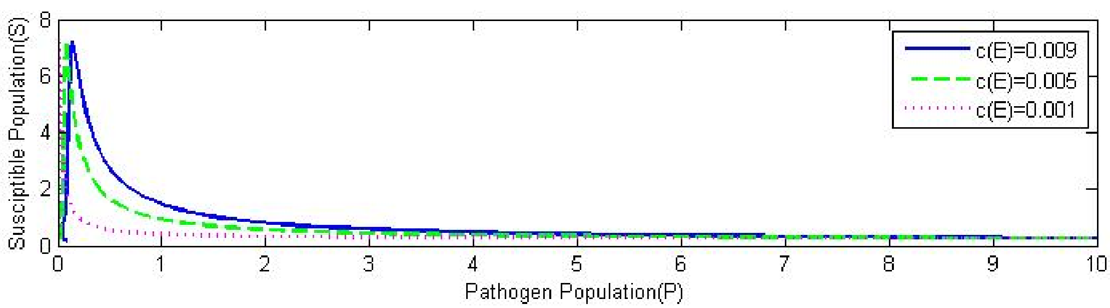

Figure 7.6: Phase plane between susceptible and pathogen population with the variation of carrying capacity of environment distinct values of *c(E)*

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It shows that the rise in the environmental factors like household emission, water-logging etc. will rise the cumulative density of the environment which in turn will increase the pathogen and vector population. Figure 7.6 shows the phase plane between susceptible and pathogen population with the variation of carrying capacity of the environment.

8. Appendix I

We now transform system using $S = S^0 + x_1, I = I^0 + x_2, V = V^0 + x_3, P = P^0 + x_4, E = E^0 + x_5$ around $E_0 = \left(S^0, 0, 0, P^0, E^0\right)$, we have

$$\begin{aligned}
\frac{dx_1}{dt} &= -(\lambda_1 x_2 + \beta_1 x_3) S^0 - x_1 (\lambda_1 x_2 + \beta_1 x_3) - \mu_1 x_1,\\
\frac{dx_2}{dt} &= (\lambda_1 x_2 + \beta_1 x_3) S^0 + x_1 (\lambda_1 x_2 + \beta_1 x_3) - (\alpha + \mu_1) x_2,\\
\frac{dx_3}{dt} &= \lambda_2 \frac{b_2}{\mu_2} x_2 - \lambda_2 x_2 x_3 - x_3 \left(\lambda_2 I^0 + \mu_2\right),\\
\frac{dx_4}{dt} &= \eta \left(x_2 P^0 + x_2 x_4\right) + x_4 \left(\theta\left(1 - \frac{2}{c(E)}\left(P^0 + x_4\right)\right) - \gamma\right),\\
\frac{dx_5}{dt} &= -\theta_1 x_5.
\end{aligned}$$

Consider the positive definite

$$V_1 = \frac{1}{2}\left(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2\right),$$

$$\begin{aligned}
\frac{dV_1}{dt} &= \left[-\lambda_1 x_1 x_2 S^0 - \lambda_1 x_2 x_1^2 - \beta_1 x_3 x_1 S^0 - \beta_1 x_3 x_1^2 - \mu_1 x_1^2\right] + \left[\lambda_1 x_2^2 S^0 + \lambda_1 x_1 x_2^2 + \beta_1 x_2 x_3 S^0 + \beta_1 x_1 x_2 x_3 \right.\\
&\quad \left. - (\alpha + \mu_1) x_2^2\right] + \left[\lambda_2 \left(\frac{b_2}{\mu_2} x_3^2 + I^0 x_3^2 + x_2 x_3^2\right) - \mu_2 x_3^2\right] + \left[\eta x_2 x_4^2 + \eta P^0 x_4^2 + \theta x_4^2 - \frac{2}{c(E)}\theta\left(P^0 + x_4\right) x_4^2 - \gamma x_4^2\right]\\
&\quad + \left[-\theta_1 x_5^2\right]\\
&= \lambda_1 x_1 x_2 \left(S^0 + x_1\right) - \beta_1 x_1 x_3 \left(S^0 + x_1\right) - \mu_1 x_1^2 + \lambda_1 x_1^2 \left(S^0 + x_1\right) - (\alpha + \mu_1) x_2^2 + \lambda_2 \frac{b_2}{\mu_2} x_3^2 + \lambda_2 x_3^2 \left(I^0 + x_2\right)\\
&\quad - \mu_2 x_3^2 + \eta x_4^2 \left(P^0 + x_4\right) + \theta x_4^2 - \frac{2}{c(E)}\theta\left(P^0 + x_4\right) x_4^2 - \gamma x_4^2 - \theta_1 x_5^2.
\end{aligned}$$

Using the region Ω and the inequality $\pm 2ab \leqslant \left(a^2 + b^2\right)$ on the right side of the above equation, we have

$$\begin{aligned}
\frac{dV_1}{dt} &\leqslant \frac{\lambda_1 b_1}{2\mu_1} x_1^2 + \frac{\lambda_1 b_1}{2\mu_1} x_2^2 + \frac{\beta_1 b_1}{2\mu_1} x_1^2 + \frac{\beta_1 b_1}{2\mu_1} x_3^2 - \mu_1 x_1^2 + \lambda_1 \frac{b_1}{\mu_1} x_2^2 + \frac{\beta_1 b_1}{2\mu_1} x_2^2 + \frac{\beta_1 b_1}{2\mu_1} x_3^2 - (\alpha + \mu_1) x_2^2 + \frac{\lambda_2 b_2}{2\mu_2} x_3^2 + \frac{\lambda_2 b_1}{\mu_1} x_3^2\\
&\quad - \mu_2 x_3^2 + \eta m x_4^2 + \theta x_4^2 - \frac{2}{c(E)}\theta m x_4^2 - \gamma x_4^2 - \theta x_5^2\\
&= -\left[a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 + a_4 x_4^2 + a_5 x_5^2\right],
\end{aligned}$$

where

$$b_1 = \mu_1 - \frac{\lambda_1 b_1}{2\mu_1} - \frac{\beta_1 b_1}{2\mu_1}, \qquad b_2 = (\alpha + \mu_1) - \frac{3}{2}\frac{\lambda_1 b_1}{\mu_1} - \frac{\beta_1 b_1}{2\mu_1},$$

$$b_3 = \mu_2 - \beta_1 \frac{b_1}{\mu_1} - \lambda_2 \left(\frac{b_2}{\mu_2} + \frac{b_1}{\mu_1}\right), \qquad b_4 = \frac{2}{c(E)}\theta m + \gamma - \eta m - \theta, \qquad b_5 = \theta.$$

Now $\frac{dV_2}{dt}$ is negative definite, if each $b_i > 0 \; \forall \; i = 1, 2 ... 5$.
Since, $\theta > 0$ (assumed). Hence, by using Lyapunov's second method of stability, the required conditions can be obtained.

9. Appendix II

We now transform system using $S = \bar{s} + y_1, V = \bar{V} + y_2, I = \bar{I} + y_3, P = \bar{P} + y_4, E = \bar{E} + y_5$ around $E_1 = \left(\bar{S}, 0, 0, 0, \bar{E}\right)$, we have

$$\begin{aligned}
\frac{dy_1}{dt} &= -(\lambda_1 y_2 + \beta_1 y_3) \bar{S} - y_1 (\lambda_1 y_2 + \beta_1 y_3) - \mu_1 y_1,\\
\frac{dy_2}{dt} &= (\lambda_1 y_2 + \beta_1 y_3) \bar{S} + y_1 (\lambda_1 y_2 + \beta_1 y_3) - (\alpha + \mu_1) y_2,
\end{aligned}$$

$$\frac{dy_3}{dt} = \lambda_2 \frac{b_2}{\mu_2} y_2 - \lambda_2 y_2 y_3 - y_3 \left(\lambda_2 \bar{I} + \mu_2\right),$$
$$\frac{dy_4}{dt} = \eta \left(y_2 \bar{P} + y_2 y_4\right) + y_4 \left(\theta \left(1 - \frac{2}{c(E)} \left(\bar{P} + y_4\right)\right) - \gamma\right),$$
$$\frac{dy_5}{dt} = -\theta_1 y_5.$$

Consider the positive definite

$$V_2 = \frac{1}{2} \left(B_1 y_1^2 + B_2 y_2^2 + B_3 y_3^2 + B_4 y_4^2 + B_5 y_5^2\right),$$

$$\begin{aligned}\frac{dV_2}{dt} &= B_1 \left[-\lambda_1 y_1 y_2 \bar{S} - \lambda_1 y_2 y_1^2 - \beta_1 y_3 y_1 \bar{S} - \beta_1 y_3 y_1^2 - \mu_1 y_1^2\right] + B_2 \left[\lambda_1 y_2^2 \bar{S} + \lambda_1 y_1 y_2^2 + \beta_1 y_2 y_3 \bar{S} + \beta_1 y_1 y_2 y_3 \right. \\ &\quad \left. - (\alpha + \mu_1) y_2^2\right] + B_3 \left[\lambda_2 \left(\frac{b_2}{\mu_2} y_3^2 + \bar{I} y_3^2 + y_2 y_3^2\right) - \mu_2 y_3^2\right] + B_4 \left[\eta y_2 y_4^2 + \eta \bar{P} y_4^2 + \theta y_4^2 - \frac{2}{c(E)} \theta y_4^3 - \gamma y_4^2\right] \\ &\quad + B_5 \left[-\theta_1 y_5^2\right] \\ &= B_1 \left[-\lambda_1 y_1 y_2 \left(\bar{S} + y_1\right) - \beta_1 y_1 y_3 \left(\bar{S} + y_1\right) - \mu_1 y_1^2\right] + B_2 \left[\lambda_1 y_2^2 \left(\bar{S} + y_1\right) + \beta_1 y_3 y_2 \left(\bar{S} + y_1\right) - (\alpha + \mu_1) y_2^2\right] \\ &\quad + B_3 \left[\lambda_2 \frac{b_2}{\mu_2} y_3^2 + y_3^2 \left(\bar{I} + y_2\right) - \mu_2 y_3^2\right] + B_4 \left[\eta y_2 y_4 \left(\bar{P} + y_4\right) - \frac{2}{c(E)} \theta y_4^2 \left(\bar{P} + y_4\right) - \gamma y_4^2\right] + B_5 \left[-\theta_1 y_5^2\right].\end{aligned}$$

Using the region Ω and the inequality $\pm 2ab \leqslant \left(a^2 + b^2\right)$ on the right side of the above equation, we have

$$\begin{aligned}&\leqslant B_1 \left[\frac{\lambda_1 b_1}{2\mu_1} y_1^2 + \frac{\lambda_1 b_1}{2\mu_1} y_2^2 + \frac{\beta_1 b_1}{2\mu_1} y_1^2 + \frac{\beta_1 b_1}{2\mu_1} y_3^2 - \mu_1 y_1^2\right] + B_2 \left[\lambda_1 \frac{b_1}{\mu_1} y_2^2 + \frac{\beta_1 b_1}{2\mu_1} y_2^2 + \frac{\beta_1 b_1}{2\mu_1} y_3^2 - (\alpha + \mu_1) y_2^2\right] \\ &\quad + B_3 \left[\frac{\lambda_2 b_2}{\mu_2} y_3^2 + \frac{b_1}{\mu_1} y_3^2 - \mu_2 y_3^2\right] + B_5 \left[\eta m y_2 y_4 - \frac{2}{c(E)} \theta m y_4^2 - \gamma y_4^2\right] + B_5 \left[-\theta y_5^2\right] \\ &= -\left[b_{11} y_1^2 + b_{33} y_3^2 + \left(b_{22} y_2^2 - b_{24} y_2 y_4 + b_{44} y_4^2\right) + b_{55} y_5^2\right],\end{aligned}$$

where

$$b_{11} = B_1 \left(\mu_1 - \frac{\lambda_1 b_1}{2\mu_1} - \frac{\beta_1 b_1}{2\mu_1}\right), \qquad b_{22} = B_2 \left((\alpha + \mu_1) - \lambda_1 \frac{b_1}{\mu_1} - \frac{\beta_1 b_1}{2\mu_1}\right) - B_1 \frac{\lambda_1}{2} \frac{b_1}{\mu_1},$$
$$b_{33} = B_3 \left(\mu_2 - \lambda_2 \frac{b_2}{\mu_2} - \frac{b_1}{\mu_1}\right) - \frac{\beta_1 b_1}{2\mu_1} (B_1 + B_2), \qquad b_{44} = B_4 \left(\frac{2}{c(E)} \theta m + \gamma\right),$$
$$b_{55} = \theta_1 B_5, \qquad b_{24} = B_4 \eta m.$$

It can be shown by using Sylvester criteria that $\frac{dV_2}{dt}$ is negative definite if the following conditions are satisfied:

$$B_1 \left(\mu_1 - \frac{\lambda_1 b_1}{2\mu} - \frac{\beta_1 b_1}{2\mu_1}\right) > 0,$$
$$B_3 \left(\mu_2 - \lambda_2 \frac{b_2}{\mu_2} - \frac{b_1}{\mu_1}\right) - \frac{\beta_1 b_1}{2\mu_1} (B_1 + B_2) > 0,$$
$$\left(B_2 \left((\alpha + \mu_1) - \lambda_1 \frac{b_1}{\mu_1} - \frac{\beta_1 b_1}{2\mu_1}\right) - B_1 \frac{\lambda_1 b_1}{2\mu_1}\right) \left(\frac{2}{c(E)} \theta m + \gamma\right) > \frac{1}{2} B_4 (\eta m)^2,$$
$$B_5 \theta_1 > 0, \quad \text{and} \quad B_4 \eta m > 0.$$

Suppose that $B_1 = B_4 = B_5 = 1$ then, we have

$$2\mu_1^2 > b_1 (\lambda_1 + \beta_1),$$
$$B_2 > \frac{\mu_1}{2\mu_1 (\alpha + \mu_1) - b_1 (2\lambda_1 + \beta_1)} \left(\frac{c(E) (\eta m)^2}{2\theta m + \gamma c(E)} + \lambda_1 \frac{b_1}{\mu_1}\right),$$
$$B_3 > \left(\frac{\mu_1 \mu_2}{\mu_1 \mu_2^2 - (\lambda_2 b_2 \mu_1 + b_1 \mu_2)}\right) \left(1 + \frac{\mu_1}{2\mu_1 (\alpha + \mu_1) - b_1 (2\lambda_1 + \beta_1)} \left(\frac{c(E) (\eta m)^2}{2\theta m + \gamma c(E)} + \lambda_1 \frac{b_1}{\mu_1}\right)\right) \frac{\beta_1 b_1}{2\mu_1}.$$

provided $2\mu_1 (\alpha + \mu_1) - b_1 (2\lambda_1 + \beta_1) > 0$ and $\mu_1 \mu_2^2 - (\lambda_2 \beta_2 \mu_1 + b_1 \mu_2) > 0$.

Finally, the required conditions for non-linear stability are:

i) $2\mu_1 (\alpha + \mu_1) > b_1 (2\lambda_1 + \beta_1)$,
ii) $\mu_1 \mu_2^2 > (\lambda_2 \beta_2 \mu_1 + b_1 \mu_2)$,
iii) $2\mu_1^2 > b_1 (\lambda_1 + \beta_1)$,
iv) $m > 0$.

10. Appendix III

We now transform system using $S = S^* + z_1, V = V^* + z_2, I = I^* + z_3, P = P^* + z_4, E = E^* + z_5$ around $E_2 = (S^*, V^*, I^*, P^*, E^*)$, we have

$$\begin{aligned}
\frac{dz_1}{dt} &= -(\mu_1 + \lambda_1 I^* + \beta_1 V^*) z_1 - (\lambda_1 z_2 + \beta_1 z_3) S^* - (\lambda_1 z_2 + \beta_1 z_3) z_1,\\
\frac{dz_2}{dt} &= (\lambda_1 z_2 + \beta_1 z_3) S^* + z_1 (\lambda_1 I^* + \beta_1 V^*) + z_1 (\lambda_1 z_2 + \beta_1 z_3) - (\alpha + \mu_1) z_2,\\
\frac{dz_3}{dt} &= \lambda_2 \frac{b_2}{\mu_2} z_2 - \lambda_2 (V^* z_2 + z_2 z_3) - (\lambda_2 I^* + \mu_2) z_2,\\
\frac{dz_4}{dt} &= \eta (I^* z_4 + P^* z_2 + z_2 z_4) + z_4 \left(\theta\left(1 - \frac{2}{c(E)} (p^* + z_4)\right) - \gamma\right),\\
\frac{dz_5}{dt} &= -\theta_1 z_5.
\end{aligned}$$

Consider the positive definite

$$V_3 = \frac{1}{2}\left(C_1 z_1^2 + C_2 z_2^2 + C_3 z_3^2 + C_4 z_4^2 + C_5 z_5^2\right),$$

$$\begin{aligned}
\frac{dV_3}{dt} &= C_1 \left[-\mu_1 z_1^2 - z_1^2 \lambda_1 I^* - z_1^2 \beta_1 V^* - \lambda_1 z_1 z_2 S^* - \beta_1 z_3 z_1 S^* - \lambda_1 z_2 z_1^2 - \beta_1 z_3 z_1^2\right] + C_2 \left[\lambda_1 z_2^2 S^* + \beta_1 z_2 z_3 S^*\right.\\
&\left. + \lambda_1 z_1 z_2 I^* + \beta_1 z_1 z_2 V^* + \lambda_1 z_1 z_2^2 + \beta_1 z_1 z_2 z_3 - (\alpha + \mu_1) z_2^2\right] + C_3 \left[\lambda_2 \frac{b_2}{\mu_2} z_2 z_3 - \lambda_2 z_2 z_3 V^* - \lambda_2 z_2 z_3^2 - \lambda_2 z_3^2 I^* - \mu_2 z_3^2\right]\\
&+ C_4 \left[\eta z_4^2 + \eta P^* z_4 z_2 + \eta z_2 z_4^2 + \left(\theta\left(1 - \frac{2}{c(E)} (P^* + z_4)\right) - \gamma\right) z_4^2\right] + C_5 \left[-\theta_1 z_5^2\right]\\
&= C_1 \left[-\mu_1 z_1^2 - \lambda_1 (I^* + z_2) z_1^2 - \beta_1 (V^* + z_3) z_1^2 - (\lambda_1 S^* + \beta_1 V^*) z_1 z_3\right] + C_2 \left[\lambda_1 (S^* + z_1) z_2^2 + \beta_1 z_2 z_3\right.\\
&\left.(S^* + z_1) + (\lambda_1 I^* + \beta_1 V^*) z_1 z_2 - (\alpha + \mu_1) z_2^2\right] + C_3 \left[\lambda_2 \frac{b_2}{\mu_2} z_2 z_3 - \lambda_2 z_2 z_3 (V^* + z_3) - \lambda_2 z_3^2 I^* - \mu_2 z_3^2\right]\\
&+ C_4 \left[\eta I^* z_4^2 + \eta z_2 z_4 (P^* + z_4) + \left(\theta\left(1 - \frac{2}{c(E)} (P^* + z_4)\right) - \gamma\right) z_4^2\right] + C_5 \left[-\theta_1 z_5^2\right].
\end{aligned}$$

Using the region Ω and the inequality $\pm 2ab \leqslant \left(a^2 + b^2\right)$ on the right side of the above equation, we have

$$\begin{aligned}
\frac{dV_3}{dt} &\leqslant C_1 \left[-\mu_1 z_1^2 - \lambda_1 \frac{b_1}{\mu_1} z_1^2 - \beta_1 \frac{b_2}{\mu_2} z_1^2 - \lambda_1 z_1 z_2 S^* - (\lambda_1 S^* + \beta_1 S^*) z_1 z_3\right] + C_2 \left[\lambda_1 \frac{b_1}{\mu_1} z_2^2 - \right.\\
&\left.\beta_1 \frac{b_1}{\mu_1} z_2 z_3 + (\lambda_1 I^* + \beta_1 V^*) z_1 z_2 - (\alpha + \mu_1) z_2^2\right] + C_3 \left[\lambda_2 \frac{b_2}{\mu_2} z_2 z_3 - \lambda_2 \frac{b_2}{\mu_2} z_2 z_3 - \lambda_2 I^* z_3^2 - \mu_2 z_3^2\right]\\
&C_4 \left[\eta I^* z_4^2 + \eta m z_2 z_4 + \left(\theta\left(1 - \frac{2m}{c(E)}\right) - \gamma\right) z_4^2\right] + C_5 \left[-\theta_1 z_5^2\right]\\
&= -\left[\left(\frac{1}{2} c_{11} z_1^2 - c_{12} z_1 z_2 + \frac{1}{3} c_{22} z_2^2\right) + \left(\frac{1}{2} c_{11} z_1^2 - c_{13} z_1 z_3 + \frac{1}{2} c_{33} z_3^2\right)\right.\\
&\left. + \left(\frac{1}{3} c_{22} z_2^2 - c_{23} z_2 z_3 + \frac{1}{2} c_{33} z_3^2\right) + \left(\frac{1}{3} c_{22} z_2^2 - c_{24} z_2 z_4 + c_{44} z_4^2\right) + c_{55} z_5^2\right],
\end{aligned}$$

where

$$\begin{aligned}
c_{11} &= C_1 \left(\mu_1 + \lambda_1 \frac{b_1}{\mu_1} + \beta_1 \frac{b_2}{\mu_2}\right), & c_{12} &= C_2 (\lambda_1 I^* + \beta_1 V^*) - C_1 \lambda_1 S^*,\\
c_{22} &= C_2 \left((\alpha + \mu_1) - \lambda_1 \frac{b_1}{\mu_1}\right), & c_{13} &= C_1 (-\lambda_1 S^* - \beta_1 S^*),\\
c_{33} &= C_3 (\lambda_2 I^* + \mu_2), & c_{23} &= C_2 \beta_1 \frac{b_1}{\mu_1},
\end{aligned}$$

$$
\begin{aligned}
c_{44} =& C_4\left(\gamma + \theta\frac{2m}{c(E)} - \theta - \eta I^*\right), \qquad\qquad c_{24} = C_4\eta m,\\
c_{55} =& C_5\theta_1.
\end{aligned}
$$

It can be shown by using Sylvester criteria that $\frac{dV_3}{dt}$ is negative definite if the following conditions are satisfied:

$$
\begin{aligned}
& C_1\left(\mu_1 + \lambda_1\frac{b_1}{\mu_1} + \beta_1\frac{b_2}{\mu_2}\right)C_2\left((\alpha+\mu_1) - \lambda_1\frac{b_1}{\mu_1}\right) > \frac{3}{2}\left[C_2\left(\lambda_1 I^* + \beta_1 V^*\right) - C_1\lambda_1 S^*\right]^2,\\
& C_3\left(\mu_1 + \lambda_1\frac{b_1}{\mu_1} + \beta_1\frac{b_2}{\mu_2}\right)(\lambda_2 I^* + \mu_2) > C_1\left(\lambda_1 S^* + \beta_1 S^*\right)^2,\\
& C_3\left((\alpha+\mu_1) - \lambda_1\frac{b_1}{\mu_1}\right)(\lambda_2 I^* + \mu_2) > C_2\frac{3}{2}\left(\beta_1\frac{b_1}{\mu_1}\right)^2,\\
& C_2\left((\alpha+\mu_1) - \lambda_1\frac{b_1}{\mu_1}\right)\left(\gamma + \theta\frac{2m}{c(E)} - \theta - \eta I^*\right) > C_4\frac{3}{4}(\eta m)^2,\\
& C_5\theta_1 > 0.
\end{aligned}
$$

Suppose that $C_1 = C_2 = C_5 = 1$, we get

$$
C_3 > max\left(L_1, L_2\right), \qquad\qquad C_4 > \frac{4\left(\mu_1(\alpha+\mu_1) - \lambda_1 b_1\right)\left(c\gamma + 2\theta m - c\theta - \eta c I^*\right)}{3\mu_1 c(E)(\eta m)^2},
$$

where

$$
L_1 = \frac{\mu_1\mu_2\left(\lambda_1 S^* + \beta_1 S^*\right)^2}{\left(\mu_2\mu_1^2 + \lambda_1\mu_2 b_1 + \beta_1\mu_1 b_2\right)(\lambda_2 I^* + \mu_2)} \qquad and \qquad L_2 = \frac{3\left(\beta_1 b_1\right)^2}{2\mu_1\left(\mu_1(\alpha+\mu_1) - \lambda_1 b_1\right)(\lambda_2 I^* + \mu_2)}.
$$

Finally, conditions required for the non-linear stability are:

$$
\begin{aligned}
& \left(\mu_1 + \lambda_1\frac{b_1}{\mu_1} + \beta_1\frac{b_2}{\mu_2}\right)\left((\alpha+\mu_1) - \lambda_1\frac{b_1}{\mu_1}\right) > \frac{3}{2}\left[\left(\lambda_1 I^* + \beta_1 V^*\right) - \lambda_1 S^*\right]^2;\\
& \mu_1(\alpha+\mu_1) > \lambda_1 b_1;\\
& c(E)\gamma + 2\theta m > c(E)\theta + \eta c(E) I^*.
\end{aligned}
$$

Clearly, by Lyapunovs direct method it is observed that endemic equilibrium point E_2 is non-linearly asymptotically stable under the set of above conditions.

11. Conclusion

In this paper, a nonlinear mathematical model is proposed to study the effect of free living pathogen on the vector borne diseases. In modeling the process, the total human population is divided into subclasses of susceptible host, infected host and recovered host and the vector population is divided into two subclasses of susceptible vector and infected vector. It is assumed that the disease spreads by direct contact of susceptible host with infected host and indirectly through biting of infected vector. The density of pathogen population is assumed to be directly proportional to the infected host population. The model exhibits three equilibria; namely disease free, pathogen free and endemic equilibrium. The model has been analyzed using stability theory and numerical simulation. Certain inferences have been drawn regarding the spread of the disease by establishing local and global stability results and numerical simulation. The analysis of the model shows that if suitable strategies such as; reducing or removing areas where vectors can easily breed, use of pesticides for vector control; are applied to curb the vector population, the infected host population declines which consequently reduces the pathogen in the environment. The study shows that the presence of environmental factors support the growth of pathogen population (Figure 7.4). The human population related environmental factors can be minimised by adopting the strategies which may include: elimination of carrier breeding sites, larvaciding, adulticiding, keeping surroundings clean and hygienic.

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SOME NEW FIXED POINT THEOREMS FOR ITERATED CONTRACTION MAPS IN INTUITIONISTIC FUZZY METRIC SPACE

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Abstract

We propose iterated contraction maps in intuitionistic fuzzy metric spaces and establish some novel fixed point theorems for intuitionistic fuzzy iterated contraction maps in intuionistic fuzzy metric spaces in this work.
2010 Mathematics Subject Classification: 46N20, 46S40, 47H10.
Keywords and Phrases: Intuitionistic fuzzy metric space, Iterated contraction map, fixed point.

1. Introduction

Zedah work in 1995 [6] provided impetus for further work in fuzzy sets. In 1969, Rheinboldt [3] began research into iterated contraction. Iterated contraction is a valuable notion for studying certain repetitive processes and has a wide range of applications in metric spaces. In 1994, George and Veeramani [2] significantly modified Kramosil and Michalek's idea of fuzzy metric space, constructed a Hausdorff topology, and proved certain previously known findings.

Atanassov [1] introduces the intuitionistic fuzzy set. After that so many authors proved various fixed point results in intuitionistic fuzzy metric spaces. Xia, and Tang [4] and [5] established a fixed point theorem for iterated contraction maps in fuzzy metric space, we followed him and apply in intuitionistic fuzzy metric space and proved fixed point in similar way.

2. Preliminaries

Definition 2.1 ([1]). *A intutionistic fuzzy metric space is an ordered five tuples $(X, M, N, *, \diamond)$ such that X is a nonempty set $*$ and $\diamond$ are the continuous t-norm and t-conorm and (M, N) is a intutionistic fuzzy set on $X \times X \times (0, \infty) \to [0, 1]$ satisfies the following conditions*

$$\text{for all } x, y, z \in X \text{ and } s, t > 0,$$

(IFM 1) $M(x, y, t) \geq 0$ *for all* $t > 0$,
(IFM 2) $M(x, y, t) = 1$ *iff* $x = y, t > 0$,
(IFM 3) $M(x, y, t) = M(y, x, t)$,
(IFM 4) $M(x, z, t + s) \geq M(x, y, t) \quad * M(y, z, s)$,
(IFM 5) $M(x, y,) : (0, \infty) \to [0, 1)$ *is continuous.*
(IFM 6) $N(x, y, t) < 1$ *for all* $t > 0$,
(IFM 7) $N(x, y, t) = 0$ *iff* $x = y, t > 0$,
(IFM 8) $N(x, y, t) = N(y, x, t)$,
(IFM 9) $N(x, z, t + s) \leq N(x, y, t) \diamond N(y, z, s)$,
(IFM 10) $N(x, y, -) : (0, \infty) \to [0, 1)$ *is continuous.*
Then $(M, N,)$ *is called a Intuitionistic fuzzy metric on* X''.

Definition 2.2 ([3]). *A mapping $T : X \to X$ in a Intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is intuitionistic Fuzzy T-contraction if for $T : X \to X$ there exist $k \in (0, 1)$ with*

$$M(Tx, Ty, t) \geq M\left(x, y, \frac{t}{k}\right) \text{ and } N(Tx, Ty, t) \leq N\left(x, y, \frac{t}{k}\right),$$

for all $x, y \in X$ and $t > 0, 0 < k < 1$, is called a contraction map.

3. Main Results

Definition 3.1. *If $(X, M, N, *, \diamond)$ is a intuitionist fuzzy metric space such that $M(Tx, T^2x, t) \geq M\left(x, Tx, \frac{t}{k}\right)$ and $N(Tx, T^2x, t) \leq N\left(x, Tx, \frac{t}{k}\right)$ for all $x \in X$, $t > 0$, $0 < k < 1$, then T is said to be a intuitionistic fuzzy iterated contraction map.*

Remark 3.1. A Intuitionistic fuzzy contraction map is continuous and is a revised intuitionistic iterated contraction. A intuitionistic fuzzy contraction map has a unique fixed point. However, a intuitionistic fuzzy iterated contraction map may have more than one fixed point.

Let $(X\ d)$ be a metric space. Define $*y = xy$, $x \diamond y = x + y - xy$ and $d(x, y) = x - y$ for all $x, y \in X$ and > 0, $M(x, y, t) = \frac{t}{t+d(x,y)}$ and $N(x, y, t) = \frac{d(x,y)}{t+d(x,y)}$, then $(X, M, N, *, \diamond)$ is a intuitionistic fuzzy metric space.

If $T : \left[\frac{-1}{2}, \frac{1}{2}\right] \to \left[\frac{-1}{2}, \frac{1}{2}\right]$ is given by $Tx = x^2$, then T is a intuitionistic fuzzy iterated contraction but not a intuitionistic fuzzy contraction map.

The following is a fixed point theorem for intuitionistic fuzzy iterated contraction map.

Theorem 3.1. *If $T : X \to X$ is a continuous intuitionistic fuzzy iterated contractive map and the sequence of iterates $\{x_n\}$ in intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$, defined by $x_{n+1} = Tx_n, n = 0, 1, 2, \ldots$ for $x \in X$, has a subsequence converging to $y \in X$, then $y = Ty$, that is, T has a fixed point*

Proof. The sequence $\{M(x_{n+1}, x_n, t)\}$ and $\{N(x_{n+1}, x_n, t)\}$ is a non-decreasing and non-increasing sequence of reals. It is bounded above by 1 and 0 and therefore has a limit. Since the subsequence converges to y and T is continuous on X, so $T(x_{n_i})$ converges to Ty and $T^2(x_{n_i})$ converges to T^2y.

Thus $M(y, Ty, t) = \lim M(x_{n_i}, x_{n_{i-1}}, t) = \lim M(x_{n_{i-1}}, x_{n_{i-2}}, t)$.

Then

$$\begin{aligned}
1 &\geq M\left(Ty, T^2y, t\right) \geq M\left(y, Ty, \frac{t}{k}\right) \\
&\geq M\left(y, Ty, \frac{t}{k^2}\right) \\
&\geq M\left(y, Ty, \frac{t}{k^n}\right), \text{ for all } n \in N, 0 < k < t.
\end{aligned}$$

When $n \to \infty$, $M\left(y, Ty, \frac{t}{k^n}\right) \to 0$, therefore $M\left(y, Ty, \frac{t}{k}\right) = 1$; that is to say $Ty = y$.

Also

$$\begin{aligned}
0 &\leq N\left(Ty, T^2y, t\right) \leq N\left(y, Ty, \frac{t}{k}\right) \\
&\leq N\left(y, Ty, \frac{t}{k^2}\right) \\
&\leq N\left(y, Ty, \frac{t}{k^n}\right), \text{ for all } n \in N, 0 < k < t.
\end{aligned}$$

When $n \to \infty$, $M\left(y, Ty, \frac{t}{k^n}\right) \to 0$, therefore $N\left(y, Ty, \frac{t}{k}\right) = 0$ that is to say $Ty = y$. □

We give the following example to show that if T is a intuitionistic fuzzy iterated contraction that is not continuous, then T may not have a fixed point.

A continuous map T that is not a intuitionistic fuzzy iterated contraction may not have a fixed point.

Note 3.1. If T is not contraction but some powers of T is contraction, then T has a unique fixed point on a complete metric space.

Proof. If x is a fixed point of k powers of T, thus $T^k(x) = x$, $T(T^k(x)) = T(x)$ then $T^k(T(x)) = T(x)$, since T^k has a unique fixed point. Consequently $T(x) = x$. □

Note 3.2. Continuity of a intuitionistic fuzzy iterated contraction is sufficient but not necessary.

As stated in Note 3.1 that if T is not contraction still T may have a unique fixed point when some powers of T is a intuitionistic fuzzy contraction map. The same is true for intuitionistic fuzzy iterated contraction map.

Theorem 3.2. *Let $T : X \to X$ be a intuitionistic fuzzy iterated contraction map on a complete metric space X. If for some power of T, say T is a intuitionistic fuzzy iterated contraction, that is $M\left(Tx, (T)^2x, t\right) \geq M\left(x, Tx, \frac{t}{k}\right)$ and $N\left(Tx, (T)^2x, t\right) \leq N\left(x, Tx, \frac{t}{k}\right)$ and T is continuous at y, where $y = \lim(T)^n$, for any arbitrary $x \in X$. Then T has a fixed point.*

Proof. Since T is a intuitionistic fuzzy iterated contraction that is continuous at y,

$$M\left(Tx, (T)^2x, t\right) \geq M\left(x, Tx, \frac{t}{k}\right). \text{ And}$$

$$N\left(Tx,(T)^2x,t\right) \leq N\left(x,Tx,\frac{t}{k}\right)\ 0<k<t,$$

$$M\left((T)^n x,(T)^m x,t\right) \geq \prod_{i=m}^{n} M\left((T)^{i+1}x,(T)^i x,\frac{t}{n-m}\right) \text{ for all } n>m, n,m \in N.$$

$$\geq \prod_{i=m}^{n} M\left(x,Tx,\frac{t}{(n-m)*k^{i-1}}\right) \to 1(m\to\infty),$$

and

$$N\left((T)^n x,(T)^m x,t\right) \leq \prod_{i=m}^{n} N\left((T)^{i+1}x,(T)^i x,\frac{t}{n-m}\right) \text{ for all } n>m, n,m \in N,$$

$$\leq \prod_{i=m}^{n} N\left(x,Tx,\frac{t}{(n-m)*k^{i-1}}\right) \to 0(m\to\infty).$$

Therefore $\{(T)^n\}_{n=1}^{\infty}$ is a Cauchy sequence in X and X is a complete metric space. Thus Exists $y \in X$, $s,t,y=(T)^n x$, and T is continuous, consequently

$$T(y)=T\left((T)^n x\right)=(T)^{n+1}x=y.$$

It is easy to show that $d(y,fy) \leq kd(y,fy)$. Since $k \leq 1$, therefore $d(y,fy)=0$ and hence f has a fixed point. □

We give the following Note to illustrate the theorem.

Note 3.3. If T is not a fuzzy iterated contraction in Theorem 3.2, but k is a intuitionistic fuzzy iterated contraction with $Ty=y$, then T has a fixed point.

Theorem 3.3. *If $T:X\to X$ is a intuitionistic fuzzy iterated contraction map, and X is a complete metric space, then the sequence of iterates x_n converges to $y\in X$.*

In case T is continuous at, then $y=Ty$, that is T has a fixed point

Proof. Let $x_{n+1}=Tx_n$ $n=1,2,3,\ldots,x_i\in X$. It is easy to show that $\{x_n\}$ is a Cauchy sequence, since T is a intuitionistic fuzzy iterated contraction. The Cauchy sequence $\{x_n\}$ converges to $\in X$, since X is a complete metric space. Moreover, if T is continuous at y, then $x_{n+1}=Tx_n$ converges to Ty.

It follows that $y=Ty$. □

Note 3.4. A continuous iterated contraction map on a complete metric space has a unique fixed point. If an iterated contraction map is not continuous, it may have more than one fixed point.

4. Conclusion

We establish some new fixed point theorems for intuitionistic fuzzy iterated contraction maps in intuitionistic fuzzy metric spaces in this work.

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IMPACT OF INCENTIVE ON THE DIFFUSION OF AN INNOVATION: A MODELLING STUDY

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Abstract

In this paper, an attempt is made to study the impact of positive incentive on the diffusion of an innovation in the society. For this purpose, a non-linear model is proposed involving the following three dependent variables- (i) the number of non-adopters (ii) the number of adopters and (iii) the variable cumulative incentive introduced to accumulate the rate of diffusion of an innovation. The model is analyzed by using stability theory of system of ordinary differential equations and numerical simulations. Although the core concept behind the model is based upon the approach of Bass model, yet we have incorporated a number of generalizations for the better adaptability of the model in the real market scenario. A dynamic market affected by demographical changes caused due to immigration, emigration, etc. has been considered. The coefficients of internal and external influence have also been considered to be variables depending linearly on the total market population and cumulative incentive, respectively. The analysis shows that the number of adopters increases with the increase in the external influence caused by cumulative incentive as a variable. It is also shown that incentive has stabilizing effect on the system. The results are illustrated by numerical simulations.

2020 Mathematical Sciences Classification: 30D15, 30A10

Keywords and Phrases: Diffusion models, spread of innovation, mathematical model, stability analysis, communication channels.

1. Introduction

The incentives play a crucial role in speeding up the diffusion of an innovation. There may be different forms of incentives depending upon the innovation and the priorities of potential user or intermediary. These incentives can be directly given to the user or an intermediary in monetary form or discounts or cash backs. Nowadays, we see that, in countries like India, e-transaction services are being rapidly diffused in the potential market which was not looking possible in the Indian market scenario a few years back. The major force behind this is the incentive in the form of cash backs and heavy discounts offered by e-commerce websites. Incentive may also be allocated in the form of government subsidy in order to diffuse government policies in the society. The following graph shows the effect of incentive in the form of subsidy given by the Indian government to Indian citizens under the Swachh Bharat Mission (Gramin) of India launched in 2014. A remarkable increase in the Individual Household Latrine Applications (IHHL) with time can be seen in the graph (see Figure 1.1).

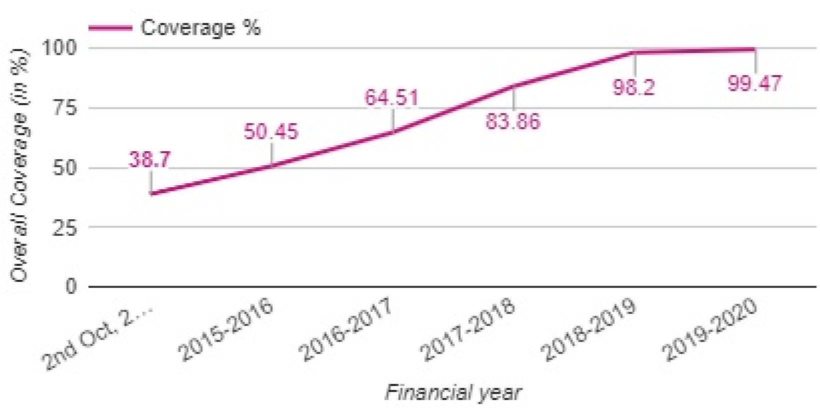

Figure 1.1: Coverage status of IHHL with years (Source: https://sbm.gov.in/sbmReport/home.aspx)

Rogers [14] described innovation diffusion as the process through which new ideas spread throughout a social system over a period of time and through certain routes. He claimed that invention, communication routes, time, and the social structure all play a role in the diffusion process [14]. Since Rogers first presented innovation diffusion theory, it has been widely applied to service sectors, retail, the pharmaceutical industry, and many other fields. Dearing [6] suggested the diffusion process that can be readily applicable in the design of interventions of health care systems. Dearing [4] reviewed diffusion theory and focused on seven concepts that have the potential to accelerate the spread of evidence-based practices, programmes, and policies in the field of social work. Dearing [7] identified three new paths for researchers to pursue diffusion research- implementation science, dissemination science, and positive deviance research. The Positive Deviance method is used by Jain [11] to uncover the successful communication techniques of rural women entrepreneurs in Uttar Pradesh, India, who gain success in extremely difficult situation. Singhal [17,18] also used positive deviance approach to study Diffusion of evidence-based practice. Balas [1] provided frameworks for information diffusion which leads to the adoption of important innovations. According to Dearing [5], when opinion leaders promote or embrace a new practise, it spreads to others rapidly, resulting in diffusion.

Negative incentives are also very helpful in the diffusion of an innovation as they deprive individuals of privileges and impose certain restrictions as penalty for not adopting the innovation. For example, in India there is an ongoing debate on the rapidly increasing population and there are suggestions to deprive those individuals of fundamental rights such as, the right to vote, who do not act in accordance with the measures for population control. Thus, the introduction of incentives can play a vital role in speeding up the diffusion of an innovation. But this characteristic has not been considered so far, in the study of innovation diffusion, where the diffusion process is governed by the framework proposed by Bass [2]. Since then, a number of mathematical models have been proposed and analyzed to study the diffusion trend by focusing on the role of transmission between adopters and non-adopters [8,9,12,13,15,21,22,23]. Centrone [3] proposed a binomial innovation diffusion model considering the demographic process of entrance and exit from the market. Wang [20] proposed a mathematical model to analyze the dynamics of users of a certain product in two different patches. Wang [19] suggested a model incorporating decision making stage and awareness stage with constant coefficient of awareness rate. Shukla [16] proposed and analyzed a non-linear mathematical model considering the adopters, non-adopters population densities and cumulative density of external influences as dependent variables to study the effect of external influences on innovation diffusion. They have treated the density of internal influences as a constant.

It is noticeable here that in most of the mathematical models of innovation diffusion, the external and internal influence factors have been assumed to be constants. But this is not realistic in the dynamic market population where the demography changes because of immigration, emigration, discontinuation of use of an innovation, etc. Mathematical modeling of the effect of incentives on the diffusion of an innovation also remains untouched in most of the existing literature. Therefore, in this paper, we suggest a non-linear mathematical model to study the effect of incentives on diffusion of innovation by considering variable coefficients of external and internal influences.

2. Mathematical Model

Suppose $N_a, N_o$and N denote the variable adopter's population, non-adopter's population and total population respectively. I represents the variable governing the cumulative density of incentive. Let p and q denote the coefficientsof external and internal influence. Let, A be the constant immigration rate and $\bar{N}$ be the non-migrating population. Using these considerations, we propose the following mathematical model -

$$\frac{dN_o}{dt} = A - \gamma(N - \bar{N}) - pN_o - q\frac{N_a N_0}{N} - eN_o + \nu N_a, \tag{2.1}$$

$$\frac{dN_a}{dt} = pN_o + q\frac{N_a N_0}{N} - eN_a - \alpha N_a - \nu N_a, \tag{2.2}$$

$$\frac{dI}{dt} = \mu N_o - \mu_o I, \tag{2.3}$$

$$N_o(0) > 0, N_a(0) \geq 0, I(0) \geq 0.$$

Here, e is the constant emigration rate of population, ν is the rate at which adopters move back to non-adopters, α is the rate at which innovation is discontinued and γ, μ, μ_o are constants of proportionality.

According to the assumptions mentioned in the introduction we assume p and q of the form $p_0 + p_1 I$ and $q_0 + q_1 N$ respectively, where p_0, p_1, q_0, q_1 are non-negative constants. Here, p_0 is constant coefficient of external influence when no incentive is there and $p_1$is the coefficient of external influence due to incentive. Also, the total population is given by $N = N_o + N_a$. Using all these assumptions and facts model (2.1) - (2.3) can be expressed in the form-

$$\frac{dN_a}{dt} = p_o(N - N_a) + p_1 I(N - N_a) + q_o\frac{N_a(N - N_a)}{N} + q_1 N_a(N - N_a) - e_1 N_a, \tag{2.4}$$

$$\frac{dN}{dt} = A_1 - (e + \gamma)N - \alpha N_a, \tag{2.5}$$

$$\frac{dI}{dt} = \mu(N - N_a) - \mu_o I, \tag{2.6}$$

$N(0) > 0, N_a(0) \geq 0, I(0) \geq 0$, where $e_1 = e + \alpha + \nu$ and $A_1 = A + \gamma\bar{N}$.

Now we determine the region of attraction for model system (2.4) - (2.6) given in the following lemma, Freedman [10].

Lemma 2.1. *The region of attraction for all solutions of model* (2.4) - (2.6) *is given by-*

$$\Omega = \left\{(N_a, N, I) : 0 \leq N_a \leq \frac{A_1}{e+\gamma}, \frac{A_1}{e+\alpha+\gamma} \leq N \leq \frac{A_1}{e+\gamma}, 0 \leq I \leq \frac{\mu A_1}{\mu_o(e+\gamma)}\right\}. \tag{2.7}$$

Proof. Since $0 \leq N_a \leq N$, we have $A_1 - (e+\alpha+\gamma)N \leq \frac{dN}{dt} \leq A_1 - (e+\gamma)N$, hence we get $0 \leq N_a \leq \frac{A_1}{e+\gamma}$ and $\frac{A_1}{e+\alpha+\gamma} \leq N \leq \frac{A_1}{e+\gamma}$.

Also, since $\frac{dI}{dt} \leq \mu N_{\max} - \mu_o I$, we have $0 \leq I \leq \frac{\mu A_1}{\mu_0(e+\gamma)}$. □

3. Equilibrium Analysis

There is only one non-negative equilibrium $E^*(N_a^*, N^*, I^*)$ of the model system (2.4) - (2.6), which can be obtained by equating the right-hand sides of the model system (2.4) - (2.6) to zero, as follows,

$$p_o(N - N_a) + p_1 I(N - N_a) + q_o\frac{N_a(N - N_a)}{N} + q_1 N_a(N - N_a) - e_1 N_a = 0, \tag{3.1}$$

$$A_1 - (e+\gamma)N - \alpha N_a = 0, \tag{3.2}$$

$$\mu(N - N_a) - \mu_o I = 0. \tag{3.3}$$

From equations (3.2) and (3.3), we get

$$I = \frac{\mu}{\mu_0}(N - N_a), N_a = \frac{A_1 - (e+\gamma)N}{\alpha}, N - N_a = \frac{(e+\alpha+\gamma)N - A_1}{\alpha}. \tag{3.4}$$

Using (3.4) in (3.1), we get

$$F(N) = \{p_0 + \frac{p_1\mu}{\mu_0}(\frac{(e+\alpha+\gamma)N - A_1}{\alpha})\}(\frac{(e+\alpha+\gamma)N - A_1}{\alpha}) + q_0(\frac{A_1 - (e+\gamma)N}{\alpha N})(\frac{(e+\alpha+\gamma)N - A_1}{\alpha})$$
$$- e_1(\frac{A_1 - (e+\gamma)N}{\alpha}) + q_1(\frac{A_1 - (e+\gamma)N}{\alpha})(\frac{(e+\alpha+\gamma)N - A_1}{\alpha}) = 0. \tag{3.5}$$

From (3.5) we get $F(\frac{A_1}{e+\alpha+\gamma}) = -\frac{e_1 A_1}{(e+\alpha+\gamma)} < 0$, $F(\frac{A_1}{e+\gamma}) = (p_0 + \frac{p_1\mu A_1}{\mu_0(e+\gamma)})\frac{A_1}{(e+\gamma)} > 0$.

Since, $F(\frac{A_1}{e+\alpha+\gamma})$ and $F(\frac{A_1}{e+\gamma})$ are of opposite sign and $F(N)$ is continuous in the interval $(\frac{A_1}{e+\alpha+\gamma}, \frac{A_1}{e+\gamma})$, hence, there exists at least one root of $F(N)$ in the interval $\frac{A_1}{e+\alpha+\gamma} < N < \frac{A_1}{e+\gamma}$.

In order to establish uniqueness of equilibrium we need to show that $F^{'}(N) > 0$ in $\frac{A_1}{e+\alpha+\gamma} < N < \frac{A_1}{e+\gamma}$.

We can rewrite $F(N)$ as

$$F(N) = (\frac{(e+\alpha+\gamma)N - A_1}{\alpha})G(N) - \frac{e_1}{\alpha}(A_1 - (e+\gamma)N) = 0. \tag{3.6}$$

where,

$$G(N) = p_0 + \frac{p_1\mu}{\mu_0}(\frac{(e+\alpha+\gamma)N - A_1}{\alpha}) + q_0(\frac{A_1 - (e+\gamma)N}{\alpha N}) + q_1(\frac{A_1 - (e+\gamma)N}{\alpha}). \tag{3.7}$$

Now differentiating (3.6), we get

$$F^{'}(N) = \frac{(e+\alpha+\gamma)}{\alpha}G(N) + (\frac{(e+\alpha+\gamma)N - A_1}{\alpha})G^{'}(N) + \frac{e_1(e+\gamma)}{\alpha},$$

or

$(A_1 - (e+\gamma)N)F^{'}(N) = \frac{(e+\alpha+\gamma)}{\alpha}G(N)(A_1 - (e+\gamma)N) + (A_1 - (e+\gamma)N)(\frac{(e+\alpha+\gamma)N - A_1}{\alpha})G^{'}(N) + \frac{e_1}{\alpha}(e+\gamma)(A_1 - (e+\gamma)N)$.

Using equation (3.6),

$(A_1 - (e+\gamma)N)F^{'}(N) = \frac{(e+\alpha+\gamma)}{\alpha}G(N)(A_1 - (e+\gamma)N) + (A_1 - (e+\gamma)N)(\frac{(e+\alpha+\gamma)N - A_1}{\alpha})G^{'}(N) + \frac{(e+\gamma)}{\alpha}((e+\alpha+\gamma)N - A_1)G(N)$.

Putting values of $G(N), G^{'}(N)$ from equation (3.7) and further simplifying, we get

$(A_1 - (e+\gamma)N)F^{'}(N) = \frac{(e+\alpha+\gamma)}{\alpha}(A_1 - (e+\gamma)N)(p_0 + \frac{p_1\mu}{\mu_0}(\frac{(e+\alpha+\gamma)N - A_1}{\alpha})) + \frac{(A_1 - (e+\gamma)N)}{\alpha}((e+\alpha+\gamma)N - A_1)\frac{p_1\mu}{\mu_0}\frac{(e+\alpha+\gamma)}{\alpha} + (\frac{(A_1 - (e+\gamma)N)}{\alpha})^2\frac{A_1 q_0}{N^2} + \frac{(e+\gamma)}{\alpha}((e+\alpha+\gamma)N - A_1)(p_0 + \frac{p_1\mu}{\mu_0}(\frac{(e+\alpha+\gamma)N - A_1}{\alpha})) + (e+\alpha+\gamma)(\frac{(A_1 - (e+\gamma)N)}{\alpha})^2 q_1 > 0.$

Thus, there exists a unique non trivial equilibrium $E^*(N_a^*, N^*, I^*)$ of model (2.4) - (2.6) in the region given by $0 \leq N_a \leq \frac{A_1}{e+\gamma}$ and $\frac{A_1}{e+\alpha+\gamma} \leq N \leq \frac{A_1}{e+\gamma}$.

4. Stability Analysis

The stability results are stated in the form of following theorems-

Theorem 4.1. *The equilibrium E^* is locally asymptotically stable without any condition.*

Proof. See Appendix A. □

Theorem 4.2. *The equilibrium E^* is non linearly stable inside the region of attraction Ω, if the following conditions are satisfied,*

$$p_1(N^* - N_a^*)\mu\alpha \leq 2\mu_0(e+\gamma)(p_0 + q_1 N_a^*), \tag{4.1}$$

$$\left(\frac{p_1 I_{\max}}{N_a^*} + \frac{q_0 (N_a)_{\max}}{N_{\min} N^*}\right)^2 \leq 2\frac{q_0}{N^*}\frac{(e+\gamma)}{\alpha}\left(\frac{p_0}{N_a^*} + q_1\right). \tag{4.2}$$

Proof. See Appendix B. □

5. Numerical Simulation

Now we conduct numerical simulation of model (2.4) - (2.6) to test the feasibility of our analysis regarding the existence and stability of the equilibrium E. For this, we assume the following set of values of parameters-
$A = 1000, e = 0.04, p_0 = 0.09, q_0 = 0.1, \mu_0 = 0.05, \alpha = 0.004, \gamma = 0.02, \nu = 0.001, p_1 = 0.00000875, \mu = 0.04, \bar{N} = 1000, q_1 = 0.000008.$

For these values of parameters, conditions (4.1) and (4.2) of stability are satisfied and the value of equilibrium point is obtained as $N_a^* = 13992.79092, N^* = 16067.14727, I^* = 1659.485079$.

Now we plot the following graphs to illustrate the analytical findings and to gain a better insight into the dynamical behaviour of the model.

Model Validation

In Figure 5.1, we see that, in this model, the density of adopter's population follows the same S-pattern as seen in the Bass Model [2]. The figure shows that the diffusion of new innovation/scheme accelerates with time and slows down near its saturation level. This trend is analogous to the diffusion of IHHL scheme with time as seen in Figure 1.1.

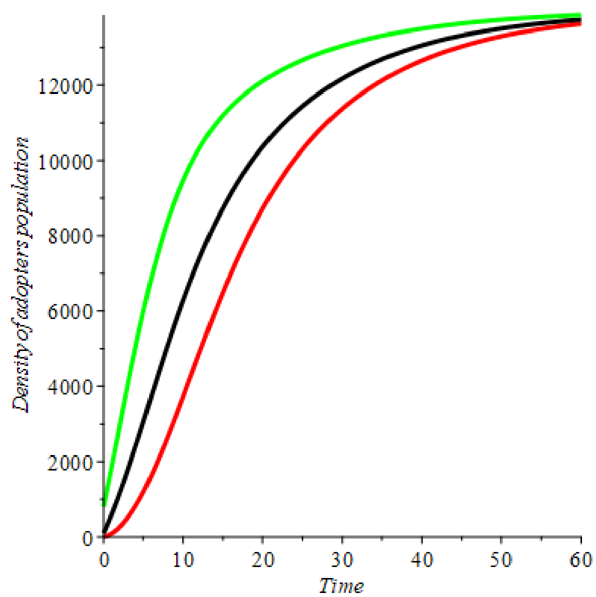

Figure 5.1: (S-curve) Density of adopters population with time for three different sets of initial conditions.

Observations

In Figure 5.2 and Figure 5.3, it is seen that the curves with different initial conditions i.e. starting from different points in the region of attraction tend to equilibrium point E^* as the time progresses. This verifies the stability of the model at equilibrium E^*. In Figure 5.4, we have plotted the effect of immigration rate on adopter's population.

In order to observe the effect, we have plotted three different curves taking three different values of parameter of immigration rate A as 1000, 1500 and 2000, while keeping the values of other parameters fixed. This figure shows that as the immigration rate increases, adopter's population also increases. In Figure 5.5, we see that adopter's population decreases as p_1 increases whereas in Figure 5.6 we see that the incentive increases with the increase in rate of immigration. Similarly in Figures 5.7, 5.8, 5.9 and 5.10 , we see that the number of adopter's increases with the increase in parameters μ, μ_0, q_0 and q_1.

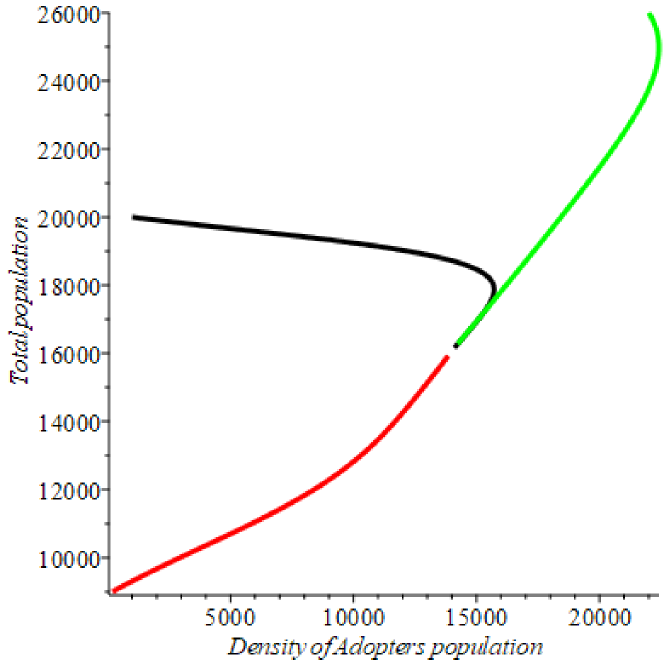

Figure 5.2: Total population with density of adopter's population.

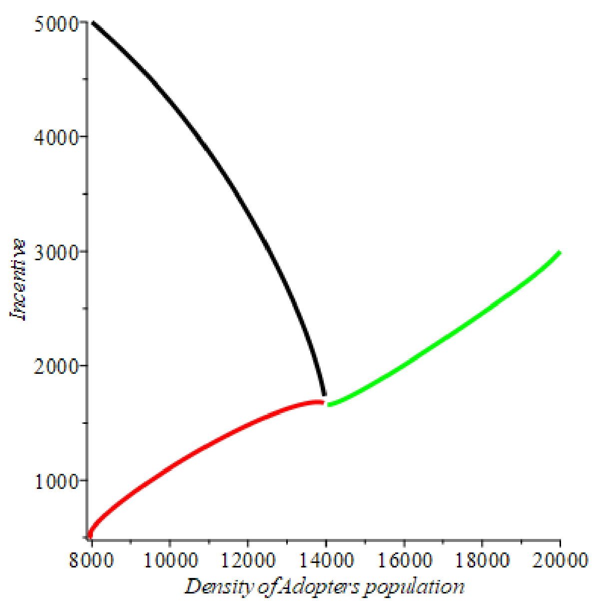

Figure 5.3: Incentive with density of adopter's population.

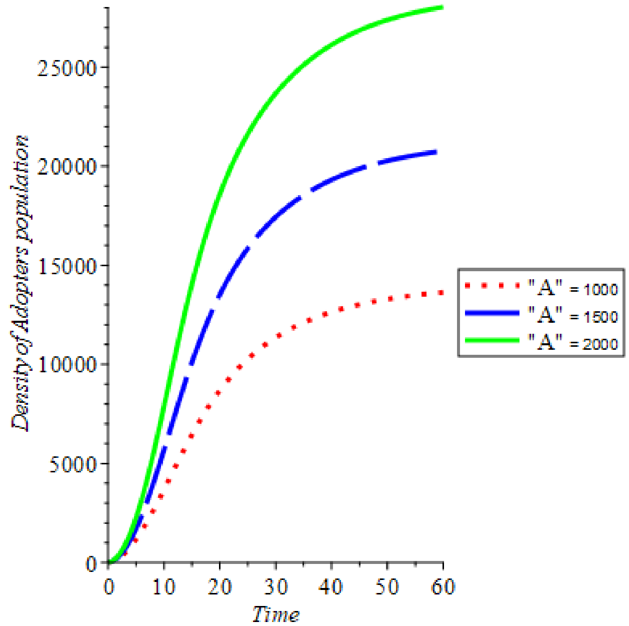

Figure 5.4: Density of adopter's population with time for different values of A.

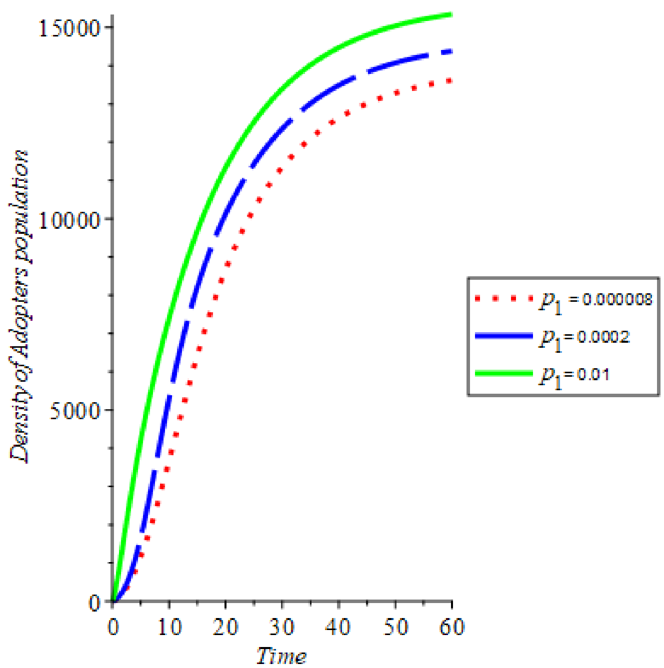

Figure 5.5: Density of adopter's population with time for different values of p_1.

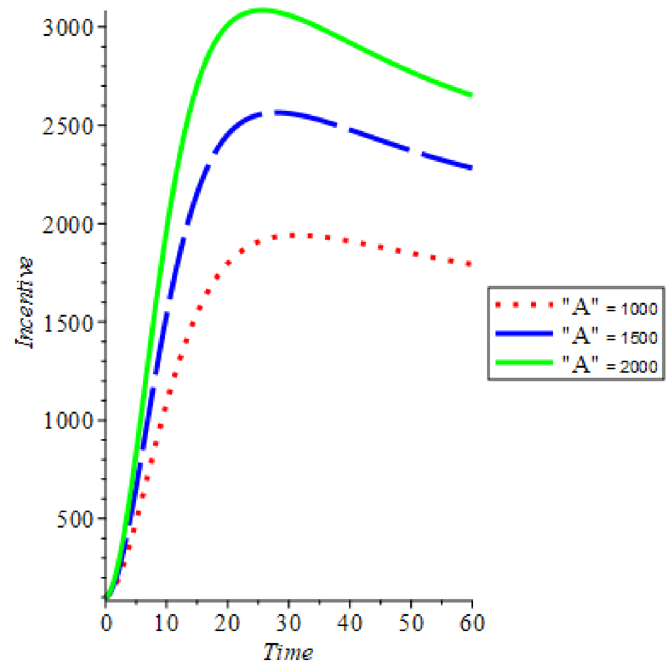

Figure 5.6: Incentive with time for different values of A.

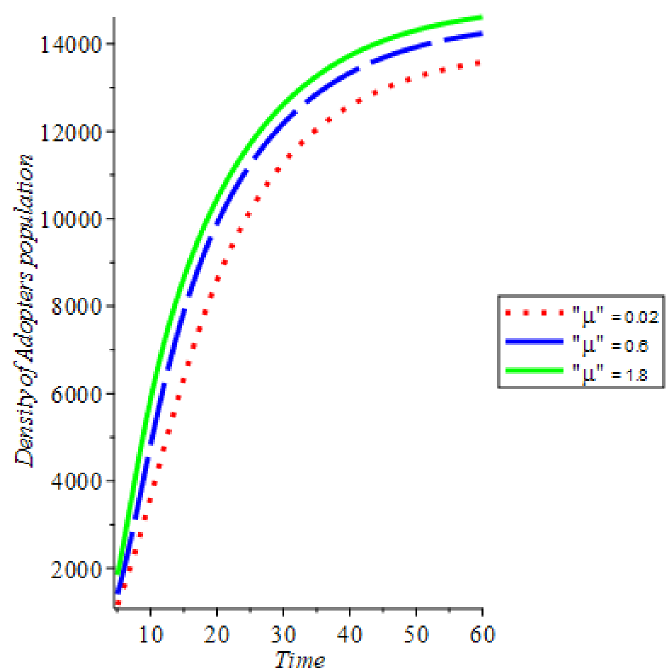

Figure 5.7: Density of adopter's population with time for different values of μ.

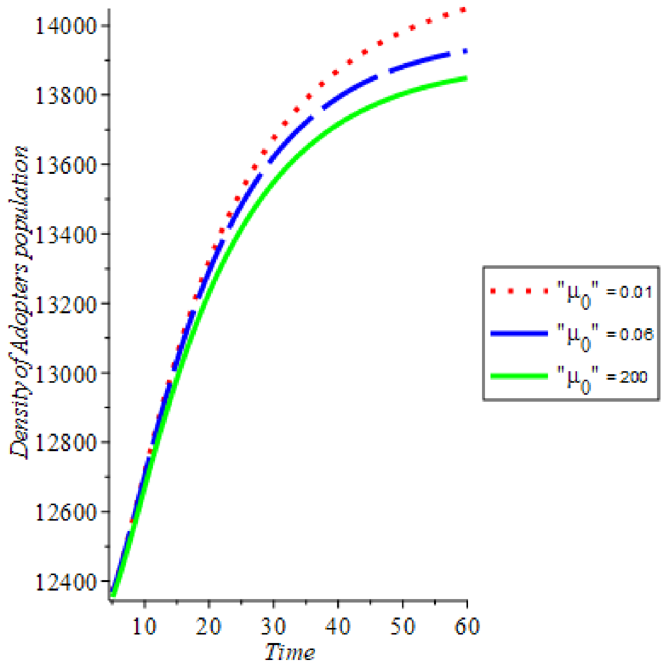

Figure 5.8: Density of adopter's population with time for different values of μ_0.

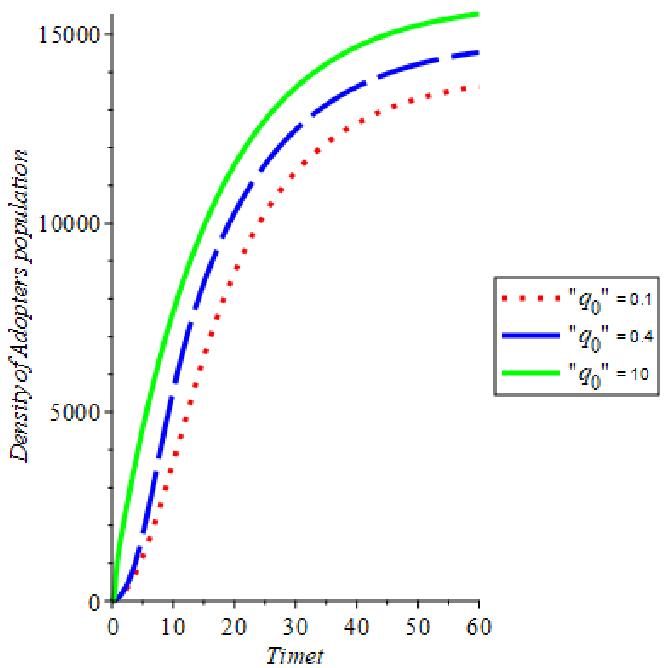

Figure 5.9: Density of adopter's population with time for different values of q_0.

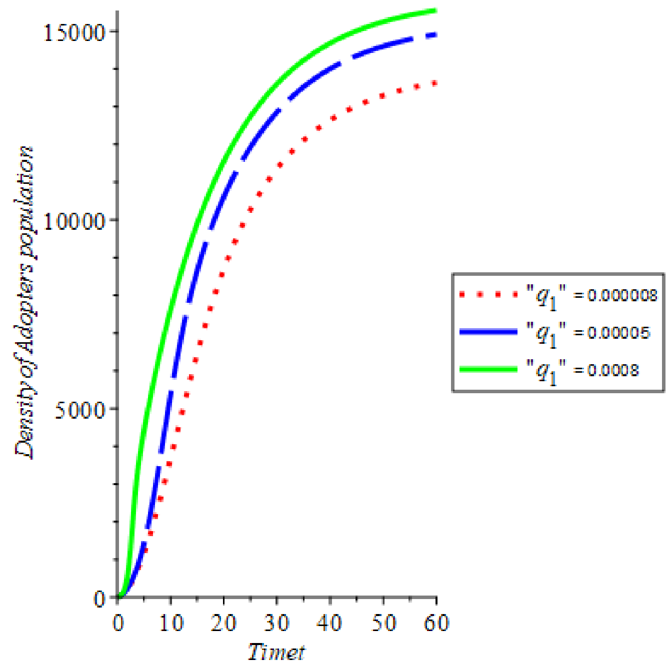

Figure 5.10: Density of adopter's population with time for different values of q_1.

6. Conclusions

In this paper, we have proposed and analyzed a nonlinear mathematical model to study the effects of (i) variable external influence which is linearly related to the incentive variable and (ii) variable internal influence which is linearly related to total market population, on diffusion of innovation under consideration. The analysis shows that there is positive effect of the variability of external influence on number of adopters and hence on diffusion process. This also shows that incentive policy has stabilizing effect on the system. From stability conditions, it is also seen that, α, the rate at which innovation is discontinued has destabilizing effect on the system. It is seen that density of adopter's population increases with increase in p_1, the coefficient of external influence due to incentive. Also μ_0, the parameter, which represents the technical difficulties in the implementation of incentive schemes has a negative effect on the density of adopter's population.

7. Appendices

7.1. Appendix A

Proof of Theorem 4.1. To examine local stability of the equilibrium $E^*(N_a^*, N^*, I^*)$, we obtain Jacobian matrix of the linearised form of model system (2.4) - (2.6) at E^* as

$$J = \begin{bmatrix} -p_0 - p_1 I^* + q_0 - \frac{2q_0 N_a^*}{N^*} - e_1 + q_1 N^* - 2q_1 N_a^* & p_0 + p_1 I^* + \frac{q_0 N_a^{*2}}{N^{*2}} & 0 \\ -\alpha & -e-\gamma & 0 \\ -\mu & \mu & -\mu_0 \end{bmatrix},$$

Using equations (3.1) and (3.2) at equilibrium $E^*(N_a^*, N^*, I^*)$,

We can express $-p_0 - p_1 I^* + q_0 - \frac{2q_0 N_a^*}{N^*} - e_1 + q_1 N^* - 2q_1 N_a^* = -\frac{p_0 N^*}{N_a^*} - \frac{p_1 N^* I^*}{N_a^*} - \frac{q_0 N_a^*}{N^*} - q_1 N_a^*$, and thus we rewrite the Jacobian matrix as,

$$J = \begin{bmatrix} -\frac{p_0 N^*}{N_a^*} - \frac{p_1 N^* I^*}{N_a^*} - \frac{q_0 N_a^*}{N^*} - q_1 N_a^* & p_0 + p_1 I^* + \frac{q_0 N_a^{*2}}{N^{*2}} & 0 \\ -\alpha & -e-\gamma & 0 \\ -\mu & \mu & -\mu_0 \end{bmatrix}.$$

The characteristic equation is of the form $(\mu_0 + \lambda)(\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})) = 0$, where $a_{11} = -\frac{p_0 N^*}{N_a^*} - \frac{p_1 N^* I^*}{N_a^*} - \frac{q_0 N_a^*}{N^*} - q_1 N_a^* < 0$, $a_{22} = -e - \gamma < 0$, $a_{12} = p_0 + p_1 I^* + \frac{q_0 N_a^{*2}}{N^{*2}} > 0$, and $a_{21} = -\alpha < 0$. Since $a_{11} + a_{22} < 0, (a_{11}a_{22} - a_{12}a_{21}) > 0$, hence all eigen values lie in the negative, left half plane. Thus, the equilibrium $E^*(N_a^*, N^*, M^*)$ is locally asymptotically stable without any condition.

7.2. Appendix B

Proof of Theorem 4.2. To establish the conditions of global stability stated in Theorem 4.2, we consider the following positive definite function about E^*,

$$V(N_a, N, I) = c_1(N_a - N_a^* - N_a^* \ln \frac{N_a}{N_a^*}) + \frac{c_2}{2}(N - N^*)^2 + \frac{c_3}{2}(I - I^*)^2. \tag{B1}$$

Differentiating with respect to t along the system (2.4) - (2.6) and applying some simple algebraic manipulations, we get

$$\begin{aligned}\frac{dV}{dt} = c_1(-\frac{p_0N}{N_aN_a^*} - \frac{p_1IN}{N_aN_a^*} - \frac{q_0}{N^*} - q_1)(N_a - N_a^*)^2 + (c_1(\frac{p_1I}{N_a^*} + \frac{p_0}{N_a^*} + \frac{q_0N_a}{NN^*} + q_1)\\ - c_2\alpha)(N - N^*)(N_a - N_a^*) + (c_1\frac{p_1}{N_a^*}(N^* - N_a^*) - c_3\mu)(I - I^*)(N_a - N_a^*) + c_3\mu(N - N^*)(I - I^*)\\ - c_2(e + \gamma)(N - N^*)^2 - c_3\mu_0(I - I^*)^2.\end{aligned} \tag{B2}$$

Taking,

$$c_1 = \mu, c_2 = \frac{p_0\mu}{N_a^*\alpha} + \frac{\mu q_1}{\alpha}, c_3 = \frac{p_1(N^* - N_a^*)}{N_a^*}, \tag{B3}$$

we get
$\frac{dV}{dt} = -\mu\frac{q_0}{N^*}(N_a - N_a^*)^2 + \mu(\frac{p_1I}{N_a^*} + \frac{q_0N_a}{NN^*})(N - N^*)(N_a - N_a^*) - \frac{1}{2}(\frac{p_0\mu}{N_a^*\alpha} + \frac{\mu q_1}{\alpha})(e + \gamma)(N - N^*)^2 - \frac{1}{2}(\frac{p_0\mu}{N_a^*\alpha} + \frac{\mu q_1}{\alpha})(e + \gamma)(N - N^*)^2 + \frac{p_1(N^* - N_a^*)}{N_a^*}\mu(N - N^*)(I - I^*) - \frac{p_1(N^* - N_a^*)}{N_a^*}\mu_0(I - I^*)^2 - \mu(\frac{p_0N}{N_aN_a^*} + \frac{p_1IN}{N_aN_a^*} + q_1)(N_a - N_a^*)^2.$

$\frac{dV}{dt}$ is negative definite if the following conditions are satisfied-

$$(\frac{p_1I}{N_a^*} + \frac{q_0N_a}{NN^*})^2 < 2\frac{q_0}{N^*}\frac{(e + \gamma)}{\alpha}(\frac{p_0}{N_a^*} + q_1), \tag{B4}$$

$$p_1(N^* - N_a^*)\mu\alpha \leq 2\mu_0(e + \gamma)(p_0 + q_1N_a^*). \tag{B5}$$

Taking supremum over N_a, M and infimum over N in inequality (B4), the inequality (B4) holds strictly if

$$(\frac{p_1I_{\max}}{N_a^*} + \frac{q_0(N_a)_{\max}}{N_{\min}N^*})^2 \leq 2\frac{q_0}{N^*}\frac{(e + \gamma)}{\alpha}(\frac{p_0}{N_a^*} + q_1). \tag{B6}$$

The inequalities (B5) and (B6) give the two conditions as stated in Theorem 4.2.

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SOME NEW BIQUADRATIC SEQUENCE SPACES OVER *n*-NORMED SPACES DEFINED BY MUSIELAK-ORLICZ FUNCTION

By

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Abstract

In this paper, we construct some new biquadratic sequence spaces using orlicz function defined by Musielak-Orlicz over n-normed spaces. We study some topological properties and inclusion relations between these spaces.

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1. Introduction

In this paper, the classes of all, entire and analytic scalar valued single sequences are denoted by ω, Γ and Λ respectively. For the set of all complex sequences (x_{mnkl}), where $m, n, k, l \in N$, the set of positive integers, we write ω^4. Then, ω^4 is a linear space under the coordinatewise addition and scalar multiplication.

A biquadratic sequence can be represented by a matrix. In case of double sequence we write in form of a square. In case of triple sequence it will be in the form of a box in three dimensions. The different type of notions of a triple sequence was introduced and investigated initially by Sahiner et al. [23], Esi [7], Esi and Catalbas [8], Datta et al. [2], Debnath et al. [6], Vandana et al.[25] and many others. In case of biquadratic sequence it will be in the form of a box in four dimensions.

The concept of 2-normed spaces was initially developed by Gähler[12] and the concept of n-normed spaces was developed by Misiak[16]. Further, Dutta[3, 4, 5], Gunawan [10], Hendra Gunawan and Mashadi[11], Mursaleen et al. [17, 18], Savas[24] and Jalal [13] also studied n-normed spaces and obtain various results in n-normed spaces. Mishra et al. [19] studied semi-normed differences and Mishra et al. [20], Rai et al. [22], Vandana et al. [26] also studied generalized difference χ^{2I} and χ^{3I} of fuzzy real numbers and χ^3 ideal fuzzy real numbers and obtain some results.

Kizmaz [14] introduced the notion of difference sequence spaces as follows

$$Z(\Delta) = \{x = x_k \in \omega : (\Delta x_k) \in Z\},$$

for $Z = l_\infty,\ c$ and c_0

$$\Delta(x) = x_k - x_{k+1}.$$

The study was further generalised by Et and Colak [9] and introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$.

The difference operator on biquadratic sequence is defined as

$$\begin{aligned}\Delta(x_{mnkl}) = x_{mnkl} - x_{(m+1)nkl} - x_{m(n+1)kl} - x_{mn(k+1)l} - x_{mnk(l+1)} \\ + x_{(m+1)(n+1)kl} + x_{(m+1)n(k+1)l} + x_{(m+1)nk(l+1)} \\ - x_{(m+1)(n+1)(k+1)l} - x_{(m+1)(n+1)k(l+1)} \\ + x_{(m+1)(n+1)(k+1)(l+1)}.\end{aligned}$$

Definitions

Definition 1.1. *A biquadratic sequence (a_{ijkl}) is said to converge at L in Pringsheim's sense if for every $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ such that*

$$|a_{ijkl} - L| < \epsilon \text{ whenever } i, j, k, l \geq N$$

and is written as $\lim_{i,j,k,l\to\infty} a_{ijkl} = L$.

Definition 1.2. *A biquadratic sequence (a_{ijkl}) is said to be Cauchy sequence if for every $\epsilon > 0$, there exists $N(\epsilon) \in \mathbb{N}$ such that*

$$|a_{ijkl} - a_{pqrs}| < \epsilon$$

whenever $i \geq p \geq N,\ j \geq q \geq N,\ k \geq r \geq N,\ l \geq s \geq N$.

Definition 1.3. *A biquadratic sequence (a_{ijkl}) is said to be bounded sequence if there exists $M > 0$, such that $|a_{ijkl}| < M$ for all $i, j, k, l \in \mathbb{N}$.*

Definition 1.4. *A biquadratic sequence X is said to be solid if $(\alpha_{ijkl}a_{ijkl}) \in X$ whenever $(a_{ijkl}) \in X$ and for all biquadratic sequence (a_{ijkl}) of scalars with $|\alpha_{ijkl}| \leq 1$, for all $i, j, k, l \in \mathbb{N}$.*

2. Preliminaries

A Orcliz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Lindenstrase and Tzaferiri[15] defined the following sequence space using the idea of Orlicz function. Let ω be the space of all real or complex sequence $x = (x_k)$ then

$$l_M = \left\{x \in \omega : \sum_{k=1}^{\infty}\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\}, \tag{2.1}$$

which is called as an Orlicz sequence space.

The space l_M is a Banach space with norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\}. \tag{2.2}$$

It is shown in [1] that every Orlicz sequence space l_M contains a subspace isomorphic to $l_p (p \geq 1)$. The Δ_2- condition is equivalent to $M(Lx) \leq kLM(x)$ for all L with $L \geq 1$.

An orlicz function M can be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt, \tag{2.3}$$

where η known as kernel of M is right differential for $t \leq 0$, $\eta(0) = 0$ and η is non-decreasing and $\eta \to \infty$ as $t \to \infty$.

A sequence $f = (f_{mnkl})$ of orlicz function is called Musielak - Orlicz function. A function $g = (g_{mnkl})$ defined by

$$g_{mnkl}(v) = sup\{|v|u - f_{mnkl}(u) : u \geq 0\} \qquad m, n, k, l = 1, 2, 3, ...$$

is called the complementry function of a Musielak- orlicz function f. For a given Musielak- orlicz function f, the Musielak- orlicz sequence space t_f is defined as follows

$$t_f = \left\{x \in \omega^4 : I_f(|x_{mnkl}|)^{\frac{1}{m+n+k+l}} \to 0 \text{ as } m, n, k, l \to \infty\right\},$$

where I_f is a convex modulus function defined by

$$I_f(x) = \sum_{m,n,k,l=1}^{\infty} f_{mnkl}(|x_{mnkl}|)^{\frac{1}{m+n+k+l}}, \quad x = (x_{mnkl}) \in t_f.$$

Definitions

2.1 Paranormed space

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called a paranorm if

(i) $p(x) \geq 0$ *for all* $x \in X$,

(ii) $p(-x) = p(x)$ *for all* $x \in X$,

(iii) $p(x + y) \leq p(x) + p(y)$ *for all* $x \in X$,

(iv) *If* (λ_n) *is a sequence of scalars with* $\lambda_n \to \lambda$ *as* $n \to \infty$ *and* x_n *is a sequence of vectors with* $p(x_n - x) \to 0$ *as* $n \to \infty$ *then* $p(\lambda_n x_n - \lambda x) \to 0$ *as* $n \to \infty$.

A paranorm p for which $p(x) = 0 \Rightarrow x = 0$ is called a total paranorm and the pair (x, p) is called a total paranorm space. It is well- known that the metric of any linear metric space is given by some total paranorm.

If $p = (p_{ijkl})$ is a biquadratic sequence of positive real number with

$$0 \leq p_{ijkl} \leq sup_{ijkl} = G, \quad K = max(1, 2^{(G-1)})$$

then

$$|a_{ijkl} + b_{ijkl}|^{p_{ijkl}} \leq K\{|a_{ijkl}|^{p_{ijkl}} + |b_{ijkl}|^{p_{ijkl}}\}$$

for all i, j, k, l and biquadratic sequence a_{ijkl}, $b_{ijkl} \in \mathbb{C}$.

In 1960, Gähler was introduced the concept of two-normed spaces, while n-normed spaces developed by Misiak [16]. Since then, many others have studied this concept and obtained various results.

2.2 *n*-normed space

Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{R}$ of dimension d, where $2 \leq n \leq d$. A real valued function $\|., ..., .\|$ on X^n satisfying the following four conditions

(i) $\|x_1, x_2, ..., x_n\| = 0$ *if and only if* $x_1, x_2, ..., x_n$ *are linearly dependent in* X,

(ii) $\|x_1, x_2, ..., x_n\|$ *is invariant under permutation*,

(iii) $\|\alpha x_1, x_2, ..., x_n\| = |\alpha|\|x_1, x_2, ..., x_n\|$ *for any* $\alpha \in \mathbb{R}$,

(iv) $\|x_1 + x_1', x_2, ..., x_n\| \leq \|x_1, x_2, ..., x_n\| + \|x_1', x_2, ..., x_n\|$, *is called n-normed on X, and the pair* $(X, \|., ..., .\|)$ *is called n-normed space over the field* $\mathbb{R}$.

Example 2.1. $(\mathbb{R}, \|., ..., .\|_E)$ where $(\mathbb{R}, \|., ..., .\|_E)$ is the volume of the n-dimensional parallelopied spanned by the vectors $x_1, x_2, ..., x_n$ which can be written as

$$\|x_1, x_2, ..., x_n\|_E = |det(x_{ij})|$$

where $x_i = (x_{i1}, x_{i2}, ..., x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, 3, ...$

Let $(X, \|., ..., .\|)$ be an n-normed space of dimension $2 \leq d \leq n$ and $\{a_1, a_2, ..., a_n\}$ be linearly independent set in X. Then the following function $\|., ..., .\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, ..., x_{n-1}\|_\infty = max\{\|x_1, x_2, ..., x_{n-1}, a_i\| : i = 1, 2, ..., n\},$$

defines an $(n-1)$ norm on X with respect to $\{a_1, a_2, ..., a_n\}$.

A sequence (x_k) in a n- normed space $(X, \|., ..., .\|)$ is said to be convergence if

$$\lim_{k,p\to\infty} \|x_k - x_p, z_1, ..., z_{n-1}\| = 0, \text{ for any } z_1, ..., z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space. The n-normed space has been studied in [1, 10, 11].

3. Construction of Biquadratic n-normed sequence spaces

Now we introduce the new class of biquadratic sequence spaces using Orlicz functions and difference operator on n-normed spaces, if M is an orlicz function and $p = (p_{ijkl})$ is a biquadratic sequence of strictly positive real numbers and $(X, \|., ..., .\|)$ is real linear n-normed space. We define the following class of sequence

$$\omega_0^4(M, \Delta, p, \|., ..., .\|) = \Big\{(a_{ijkl}) \in \omega^4 : \lim_{p,q,r,s\to\infty} \frac{1}{pqrs} \sum_{i,j,k,l=1}^{p,q,r,s} \left[M\left(\left\|\frac{\Delta a_{ijkl}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{ijkl}}$$
$$= 0, \text{ for each } z_1, z_2, ..., z_{n-1}, \text{ for some } \rho > 0\Big\},$$

$$\omega^4(M, \Delta, p, \|., ..., .\|) = \Big\{(a_{ijkl}) \in \omega^4 : \lim_{p,q,r,s\to\infty} \frac{1}{pqrs} \sum_{i,j,k,l=1}^{p,q,r,s} \left[M\left(\left\|\frac{\Delta a_{ijkl} - L}{\rho}, z_1, ..., z_{n-1}\right\|\right)\right]^{p_{ijkl}}$$
$$= 0, \text{ for each } z_1, z_2, ..., z_{n-1}, \text{ for some } \rho > 0, \ L > 0\Big\},$$

$$\omega_\infty^4(M, \Delta, p, \|., ..., .\|) = \Big\{(a_{ijkl}) \in \omega^4 : \sup_{pqrs} \frac{1}{pqrs} \sum_{i,j,k,l=1}^{p,q,r,s} \left[M\left(\left\|\frac{\Delta a_{ijkl}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{ijkl}}$$
$$< \infty, \text{ for each } z_1, z_2, ..., z_{n-1}, \text{ for some } \rho > 0\Big\}.$$

Some Special Cases

(i) If we take $M(x) = x$, we get

$$\omega_0^4(\Delta, p, \|., ..., .\|) = \Big\{(a_{ijkl}) \in \omega^4 : \lim_{p,q,r,s\to\infty} \frac{1}{pqrs} \sum_{i,j,k,l=1}^{p,q,r,s} \left[\left(\left\|\frac{\Delta a_{ijkl}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{ijkl}}$$
$$= 0, \text{ for each } z_1, z_2, ..., z_{n-1}, \text{ for some } \rho > 0\Big\},$$

$$\omega^4(\Delta, p, \|., ..., .\|) = \Big\{(a_{ijkl}) \in \omega^4 : \lim_{p,q,r,s\to\infty} \frac{1}{pqrs} \sum_{i,j,k,l=1}^{p,q,r,s} \left[\left(\left\|\frac{\Delta a_{ijkl} - L}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{ijkl}}$$
$$= 0, \text{ for each } z_1, z_2, ..., z_{n-1}, \text{ for some } \rho > 0, \ L > 0\Big\},$$

$$\omega_\infty^4(\Delta, p, \|., ..., .\|) = \Big\{(a_{ijkl}) \in \omega^4 : \sup_{pqrs} \frac{1}{pqrs} \sum_{i,j,k,l=1}^{p,q,r,s} \left[\left(\left\|\frac{\Delta a_{ijkl}}{\rho}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{ijkl}}$$
$$< \infty, \text{ for each } z_1, z_2, ..., z_{n-1}, \text{ for some } \rho > 0\Big\}.$$

(ii) If we take $p = (p_{ijkl}) = 1$, we get

$$\omega_0^4(M, \Delta, \|., ..., .\|) = \Big\{(a_{ijkl}) \in \omega^4 : \lim_{p,q,r,s\to\infty} \frac{1}{pqrs} \sum_{i,j,k,l=1}^{p,q,r,s} \left[M\left(\left\|\frac{\Delta a_{ijkl}}{\rho} z_1, z_2, ..., z_{n-1}\right\|\right)\right] = 0$$
$$\text{for each } z_1, z_2, ..., z_{n-1}, \text{ for some } \rho > 0\Big\},$$

$$\omega^4(M,\Delta,\|.,...,.\|)=\Big\{(a_{ijkl})\in\omega^4:\lim_{p,q,r,s\to}\frac{1}{pqrs}\sum_{i,j,k,l=1}^{p,q,r,s}\Big[M\Big(\Big\|\frac{\Delta a_{ijkl}-L}{\rho},z_1,z_2,...,z_{n-1}\Big\|\Big)\Big]=0$$
$$\text{for each } z_1,z_2,...,z_{n-1},\text{ for some }\rho>0,\ L>0\Big\},$$

$$\omega^4_\infty(M,\Delta,\|.,...,.\|)=\Big\{(a_{ijkl})\in\omega^4:\sup_{pqrs}\frac{1}{pqrs}\sum_{i,j,k,l=1}^{p,q,r,s}\Big[M\Big(\Big\|\frac{\Delta a_{ijkl}}{\rho},z_1,z_2,...,z_{n-1}\Big\|\Big)\Big]<\infty$$
$$\text{for each } z_1,z_2,...,z_{n-1},\text{ for some }\rho>0,\Big\}.$$

Theorem 3.1. *Let M be a orlicz function and $p=p_{ijkl}$ be bounded biquadratic sequence of strictly positive reals numbers. Then the classes of sequence*
$\omega^4_0(M,\Delta,p,\|.,...,.\|)$, $\omega^4(M,\Delta,p,\|.,...,.\|)$ and $\omega^4_\infty(M,\Delta,p,\|.,...,.\|)$ are linear spaces over the field of real numbers $\mathbb{R}$.

Proof. Let (a_{ijkl}), $(b_{ijkl})\in\omega^4_\infty(M,\Delta,p,\|.,...,.\|)$ and $\alpha,\beta\in\mathbb{R}$. Then $\exists$ a positive real number ρ_1,ρ_2 such that

$$\sup_{pqrs}\frac{1}{pqrs}\sum_{i,j,k,l=1}^{p,q,r,s}\Big[M\Big(\Big\|\frac{\Delta a_{ijkl}}{\rho_1},z_1,z_2,...,z_{n-1}\Big\|\Big)\Big]^{p_{ijkl}}<\infty$$

for each $z_1,z_2,...,z_{n-1}$, for some $\rho_1>0$,
and

$$\sup_{pqrs}\frac{1}{pqrs}\sum_{i,j,k,l=1}^{p,q,r,s}\Big[M\Big(\Big\|\frac{\Delta b_{ijkl}}{\rho_2},z_1,z_2,...,z_{n-1}\Big\|\Big)\Big]^{p_{ijkl}}<\infty$$

for each $z_1,z_2,...,z_{n-1}$ for some $\rho_2>0$.

Let $\rho=max\{2|\alpha|\rho_1,2|\beta|\rho_2\}$, then Since $\|.,...,.\|$ is a norm on X and M is non-dreasing, convex and using inequality, we have

$$\begin{aligned}\sup_{pqrs}\frac{1}{pqrs}&\sum_{i,j,k,l=1}^{p,q,r,s}\Big[M\Big(\Big\|\frac{\Delta\alpha a_{ijkl}+\Delta\beta b_{ijkl}}{\rho},z_1,z_2,...,z_{n-1}\Big\|\Big)\Big]^{p_{ijkl}}\\
&\le\sup_{pqrs}\frac{1}{pqrs}\sum_{i,j,k,l=1}^{p,q,r,s}\Big\{\Big[M\Big(\Big\|\frac{\Delta\alpha a_{ijkl}}{\rho},z_1,z_2,...,z_{n-1}\Big\|\Big)\Big]^{p_{ijkl}}\\
&\qquad+\Big[M\Big(\Big\|\frac{\Delta\beta b_{ijkl}}{\rho},z_1,z_2,...,z_{n-1}\Big\|\Big)\Big]^{p_{ijkl}}\Big\}\\
&\le K\sup_{pqrs}\frac{1}{pqrs}\sum_{i,j,k,l=1}^{p,q,r,s}\frac{1}{2^{ijkl}}\Big\{\Big[M\Big(\Big\|\frac{\Delta a_{ijkl}}{\rho_1},z_1,z_2,...,z_{n-1}\Big\|\Big)\Big]^{p_{ijkl}}\Big\}\\
&\qquad+K\sup_{pqrs}\frac{1}{pqrs}\sum_{i,j,k,l=1}^{p,q,r,s}\frac{1}{2^{ijkl}}\Big\{\Big[M\Big(\Big\|\frac{\Delta b_{ijkl}}{\rho_2},z_1,z_2,...,z_{n-1}\Big\|\Big)\Big]^{p_{ijkl}}\Big\}\\
&<\infty.\end{aligned}$$

Thus we have $\alpha(a_{ijkl})+\beta(b_{ijkl})\in\omega^4_\infty(M,\Delta,p,\|.,...,.\|)$. Hence $\omega^4_\infty(M,\Delta,p,\|.,...,.\|)$ is a linear space.

Similarly, we can show that $\omega^4_0(M,\Delta,p,\|.,...,.\|)$ and $\omega^4(M,\Delta,p,\|.,...,.\|)$ are linear spaces over the field of $\mathbb{R}$. □

Theorem 3.2. *Let M be a orlicz function and $p=(p_{ijkl})$ be bounded biquadratic sequence of strictly positive reals numbers. Then the classes of sequence*
$\omega^4_0(M,\Delta,p,\|.,...,.\|)$, $\omega^4(M,\Delta,p,\|.,...,.\|)$ and $\omega^4_\infty(M,\Delta,p,\|.,...,.\|)$ are paranormed spaces paranormed by

$$g(a_{ijkl})=\sup_i|a_{i111}|+\sup_j|a_{1j11}|+\sup_k|a_{11k1}|+\sup_l|a_{111l}|$$
$$+\,inf\Big\{\rho^{\frac{p_{ijkl}}{H}}>0:\sup_{p,q,r,s}\Big[M\Big(\Big\|\frac{\Delta a_{ijkl}}{\rho},z_1,z_2,...,z_{n-1}\Big\|\Big)\Big]^{p_{ijkl}/H}\le1\Big\},$$

where

$$H=max(1,G),\ G=\sup_{i,j,k,l\ge1}p_{ijkl}.$$

Proof. Clearly, $g(0)=0$ and $g(-(a_{ijkl}))=g((a_{ijkl}))$

Let $(a_{ijkl}),(b_{ijkl})\in\omega^4_\infty(M,\Delta,p,\|.,...,.\|)$

Then $\exists$ some $\rho_1, \rho_2 > 0$
such that

$$\sup_{p,q,r,s}\left[M\left(\left\|\frac{\Delta a_{ijkl}}{\rho_1}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{ijkl}/H} \leq 1$$

and

$$\sup_{p,q,r,s}\left[M\left(\left\|\frac{\Delta b_{ijkl}}{\rho_2}, z_1, z_2, ..., z_{n-1}\right\|\right)\right]^{p_{ijkl}/H} \leq 1.$$

Let $\rho = \rho_1 + \rho_2$, then by using Minkowski's inequality, we have

$$\begin{aligned}\sup_{p,q,r,s}&\left[M\left(\left\|\frac{\Delta a_{ijkl} + \Delta b_{ijkl}}{\rho}, z_1, z_2, .., z_{n-1}\right\|\right)\right]^{p_{ijkl}/H} \\ &\leq \left(\frac{\rho_1}{\rho_1 + \rho_2}\right)\sup_{p,q,r,s}\left[M\left(\left\|\frac{\Delta a_{ijkl}}{\rho_1}, z_1, z_2, .., z_{n-1}\right\|\right)\right]^{p_{ijkl}/H} \\ &\quad + \left(\frac{\rho_2}{\rho_1 + \rho_2}\right)\sup_{p,q,r,s}\left[M\left(\left\|\frac{\Delta b_{ijkl}}{\rho_2}, z_1, z_2, .., z_{n-1}\right\|\right)\right]^{p_{ijkl}/H}.\end{aligned}$$

Now,

$$\begin{aligned} g((a_{ijkl}) + (b_{ijkl})) \\ &= \sup_i |a_{i111} + b_{i111}| + \sup_j |a_{1j11} + b_{1j11}| + \sup_k |a_{11k1} + b_{11k1}| + \sup_l |a_{111l} + b_{111l}| \\ &\quad + inf\left\{(\rho_1 + \rho_2)^{p_{ijkl}/H} > 0 : \sup_{p,q,r,s}\left[M\left(\left\|\frac{\Delta a_{ijkl} + \Delta b_{ijkl}}{\rho_1 + \rho_2}, z_1, z_2, .., z_{n-1}\right\|\right)\right]^{p_{ijkl}/H} \leq 1\right\} \\ &= \sup_i |a_{i111}| + \sup_j |a_{1j11}| + \sup_k |a_{11k1}| + \sup_l |a_{111l}| \\ &\quad + inf\left\{(\rho_1 + \rho_2)^{p_{ijkl}/H} > 0 : \sup_{p,q,r,s}\left[M\left(\left\|\frac{\Delta a_{ijkl}}{\rho_1 + \rho_2}, z_1, z_2, .., z_{n-1}\right\|\right)\right]^{p_{ijkl}/H} \leq 1\right\} \\ &\quad + \sup_i |b_{i111}| + \sup_j |b_{1j11}| + \sup_k |b_{11k1}| + \sup_l |b_{111l}| \\ &\quad + inf\left\{(\rho_1 + \rho_2)^{p_{ijkl}/H} > 0 : \sup_{p,q,r,s}\left[M\left(\left\|\frac{\Delta b_{ijkl}}{\rho_1 + \rho_2}, z_1, ..., z_{n-1}\right\|\right)\right]^{p_{ijkl}/H} \leq 1\right\} \\ &= \sup_i |a_{i111}| + \sup_j |a_{1j11}| + \sup_k |a_{11k1}| + \sup_l |a_{111l}| \\ &\quad + inf\left\{(\rho)^{p_{ijkl}/H} > 0 : \sup_{p,q,r,s}\left[M\left(\left\|\frac{\Delta a_{ijkl}}{\rho}, z_1, z_2, .., z_{n-1}\right\|\right)\right]^{p_{ijkl}/H} \leq 1\right\} \\ &\quad + \sup_i |b_{i111}| + \sup_j |b_{1j11}| + \sup_k |b_{11k1}| + \sup_l |b_{111l}| \\ &\quad + inf\left\{(\rho)^{p_{ijkl}/H} > 0 : \sup_{p,q,r,s}\left[M\left(\left\|\frac{\Delta b_{ijkl}}{\rho}, z_1, ..., z_{n-1}\right\|\right)\right]^{p_{ijkl}/H} \leq 1\right\} \\ &= g(a_{ijkl}) + g(b_{ijkl}).\end{aligned}$$

Let $\lambda \in \mathbb{C}$ then the continuity of the follows from the following inequality

$$\begin{aligned} g(\lambda(a_{ijkl})) &= \sup_i |\lambda a_{i111}| + \sup_j |\lambda a_{1j11}| + \sup_k |\lambda a_{11k1}| + \sup_l |\lambda a_{111l}| \\ &\quad + inf\left\{(\rho)^{p_{ijkl}/H} > 0 : \sup_{p,q,r,s}\left[M\left(\left\|\frac{\Delta \lambda a_{ijkl}}{\rho}, z_1, z_2, .., z_{n-1}\right\|\right)\right]^{p_{ijkl}/H} \leq 1\right\} \\ &= |\lambda|(\sup_i |a_{i111}| + \sup_j |a_{1j11}| + \sup_k |a_{11k1}| + \sup_l |a_{111l}|) \\ &\quad + inf\left\{(|\lambda|.r)^{p_{ijkl}/H} > 0 : \sup_{p,q,r,s}\left[M\left(\left\|\frac{\Delta a_{ijkl}}{\rho}, z_1, z_2, .., z_{n-1}\right\|\right)\right]^{p_{ijkl}/H} \leq 1\right\}\end{aligned}$$

$$= |\lambda|.g(x) \quad \text{where } r = \frac{\rho}{|\lambda|}.$$

Hence $\omega^4_\infty(M, \Delta, p, \|., ..., .\|)$ is a paranormed space.

Similarly, we can show that $\omega^4_0(M, \Delta, p, \|., ..., .\|)$ and $\omega^4(M, \Delta, p, \|., ..., .\|)$ are also paranormed space. □

Theorem 3.3. *Let M be a orlicz function and $p = (p_{ijkl})$ be bounded biquadratic sequence of strictly positive reals numbers. Then the classes of sequence $\omega^4_0(M, \Delta, p, \|., ..., .\|)$, $\omega^4(M, \Delta, p, \|., ..., .\|)$ and $\omega^4_\infty(M, \Delta, p, \|., ..., .\|)$ are complete paranormed spaces paranormed by*

$$g(a_{ijkl}) = \sup_i |a_{i111}| + \sup_j |a_{1j11}| + \sup_k |a_{11k1}| + \sup_l |a_{111l}|$$
$$+ inf\Big\{\rho^{\frac{p_{ijkl}}{H}} > 0 : \sup_{p,q,r,s}\Big[M\Big(\Big\|\frac{\Delta a_{ijkl}}{\rho}, z_1, z_2, ..., z_{n-1}\Big\|\Big)\Big]^{p_{ijkl}/H} \le 1\Big\}.$$

Proof. Let (a_{ijkl}) be a Cauchy sequence in $\omega^4_\infty(M, \Delta, p, \|., ..., .\|)$.

Then $g(a^s_{ijkl} - a^t_{ijkl}) \to 0$ as $s, t \to \infty$.

For given $\epsilon > 0$, choose $r > 0$ and $x_0 > 0$ be such that $\frac{\epsilon}{rx_0}$ and $M(\frac{rx_0}{2}) \ge 1$.

Now $g(a^s_{ijkl} - a^t_{ijkl}) \to 0$ as $s, t \to \infty$ implies that there exists $m_0 \in \mathbb{N}$ such that

$$g(a^s_{ijkl} - a^t_{ijkl}) < \frac{\epsilon}{rx_0} \text{ for all } s, t \ge m_0.$$

Thus we have

$$\sup_i |a^s_{i111} - a^t_{i111}| + \sup_j |a^s_{1j11} - a^t_{1j11}| + \sup_k |a^s_{11k1} - a^t_{11k1}| + \sup_l |a^s_{111l} - a^t_{111l}|$$
$$+ inf\Big\{\rho^{\frac{p_{ijkl}}{H}} > 0 : \sup_{p,q,r,s}\Big[M\Big(\Big\|\frac{\Delta a^s_{ijkl} - \Delta a^t_{ijkl}}{\rho}, z_1, z_2, ..., z_{n-1}\Big\|\Big)\Big]^{p_{ijkl}/H} \le 1\Big\}$$
$$< \frac{\epsilon}{rx_0}.$$

This shows that (a^s_{i111}), (a^s_{1j11}), (a^s_{11k1}) and (a^s_{111l}) are Cauchy sequence of real numbers. As the set of real numbers is complete so there exists real numbers a_{i111}, a_{1j11}, a_{11k1}, a_{111l} such that

$$\lim_{s\to\infty} a^s_{i111} = a_{i111}, \ \lim_{s\to\infty} a^s_{1j11} = a_{1j11}, \ \lim_{s\to\infty} a^s_{11k1} = a_{11k1}, \ \lim_{s\to\infty} a^s_{111l} = a_{111l}.$$

Then we have

$$\sup_{i,j,k,l}\Big[M\Big(\Big\|\frac{\Delta a^s_{ijkl} - \Delta a^t_{ijkl}}{\rho}, z_1, ..., z_{n-1}\Big\|\Big)\Big] \le 1$$
$$\Rightarrow \quad \Big[M\Big(\Big\|\frac{\Delta a^s_{ijkl} - \Delta a^t_{ijkl}}{\rho}, z_1, ..., z_{n-1}\Big\|\Big)\Big] \le 1$$
$$\Rightarrow \quad \Big[M\Big(\Big\|\frac{\Delta a^s_{ijkl} - \Delta a^t_{ijkl}}{\rho}, z_1, ..., z_{n-1}\Big\|\Big)\Big] \le M(\frac{rx_0}{2})$$
$$\Rightarrow \quad \Big(\Big\|\frac{\Delta a^s_{ijkl} - \Delta a^t_{ijkl}}{\rho}, z_1, ..., z_{n-1}\Big\|\Big) \le \frac{rx_0}{2}$$
$$\Rightarrow \quad \Big(\Big\|\frac{\Delta a^s_{ijkl} - \Delta a^t_{ijkl}}{g(a^s_{ijkl} - a^t_{ijkl})}, z_1, ..., z_{n-1}\Big\|\Big) \le \frac{rx_0}{2} \quad (\text{Replace } \rho \text{ by } g(a^s_{ijkl} - a^t_{ijkl}))$$
$$\Rightarrow \quad \Big(\Big\|(\Delta a^s_{ijkl} - \Delta a^t_{ijkl}), z_1, ..., z_{n-1}\Big\|\Big) \le \frac{rx_0}{2}.g(a^s_{ijkl} - a^t_{ijkl})$$
$$\Rightarrow \quad \Big(\Big\|(\Delta a^s_{ijkl} - \Delta a^t_{ijkl}), z_1, ..., z_{n-1}\Big\|\Big) < \frac{rx_0}{2}.\frac{\epsilon}{rx_0}$$
$$\Rightarrow \quad \Big(\Big\|(\Delta a^s_{ijkl} - \Delta a^t_{ijkl}), z_1, ..., z_{n-1}\Big\|\Big) < \frac{\epsilon}{2}.$$

This implies that (Δa^s_{ijkl}) is a cauchy sequence of real numbers.

Let $\lim_{s\to\infty}(\Delta a^s_{ijkl}) = y_{ijkl}$ for all $i, j, k, l \in \mathbb{N}$.

Now,

$$\Delta a^s_{1111} = a^s_{1111} - a^s_{2111} - a^s_{1211} - a^s_{1121} - a^s_{1112} + a^s_{2211} + a^s_{2121} + a^s_{2112} - a^s_{2221} - a^s_{2212} + a^s_{2222}$$

$$\begin{aligned}\lim_{s\to\infty} a^s_{2222} &= \lim_{s\to\infty}\{\Delta a^s_{ijkl} - a^s_{1111} + a^s_{2111} + a^s_{1211} + a^s_{1121} + a^s_{1112} \\ &\quad - a^s_{2211} - a^s_{2121} - a^s_{2112} + a^s_{2221} + a^s_{2212}\} \\ &= y_{1111} - a_{1111} + a_{2111} + a_{1211} + a_{1121} + a_{1112} - a_{2211} \\ &\quad - a_{2121} - a_{2112} + a_{2221} + a_{2212}.\end{aligned}$$

Thus $\lim_{s\to\infty} a^s_{2222}$ exists. Proceeding in this way we conclude that $\lim_{s\to\infty}(a^s_{ijkl})$ exists.
Continuity of M, we have

$$\lim_{t\to\infty} \sup_{i,j,k,l}\left[M\left(\left\|\frac{\Delta a^s_{ijkl} - \Delta a^t_{ijkl}}{\rho}, z_1, ..., z_{n-1}\right\|\right)\right] \le 1.$$

Let $s \ge m_0$, then taking infimum of such $\rho's$ we have $g(a^s_{ijkl} - a^t_{ijkl}) < \epsilon$. Thus $(a^s_{ijkl} - a^t_{ijkl}) \in \omega^4_\infty(M, \Delta, p, \|., ..., .\|)$.
Hence $\omega^4_\infty(M, \Delta, p, \|., ..., .\|)$ is complete.
Similarly, we can show that $\omega^4_0(M, \Delta, p, \|., ..., .\|)$ and $\omega^4(M, \Delta, p, \|., ..., .\|)$ are also complete. □

Theorem 3.4. *Let M be a orlicz function and $p = (p_{ijkl})$ be bounded biquadratic sequence of strictly positive reals numbers. Then*
(i) $\omega^4(M, \Delta, p, \|., ..., .\|) \subset \omega^4_\infty(M, \Delta, p, \|., ..., .\|)$
(ii) $\omega^4_0(M, \Delta, p, \|., ..., .\|) \subset \omega^4_\infty(M, \Delta, p, \|., ..., .\|)$.

Proof. This proof can be completed by simply manipulation. □

Theorem 3.5. *Let M be a orlicz function and $p = (p_{ijkl})$ be bounded biquadratic sequence of strictly positive real numbers. Then the following relation holds*
(i) If $0 < inf\, p_{ijkl} \le p_{ijkl} < 1$, then $\omega^4(M, \Delta, p, \|., ..., .\|) \subseteq \omega^4(M, \Delta, \|., ..., .\|)$.
(ii) If $0 < p_{ijkl} \le sup\, p_{ijkl} < \infty$, then $\omega^4(M, \Delta, \|., ..., .\|) \subseteq \omega^4(M, \Delta, p, \|., ..., .\|)$.

Proof. (i) Let $(a_{ijkl}) \in \omega^4(M, \Delta, p, \|., ..., .\|)$, then

$$\lim_{p,q,r,s\to\infty} \frac{1}{pqrs}\sum_{i,j,k,l=1}^{p,q,r,s}\left[M\left(\left\|\frac{\Delta a_{ijkl} - L}{\rho}, z_1, ..., z_n\right\|\right)\right]^{p_{ijkl}} = 0.$$

Since $0 < \inf p_{ijkl} \le p_{ijkl} \le 1$

$$\begin{aligned}&\lim_{p,q,r,s\to\infty} \frac{1}{pqrs}\sum_{i,j,k,l=1}^{p,q,r,s}\left[M\left(\left\|\frac{\Delta a_{ijkl} - L}{\rho}, z_1, ..., z_n\right\|\right)\right] \\ &\le \lim_{p,q,r,s\to\infty} \frac{1}{pqrs}\sum_{i,j,k,l=1}^{p,q,r,s}\left[M\left(\left\|\frac{\Delta a_{ijkl} - L}{\rho}, z_1, ..., z_n\right\|\right)\right]^{p_{ijkl}}\end{aligned}$$

Thus

$$\lim_{p,q,r,s\to\infty} \frac{1}{pqrs}\sum_{i,j,k,l=1}^{p,q,r,s}\left[M\left(\left\|\frac{\Delta a_{ijkl} - L}{\rho}, z_1, ..., z_n\right\|\right)\right] = 0.$$

Therefore, $(a_{ijkl}) \in \omega^4(M, \Delta, \|., ..., .\|)$.
Hence,

$$\omega^4(M, \Delta, p, \|., ..., .\|) \subseteq \omega^4(M, \Delta, \|., ..., .\|).$$

(ii) Let $(a_{ijkl}) \in \omega^4(M, \Delta, \|., ..., .\|)$, then for each $\rho > 0$ we have

$$\lim_{p,q,r,s\to\infty} \frac{1}{pqrs}\sum_{i,j,k,l=1}^{p,q,r,s}\left[M\left(\left\|\frac{\Delta a_{ijkl} - L}{\rho}, z_1, ..., z_n\right\|\right)\right] = 0 < 1.$$

Since $1 \le p_{ijkl} \le \sup p_{ijkl} < \infty$, we have

$$\lim_{p,q,r,s\to\infty} \frac{1}{pqrs}\sum_{i,j,k,l=1}^{p,q,r,s}\left[M\left(\left\|\frac{\Delta a_{ijkl} - L}{\rho}, z_1, ..., z_n\right\|\right)\right]^{p_{ijkl}}$$

$$\leq \lim_{p,q,r,s\to\infty} \frac{1}{pqrs} \sum_{i,j,k,l=1}^{p,q,r,s} \left[M\left(\left\| \frac{\Delta a_{ijkl} - L}{\rho}, z_1, ..., z_n \right\| \right) \right]$$

$$\Rightarrow \quad \lim_{p,q,r,s\to\infty} \frac{1}{pqrs} \sum_{i,j,k,l=1}^{p,q,r,s} \left[M\left(\left\| \frac{\Delta a_{ijkl} - L}{\rho}, z_1, ..., z_n \right\| \right) \right]^{p_{ijkl}} = 0.$$

Therefore $(a_{ijkl}) \in \omega^4(M, \Delta, p, \|., ..., .\|)$ for each $p > 0$.
Hence,

$$\omega^4(M, \Delta, \|., ..., .\|) \subseteq \omega^4(M, \Delta, p, \|., ..., .\|).$$

□

Theorem 3.6. *Let $0 < \inf p_{ijkl} = p_{ijkl} < \sup p_{ijkl} < \infty$. Then for a Musielak- Orlicz function M which satisfies Δ_2 condition*

(i) $\omega_0^4(\Delta, p, \|., ..., .\|) \subset \omega_0^4(M, \Delta, p, \|., ..., .\|)$,
(ii) $\omega^4(\Delta, p, \|., ..., .\|) \subset \omega^4(M, \Delta, p, \|., ..., .\|)$,
(iii) $\omega_\infty^4(\Delta, p, \|., ..., .\|) \subset \omega_\infty^4(M, \Delta, p, \|., ..., .\|)$.

Proof. This proof can be completed by simply manipulation. □

Theorem 3.7. *The sequence space $\omega_\infty^4(M, \Delta, p, \|., ..., .\|)$ is solid.*

Proof. Let $(a_{ijkl}) \in \omega_\infty^4(M, \Delta, p, \|., ..., .\|)$, then

$$\sup_{p,q,r,s} \frac{1}{pqrs} \sum_{i,j,k,l=1}^{p,q,r,s} \left[M\left(\left\| \frac{\Delta a_{ijkl} - L}{\rho}, z_1, ..., z_n \right\| \right) \right]^{p_{ijkl}} < \infty.$$

Let (a_{ijkl}) be a biquadratic sequence of scalars such that $|\alpha_{ijkl}| \leq 1$ for all $i, j, k, l \in \mathbb{N}$.
Thus, we have

$$\sup_{p,q,r,s} \frac{1}{pqrs} \sum_{i,j,k,l=1}^{p,q,r,s} \left[M\left(\left\| \frac{\Delta \alpha_{ijkl} a_{ijkl}}{\rho}, z_1, ..., z_n \right\| \right) \right]^{p_{ijkl}}$$
$$\leq \sup_{p,q,r,s} \frac{1}{pqrs} \sum_{i,j,k,l=1}^{p,q,r,s} \left[M\left(\left\| \frac{\Delta a_{ijkl}}{\rho}, z_1, ..., z_n \right\| \right) \right]^{p_{ijkl}}$$
$$< \infty.$$

This shows that $(\alpha_{ijkl} a_{ijkl}) \in \omega_\infty^4(M, \Delta, p, \|., ..., .\|)$ for all sequence of scalars (α_{ijkl}) with $|\alpha_{ijkl}| \leq 1$.
Hence the space $\omega_\infty^4(M, \Delta, p, \|., ..., .\|)$ is solid sequence space. □

Theorem 3.8. *The sequence space $\omega_\infty^4(M, \Delta, p, \|., ..., .\|)$ is monotone.*

Proof. The result is obvious. □

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AN ANALYTIC AND COMPARATIVE STUDY OF β_1 AND α_2 NEAR-RINGS

By

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Abstract

This paper aims to establish the comparative study of β_1 and α_2 near-rings symmetrically and non-symmetrically. Moreover, some theorems and examples have shown the necessity of the comparative research of near-rings.

Keywords and Phrases: β_1 near-ring, α_2 near-ring, Idempotent, Nilpotent, near-field.

1. Introduction

This paper is inspired by the work of S.Uma and G.Sugantha about the study of β_1 near-ring and α_2 near-ring respectively. Here N_r stands for right near-ring i.e. $(N_r, +, \cdot)$ and If N_r is an additive group (need not to be commutative), and is multiplicative semi-group also it satisfies right distributivity, is known as right near-ring. A Regular near-ring [2] is the ring that satisfies the following property:

$$\text{for every } x \in N_r \ \exists\ y \in N_r \text{ s.t. } x = xyx.$$

A mapping ϕ defined from N_r to N_r is called a Mate Function [5] for N_r if for every $x \in N_r$, $x = x\,\phi(x)x$.
Analogous to the comparative study of β_1 and α_2 near-rings, we are presenting some examples and also obtaining their complete characterization [3]. For the basic concepts, notations, and terms used, we refer to [1, 2].

2. Notations

Throughout the paper these are some notations that will be used:

1. N denotes the set of all Nilpotents of N_r.
2. $Z\,(N_r) = \{\, a \in N_r\ ;\ ax = xa\ \forall\ x \in N_r\}$ denotes Centre of N_r.
3. All distributive elements of N_r is represented by $N_d = \{\, a \in N_r\ ;\ a(x + y) = ax + ay\ \forall\ x, y \in N_r\}$.
4. Set of all idempotents of N_r is represented by E.
5. Zero-symmetric part of N_r is denoted by $N_0 = \{\, a \in N_r\ ;\ a\,0 = 0\}$.

3. Preliminaries

Here are some important definitions and lemmas which were reviewed and useful in the advanced study of related topics.

Lemma 3.1 (Lemma 2.5 of [7]). *For an ideal I of N_r, $N_rI \subseteq I$ and hence $N_rI\,N_r \subseteq I$ if N_r is a zero symmetric near-ring.*

Proof. If any r $\in I$ and $a, b \in N_r$ then $a(b + r) - ab \in I$. We have N_r is a zero-symmetric near-ring so put $b = 0 \implies a(0 + r) - a0 = ar \in I$. Hence $N_rI \subseteq I$. Also $IN_r \subseteq I$. Hence $N_rIN_r \subseteq IN_r \subseteq I \implies N_rIN_r \subseteq I$. □

Lemma 3.2 (Problem14 of [1]). *For all $a \in N_r$, $a^2 = 0 \implies a = 0$ iff there is no non-zero nilpotent elements in N_r.*

Definition 3.1 (Definition 1.31 of [2]). *A sub near-ring R of N_r is called invariant near-ring If $R\,N_r \subseteq R$ & $N_r\,R \subseteq R$.*

Lemma 3.3 (Lemma 2.6 of [7]). *Idempotents are central if $E \neq 0$ and N_r is a sub commutative near-ring.*

Lemma 3.4 (Theorem 8.3 of [2]). *If $N_d \neq \{0\}$ and $N_rx = N_r\ \forall\ x \in N_r$ - $\{0\}$ then a zero-symmetric near-ring N_r is a near-field.*

Lemma 3.5 (Lemma 3.2 of [5]). *If ϕ is a mate function for N_r then every $x \in N_r$, $x\phi(x)$, $\phi(x)x \in E$ and $N_rx = N_r\phi(x)x$ and $xN_r = x\phi(x)N_r$.*

Definition 3.2 (Definition 9.4 of [2]). *If $abc = acb\ \forall$ a, b, c $\in N_r$ then the near-ring N_r is called weak commutative.*

Lemma 3.6 (Proposition 2.9 of [6]). *Let N_r be any Pseudo commutative near-ring and e is its right identity then N_r is weak commutative.*

Theorem 3.1 (Theorem 1.62 of [2]). *Let N_r be a near-ring and G be a sub direct product of sub directly irreducible near-rings $N_{ri}'s$ then $N_r \cong G$.*

Theorem 3.2 (Theorem5.9 of [4]). *Every zero-symmetric β_1 near-ring with a mate function $'f'$ has (*, IFP).*

Theorem 3.3 ([2]).

1. Every sub-directly irreducible zero symmetric near-ring N_r without non-zero nilpotent is integral.
2. Let a be an idempotent of N_r and $a \neq 0$ then $a = e$ where e is the right identity.

4. Definition of β_1 and α_2 near-rings with Examples

These are detailed definitions of β_1 and α_2 near-rings with some useful Examples.

Definition 4.1. *A right near-ring N_r is said to be β_1 near-ring*

$$\text{if } xN_ry = N_rxy \ \forall\ x, y \in N_r.$$

Definition 4.2. *A right near-ring N_r is said to be α_2 near-ring*

$$\textit{if } \forall\ y \in N_r \text{ - } \{0\}\ \exists\ x \in N_r \text{ - } \{0\}\ \textit{s.t. } x = xyx.$$

Example 4.1. Let (N_r = {*1,3,5,7*}, $\times_8$, $\cdot$) be a near-ring where {$N_r, \times_8$} be a particular Kleins four group then

(a) (N_r, $\times_8$, $\cdot$) is the β_1 near-ring where second operation ($\cdot$) is defined according Pilz ([2], scheme 4, p.408)

.	1	3	5	7
1	1	1	1	1
3	1	1	3	3
5	1	3	7	5
7	1	3	5	7

(b) (N_r, $\times_8$, $\cdot$) is the α_2 near-ring where second operation ($\cdot$) is defined according Pilz ([2], scheme 18, p.408)

.	1	3	5	7
1	1	1	1	1
3	3	3	3	3
5	1	1	5	5
7	3	3	7	7

(c) (N_r, $\times_8$, $\cdot$) is not a β_1 near-ring where second operation ($\cdot$) is defined according Pilz ([2], scheme 8, p.408)

.	1	3	5	7
1	1	1	1	1
3	1	1	1	3
5	1	3	5	5
7	1	3	5	7

(d) (N_r, $\times_8$, $\cdot$) is neither regular nor α_2 near-ring where second operation ($\cdot$) is defined according to Pilz ([2], scheme 2, p.408)

.	1	3	5	7
1	1	1	1	1
3	1	1	3	3
5	1	3	5	5
7	1	3	7	7

Example 4.2. Let (N_r = {*0,1,2,3,4,5*}, $+_6$, $\cdot$) be a near-ring where (N_r,$+_6$) be the group of integers modulo *6* then

(a) ($N_r, +_6, \cdot$) is zero-symmetric β_1 near-ring with no identity where second operation ($\cdot$) is defined according Pilz ([2], scheme 36, p. 409)

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	4	2	0	4	2
2	0	2	4	0	2	4
3	0	0	0	0	0	0
4	0	4	2	0	4	2
5	0	2	4	0	2	4

(b) ($N_r, +_6, \cdot$) is a near-ring which is not regular, also it is not α_2 near-ring where second operation ($\cdot$) is defined according Pilz ([2], scheme 34, p.409)

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	5	1	0	5	1
2	0	4	2	0	4	2
3	0	3	3	0	3	3
4	0	2	4	0	2	4
5	0	1	5	0	1	5

5. Properties of β_1 and α_2 near-rings

5.1. Similar Properties of β_1 and α_2 near-rings

In this section, we have studied some similar properties of β_1 and α_2 near-rings [4, 7].
Throughout this section N_r denotes the right near-ring and $N_r^* = N_r - \{0\}$

Proposition 5.1. *Every Isomorphic Image of β_1 near-ring is also a β_1 near-ring. Moreover, every homomorphic image of β_1 near-ring is also β_1 near-ring.*

Proof. The proof is straightforward. □

Proposition 5.2. *Every Isomorphic Image of α_2 near-ring is also an α_2 near-ring. However, every homomorphic image of α_2 near-ring need not be an α_2 near-ring.*

Proof. The proof is straightforward. □

Theorem 5.1. *Every β_1 near-ring with a mate function is sub directly irreducible near-ring iff it is a near-field.*

Proof. Let N_r be a β_1 near-ring with a mate function f and A be the intersection of an arbitrary family of non-zero ideals of N_r. Since N_r is subdirectly irreducible near-ring $\implies$ A $\neq \{0\}$. Now if any $x \in$ A
s.t. $x \neq 0 \implies xe = 0\ \forall\ e \in E$ (5.1)
by Theorem 3.2,

$$ex = 0 \Longrightarrow ef(x)x = f(x)x \in E.$$

Thus $xf(x)x = 0$ [by(5.1)] $\implies x = 0$, which is not possible. So, no non-zero idempotent of N_r is a zero-divisor (5.2)
Let G be any non-zero N_r subgroup of N_r and let any non-zero $x \in G$. Thus $\forall\ a, a_1 \in N_r$, $(a - a_1 x)f(x)x = 0$ [by(5.2)]

$$\Longrightarrow a - a_1 x = 0 \Longrightarrow a = a_1 x \in N_r G \subset G.$$

Thus $N_r \subset G$. Consequently, N_r has no non trivial N_r-subgroups (5.3)
Let $a \in N_r - \{0\}$. Then by (5.3), $N_r a = N_r$. Now, we have if $E \subset Z(N_r)$ and $Z(N_r) \subset N_d$. Therefore, $N_d \neq \{0\}$. Thus N_r is a near-field. The converse of this theorem is straight forward. □

Theorem 5.2. *Every α_2 near-ring with sub commutativity is sub directly irreducible near-ring iff it is a near-field.*

Proof. Let N_r be an α_2 near-ring with sub commutativity then we have the set of all idempotent is nontrivial Let a be an idempotent of N_r and $a \neq 0$ then $a = e$ where e is the right identity. (By Theorem 3.3) Also $ae = ea = e \implies a = e$ because N_r is sub commutative and $E \neq \{0\}$. Thus N_r has a unique non-zero idempotent e which implies e represents the identity element of N_r. Again since N_r is α_2 near-ring $\forall\ a \in N_r^*$ then $\exists\ x \in N_r^*$ s.t. $x = xax \implies ax$ and xa are idempotent and hence $ax = xa = e$.
Thus N_r is a near-field.
Conversely, if N_r is a near-field then N_r will be an integral near-ring and a sub commutative α_2 near-ring is zero-symmetric reduced sub directly irreducible near-ring. Hence, N_r is sub directly irreducible near-ring. □

Proposition 5.3. *β_1 near-ring need not be regular near-ring.*

Proof. Let (N_r = {1,3,5,7}, $\times_8$, $\cdot$) be a near-ring where $\{N_r, \times_8\}$ be a particular Klein's four group then (N_r, $\times_8$, $\cdot$) is the β_1 near-ring where second operation ($\cdot$) is defined according Pilz ([2], scheme 4, p.408),

.	1	3	5	7
1	1	1	1	1
3	1	1	3	3
5	1	3	7	5
7	1	3	5	7

but it is not regular near-ring. □

Proposition 5.4. *α_2 near-ring need not be regular near-ring.*

Proof. Let (N_r = {1,3,5,7}, $\times_8$, $\cdot$) be a near-ring where (N_r, $\times_8$) be a particular Klein's four group then (N_r, $\times_8$, $\cdot$) is the α_2 near-ring where second operation ($\cdot$) is defined according Pilz ([2], scheme 4, p.408),

.	1	3	5	7
1	1	1	1	1
3	3	3	3	3
5	1	1	5	5
7	3	3	7	7

but it is not regular near-ring. □

Proposition 5.5. *Let N_r be a β_1 near-ring and G be a sub direct product of sub directly irreducible near-rings $N_{ri}'s$ then $N_r \cong G$.*

Proof. Let N_r be a β_1 near-ring then we have, N_r is isomorphic to a sub direct product of sub directly irreducible near-rings $N_{ri}'s$ [By Theorem 3.1] and under the projection mapping π_i every $N_{ri}'s$ is a homomorphic image of N_r [by Proposition 5.1].
We have every homomorphic image of β_1 near-ring is again β_1 near-ring. Hence, every β_1 near-ring is isomorphic to a sub direct product of sub directly irreducible near-rings. □

Proposition 5.6. *Every sub commutative α_2 integral near-ring is isomorphic to zero-symmetric reduced sub directly irreducible near-ring.*

Proof. Let N_r be an α_2 near-ring with sub commutativity then the set of all idempotent is nontrivial. By Theorem 3.3(2), each non-zero idempotent is a right identity of N_r. Let any two non-zero idempotent $a, e \in N_r$ then $ae = ea \implies a = e$ because N_r is sub commutative and $E \neq \{0\}$. Thus N_r has a unique idempotent e, and $e \neq 0$ which implies e represents the identity element of N_r. Again since N_r is α_2 near-ring $\forall\ a \in N_r^*$ then $\exists\ x \in N_r^*$ s.t. $x = xax. \implies ax$ and xa are idempotent and hence $ax = xa = e \implies N_r$ is a near-field. Therefore by Theorem 5.2, we will get the required result. □

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5.2. Non-Similar Properties of β_1 and α_2 near-rings

In this section, we have studied some non-similar properties of β_1 and α_2 near-rings [4, 7]. Throughout this section N_r denotes the right near-ring and $N_r^* = N_r - \{0\}$

Proposition 5.7. *Every β_1 near-ring which has identity* 1*, is zero-symmetric.*

Proof. Let N_r be a β_1 near-ring which has identity one. Then for all x, y in N_r, $xN_ry = N_rxy$.
Put $y = 1$ then $xN_r1 = N_rx1\ \forall\ x \in N_r$. When $x = 0$, $0N_r = N_r0 = \{0\}$.
$\implies 0x = 0$ hence $\forall\ x \in N_r$ it follow that N_r is zero-symmetric. □

Proposition 5.8. *Let N_r be an α_2 near-ring and I be any ideal of a zero-symmetric α_2 near-ring N_r then I is also an α_2 sub near-ring.*

Proof. Let I be an ideal of the zero-symmetric α_2 near-ring N_r. Let $a \in I^*$. Since N_r is an α_2 near-ring then $\exists$ $b \in N_r{}^*$ s.t. $bab = b$. Now $bab \in N_r{}^*I^*N_r{}^* \subseteq I^*$ [By Lemma 3.1]. Hence I is also an α_2 sub near-ring. □

Proposition 5.9. *Every regular near-ring need not be β_1 a near-ring.*

Proof. Let (N_r = {1,3,5,7}, $\times_8$, $\cdot$) be a near-ring where $\{N_r, \times_8\}$ be a particular Klein's four group then (N_r, $\times_8$, $\cdot$) is the Regular near-ring where second operation ($\cdot$) is defined according to to Pilz ([2], scheme 4, p.408),

.	1	3	5	7
1	1	1	1	1
3	3	3	3	3
5	1	1	5	5
7	3	3	7	7

but it is not β_1 near-ring. □

Proposition 5.10. *Every Regular near-ring is an α_2 near-ring.*

Proof. Let N_r be a regular near-ring. Hence for any $x \in N_r$ there exist $x, y \in N_r$ such that $xyx = x$.
Let $a = yxy$ then $axa = (yxy)x(yxy) = y(xyx)yxy = y(x)yxy = y(xyx)y = yxy = a \implies axa = a$ Hence N_r is an α_2 near-ring, which implies that every regular near-ring is an α_2 near-ring. □

Theorem 5.3. *Every β_1 near-ring with a mate function is isomorphic to a subdirect product of near-fields.* *Proof.* Let N_r be a β_1 near-ring. By Proposition 5.5, N_r is isomorphic to a subdirect product of sub directly irreducible β_1 near-rings $N_{ri}'s$. Since N_r has a mate function it follows that each N_{ri} also has a mate function. Again by Theorem 5.1, we have every β_1 near-ring with a mate function is sub directly irreducible near-ring if it is a near-field. Hence N_r is isomorphic to a subdirect product of near-fields. □	**Theorem 5.4.** *Every α_2 near-ring need not be isomorphic to a subdirect product of near-fields.* *Proof.* Let N_r be an α_2 near-ring then by Theorem 5.2, we have every α_2 near-ring N_r with sub commutativity is sub directly irreducible near-ring iff it is a near-field, and every α_2 near-ring is isomorphic to zero symmetric reduced sub directly irreducible near-ring if it is near-field. Hence in this case N_r is isomorphic to a subdirect product of near-fields but if N_r is not an integral near-ring then N_r need not be isomorphic to a subdirect product of near-fields. □

6. Conclusion

This comparative study can help us define the organizational structure of both the near-rings as well as give the differential points between them. By this comparative study, we will be able to establish the connection between these rings, also subtle differences or unexpected similarities can be illuminated. As a result, A Comprehensive investigation is shown in this paper about both the rings which can lead us to look at a broad approach of comparing and contrasting.

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ERROR ESTIMATES IN PICARD'S METHOD OF SUCCESSIVE APPROXIMATIONS FOR A PARTICULAR SECOND ORDER INITIAL VALUE PROBLEM

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Abstract

In this paper, we extend the method of successive approximations to second order Initial Value Problems (*IVPs*) of the type $y'' = f(x, y),\ y(x_0) = y_0,\ y'(x_0) = y_1$, without converting it to a system of first order differential equations. We obtain an upper bound in the closed form for the difference between two successive iterates. Further, we calculate an error bound for the solution and see that we get a tighter bound for the second order *IVP* as compared to its first order counterpart.

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1. Introduction

Initial Value Problems (*IVPs*) are differential equations with some initial conditions that find a wide range of applications in physics and modeling sciences. While every book on ordinary differential equations explains the existence and uniqueness of solutions and methods to compute solutions, it is noteworthy to mention certain developments in finding solutions to *IVPs*. A new family of explicit schemes for numerical solutions of *IVPs* are constructed in Ramos et al. [12] by considering suitable rational approximations to the theoretical solution. By varying the step size, Arefin et al. [5] analyzes the solutions of *IVPs* by discussing Euler's, Modified Euler and Runge-Kutta method. Further, Islam [9] presents Euler method and fourth-order Runge-Kutta method for solving *IVPs* by investigating and computing errors between the two methods. With the theory for *IVP* in place, Kumar [10] has illustrated the application of *IVPs* on a contour integral for Drivastava-Daoust functions of two variables. Having a lot of literature on solving *IVPs*, it is important to analyze the success of these numerical methods. The popular Euler's method is studied for its accuracy by obtaining the error bounds by Akanbi [4]. With the help of adjoint sensitivity software, Cao and Petzold [6] propose a general method for a posteriori error estimation in the solution of *IVPs*. By comparing the adjoint method and the classical approach based on the first variational equation, Lang and Verwer[11] address the global error estimation and control for *IVPs*. Using the preconditioned defect estimates and optimization techniques, Fazal and Neumaier [8] compute the error bounds for approximate solutions of *IVPs*.

The method of successive approximations by Picard for first order *IVPs* of the type $y' = f(x, y),\ y(x_0) = y_0$ found in Agarwal and O'Regan [1], and Deo et al. [7] has been extended to *IVPs* of the type $y'' = f(x, y),\ y(x_0) = y_0,\ y'(x_0) = y_1$, without converting these to a system of first order differential equations. The existence of solutions to such second-order *IVPs* can be found in Schrader [14]. The solutions of such second order *IVPs* can be computed using the approach in Schrader [13]. Having calculated the first few terms and the difference between successive terms, a pattern is observed for the difference between successive terms which is proved by induction. Finally, an error estimate for solutions has been calculated.

2. Preliminary Results

Consider the first order *IVP*,

$$y' = f(x, y),\ y(x_0) = y_0. \tag{2.1}$$

Equation (2.1) is a first order *IVP* and is equivalent to the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\mathrm{d}t. \tag{2.2}$$

For this integral equation (2.2), picking $y_0(x) = y_0$, the successive iterates are computed as

$$y_{m+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_m(t))\mathrm{d}t,\ m = 0, 1, 2, \cdots . \tag{2.3}$$

For the above first order *IVP* we have Picard's local existence theorem as follows:

Theorem 2.1 ([2]). *Let us suppose that there is a continuous and bounded function $f(x,y)$ defined on the closed rectangle $\bar{S}:|x-x_0|\leq a,\ |y-y_0|\leq b$. Let M be the bound of f on $\bar{S}$. Further, let f satisfy the Lipschitz condition on $\bar{S}$ with Lipschitz constant L. If $y_0(x)$ is a continuous function on $\bar{S}$, and if $y(x)$ is the unique solution of the IVP* (2.1), *then $\{y_m(x)\}$, which is the sequence obtained from Picard's iterative scheme* (2.3) *converges to $y(x)$. This solution holds true in*

$$J_h:|x-x_0|\leq h=\min\left\{a,\frac{b}{M}\right\}.$$

Theorem 2.2 ([3]). *Let the conditions of* Theorem 2.1 *be satisfied.*

Then for all $x\in J_h$ we have the following error estimate:

$$|y(x)-y_m(x)|\leq N\exp(Lh)\min\left\{1,\frac{(Lh)^m}{m!}\right\},m=0,1,2,\cdots, \tag{2.4}$$

where $\max\limits_{x\in J_h}|y_1(x)-y_0(x)|\leq N$.

The difference between two successive iterates is bounded and given by

$$|y_m(x)-y_{m-1}(x)|\leq N\frac{(L|x-x_0|)^{m-1}}{(m-1)!},m=1,2,\cdots. \tag{2.5}$$

3. Generalizations to second order *IVPs* of the type $\mathbf{y''=f(x,y),\ y(x_0)=y_0,\ y'(x_0)=y_1}$

We shall begin this section by establishing a result converting the second order ordinary differential equation to an integral equation.

Theorem 3.1. *The solution of the second order IVP*

$$y''=f(x,y),\ y(x_0)=y_0,\ y'(x_0)=y_1, \tag{3.1}$$

is equivalent to the solution of the Volterra integral equation

$$y(x)=y_0+(x-x_0)y_1+\textstyle\int_{x_0}^{x}(x-t)f(t,y(t))\mathrm{d}t. \tag{3.2}$$

Proof. We can rewrite equation (3.1) as $y''(t)=f(t,y(t)),\ y(x_0)=y_0,\ y'(x_0)=y_1$.
Integrating this equation with respect to t between the limits x_0 and x, and using the initial conditions, together with the fundamental theorem of calculus, we get

$$y'(x)=y_1+\textstyle\int_{x_0}^{x}f(t,y(t))dt.$$

Integration of this equation together with the initial conditions, and fundamental theorem of calculus, yields

$$y(x)=y_0+(x-x_0)y_1+\textstyle\int_{x_0}^{x}\int_{x_0}^{t_2}f(t_1,y(t_1))dt_1dt_2. \tag{3.3}$$

We will convert the double integral in equation (3.3) to a single integral. To do this, consider

$$I_1(x)=\textstyle\int_{x_0}^{x}f(t,y(t))dt, \tag{3.4}$$

and,

$$I_2(x)=\textstyle\int_{x_0}^{x}(x-t)f(t,y(t))dt. \tag{3.5}$$

Differentiating equation (3.5) with respect to x by using Leibnitz rule for differentiation under the integral sign, we get

$$\frac{dI_2(x)}{dx}=\int_{x_0}^{x}\frac{\partial}{\partial x}(x-t)f(t,y(t))dt+(x-x)f(x,y(x))\frac{dx}{dx}-(x-x_0)f(x_0,y(x_0))\frac{dx_0}{dx},$$

which becomes

$$\frac{dI_2(x)}{dx}=\int_{x_0}^{x}f(t,y(t))dt=I_1(x). \tag{3.6}$$

Differentiating equation (3.6) using Leibnitz rule, and using equation (3.4), we get

$$\frac{d^2I_2(x)}{dx^2}=f(x,y(x)). \tag{3.7}$$

We note that $I_2(x_0)=0$ and $I_2'(x_0)=0$.
We can rewrite equation (3.4) as

$$I_1(x)=\textstyle\int_{x_0}^{x}f(t_1,y(t_1))dt_1. \tag{3.8}$$

Using equations (3.6) and (3.8), we have

$$I_2(x) = \int_{x_0}^{x} I_1(t_2)dt_2 = \int_{x_0}^{x}\int_{x_0}^{t_2} f(t_1, y(t_1))dt_1dt_2. \tag{3.9}$$

From equations (3.5) and (3.9), we see that

$$\int_{x_0}^{x}\int_{x_0}^{t_2} f(t_1, y(t_1))dt_1dt_2 = \int_{x_0}^{x}(x-t)f(t, y(t))dt. \tag{3.10}$$

Using equation (3.10) in equation (3.3), we get the desired equation (3.2).

The Picard's iterative scheme for (3.1) is given by

$$y_{m+1}(x) = y_0 + (x - x_0)y_1 + \int_{x_0}^{x}(x-t)f(t, y_m(t))\mathrm{d}t,\ m = 0, 1, 2, \dots\ . \tag{3.11}$$

Here we generalize Picard's local existence theorem to the second order *IVP* (3.1).

Theorem 3.2. *Let us suppose that there is a continuous and bounded function $f(x, y)$ defined on the closed rectangle $\bar{S} :| x - x_0 |\leq a,\ | y - y_0 |\leq b$. Let M be the bound of f on $\bar{S}$. Further, let f satisfy the Lipschitz condition on $\bar{S}$ with Lipschitz constant L. If $y_0(x)$ is a continuous function on $\bar{S}$, and if $y(x)$ is the unique solution of the IVP* (3.1), *then $\{y_m(x)\}$, which is the sequence obtained from Picard's iterative scheme* (3.11) *converges to $y(x)$. This solution holds true in*

$$J_h :| x - x_0 |\leq h = \min\left\{a, \frac{b}{M_1}\right\},$$

where $M_1 =| y_1 | + \frac{Ma}{2}$.

Proof. Consider the sequence $\{y_{m+1}(x)\}$, obtained from (3.11). We shall prove that each term of this sequence exists and is continuous in $J_h :| x - x_0 |\leq h = \min\left\{a, \frac{b}{M_1}\right\}$.

By the assumption on $y_0(x)$, the function $F_0(x) = f(x, y_0(x))$ is continuous in J_h, and hence $y_1(x)$ defined as $y_1(x) = y_0 + (x - x_0)y_1 + \int_{x_0}^{x}(x-t)f(t, y_0(t))\mathrm{d}t$ is continuous in J_h.

Further,

$$\begin{aligned}| y_1(x) - y_0 | &= |y_0 + (x - x_0)y_1 + \int_{x_0}^{x}(x-t)f(t, y_0(t))\mathrm{d}t - y_0| \\ &\leq| x - x_0 || y_1 | + M\int_{x_0}^{x} | x - t | \mathrm{d}t \\ &\leq| y_1 | h + \frac{Mh^2}{2} \\ &\leq h\left[| y_1 | + \frac{Ma}{2}\right] \\ &\leq b.\end{aligned}$$

Assuming $| y_{m-1}(x) - y_0 |\leq b\ \ \forall m \geq 2$, it is sufficient to prove that $| y_m(x) - y_0 |\leq b$.

The function $F_{m-1}(x) = f(x, y_{m-1}(x))$ is also continuous in J_h as $y_{m-1}(x)$ is continuous there.

Moreover,

$$\begin{aligned}| y_m(x) - y_0 | &= |y_0 + (x - x_0)y_1 + \int_{x_0}^{x}(x-t)f(t, y_{m-1}(t))\mathrm{d}t - y_0| \\ &\leq| x - x_0 || y_1 | + M\int_{x_0}^{x} | x - t | \mathrm{d}t \\ &\leq| y_1 | h + \frac{Mh^2}{2} \\ &\leq h\left[| y_1 | + \frac{Ma}{2}\right] \\ &\leq b.\end{aligned}$$

Since $y_0(x)$ and $y_1(x)$ are continuous in J_h, and J_h is closed and bounded, there exists an $N > 0$, such that $\sup_{x\in J_h} | y_1(x) - y_0(x) |\leq N$. We now turn to obtaining a bound for the difference between two successive iterates, which we have found to be

$$| y_{m+1}(x) - y_m(x) |\leq N\frac{L^m | x - x_0 |^{2m}}{(2m)!}, m = 0, 1, 2, \dots\ . \tag{3.12}$$

The estimate of bound for difference between successive iterates which is given by (3.12), is now proved via induction.

The case for $m = 0$ is argued earlier.

So we now assume the result to be true for $m = k \geq 2$, i.e., we assume the inductive step as $| y_{k+1}(x) - y_k(x) |\leq N\frac{L^k|x-x_0|^{2k}}{(2k)!},\ k = 0, 1, 2, \dots\ .$

We need to prove the result to be true for $m = k+1$. That is, we need to prove
$| y_{k+2}(x) - y_{k+1}(x) | \leq N\frac{L^{k+1}|x-x_0|^{2(k+1)}}{(2k+2)!},\ k = 0, 1, 2, \ldots$.

The main ideas in establishing a bound for the difference between successive iterates begins by substitution of the values of the solutions y_{k+2} and y_{k+1}. Further f is Lipschitz, i.e., $| f(x, y_1) - f(x, y_2) | \leq L | y_1 - y_2 |$ in $\bar{S}$. The Lipschitz condition on f then brings us to the induction hypothesis step. The induction is then completed by using the binomial theorem twice.

Now,

$$
\begin{aligned}
| y_{k+2}(x) - y_{k+1}(x) | &= \Big|\Big(y_0 + (x-x_0)y_1 + \textstyle\int_{x_0}^{x}(x-t)f(t, y_{k+1}(t))\mathrm{d}t\Big) \\
&\quad - \Big(y_0 + (x-x_0)y_1 + \textstyle\int_{x_0}^{x}(x-t)f(t, y_k(t))\mathrm{d}t\Big)\Big| \\
&\leq \textstyle\int_{x_0}^{x} | x-t || f(t, y_{k+1}(t)) - f(t, y_k(t)) | \mathrm{d}t \\
&\leq L\textstyle\int_{x_0}^{x} | x-t || y_{k+1}(t) - y_k(t) | \mathrm{d}t \\
&\leq \frac{NL^{k+1}}{(2k)!}\textstyle\int_{x_0}^{x} | x-t | | t-x_0 |^{2k}\mathrm{d}t \\
&= \frac{NL^{k+1}}{(2k)!}\textstyle\int_{x_0}^{x} | x-t | \Big|t^{2k} - \binom{2k}{1}t^{2k-1}x_0 + \binom{2k}{2}t^{2k-2}x_0^2 - \binom{2k}{3}t^{2k-3}x_0^3 \\
&\quad + \cdots + x_0^{2k}\Big|\mathrm{d}t \\
&= \frac{NL^{k+1}}{(2k)!}\Big|\Big(\frac{xx_0^{2k+1}}{2k+1} - xx_0^{2k+1} + kxx_0^{2k+1} - \frac{(2k)(2k-1)xx_0^{2k+1}}{6} \\
&\quad - \frac{x_0^{2k+2}}{2k+2} + \frac{2kx_0^{2k+2}}{2k+1} - \frac{(2k-1)x_0^{2k+2}}{2} + \frac{(2k)(2k-2)x_0^{2k+2}}{6} + \cdots - \frac{x_0^{2k+2}}{2}\Big) \\
&\quad - \Big(\frac{x^{2k+2}}{2k+1} - x_0x^{2k+1} + kx^{2k}x_0^2 - \frac{(2k)(2k-1)x^{2k-1}x_0^3}{6} - \frac{x^{2k+2}}{2k+2} \\
&\quad + \frac{2kx^{2k+1}x_0}{2k+1} - \frac{(2k-1)x^{2k}x_0^2}{2} + \frac{(2k)(2k-2)x^{2k-1}x_0^3}{6} + \cdots - \frac{x^2x_0^{2k}}{2}\Big)\Big| \\
&= \frac{NL^{k+1}}{(2k)!}\left(\frac{x^{2k+2} - (2k+2)x^{2k+1}x_0 + \frac{(2k+1)(2k+2)x^{2k}x_0^2}{2} - \cdots + x_0^{2k+2}}{(2k+1)(2k+2)}\right) \\
&\leq \frac{NL^{k+1} | x-x_0 |^{2k+2}}{(2k)!(2k+1)(2k+2)} \\
&= N\frac{L^{k+1} | x-x_0 |^{2(k+1)}}{(2k+2)!}.
\end{aligned}
$$

By principle of mathematical induction we have established (3.12), $\forall m = 0, 1, 2, \cdots, \forall x \in J_h$.

Next,

$$
\begin{aligned}
\textstyle\sum_{m=0}^{\infty}\frac{NL^m | x-x_0 |^{2m}}{(2m)!} &= N\textstyle\sum_{m=0}^{\infty}\frac{(\sqrt{L} | x-x_0 |)^{2m}}{(2m)!} \\
&\leq N\textstyle\sum_{m=0}^{\infty}\frac{(\sqrt{L}h)^{2m}}{(2m)!} \\
&= N\exp\left(\frac{Lh}{2}\right) \\
&< \infty.
\end{aligned}
$$

By Weierstrass M-test, the series $\sum_{m=0}^{\infty}(y_{m+1}(x) - y_m(x))$ converges absolutely and uniformly to a continuous function in interval J_h.
i.e., $y(x) = \lim_{m\to\infty} y_m(x)$.

This $y(x)$ is a solution of the integral equation (3.2).

The uniqueness of solution easily follows from Gronwall's inequality which can be found in Walter [15].

For if $z(x)$ is another solution in J_h, and $(x, z(x)) \in \bar{S}$, then by using the definitions of $y(x)$ and $z(x)$, we get

$$\begin{aligned}
\mid y(x) - z(x) \mid &= \Big|\Big(y_0 + (x - x_0)y_1 + \textstyle\int_{x_0}^{x}(x - t)f(t, y(t))\mathrm{d}t\Big) \\
&\quad - \Big(y_0 + (x - x_0)y_1 + \textstyle\int_{x_0}^{x}(x - t)f(t, z(t))\mathrm{d}t\Big)\Big| \\
&\leq \textstyle\int_{x_0}^{x} \mid x - t \mid\mid f(t, y(t)) - f(t, z(t)) \mid \mathrm{d}t \\
&\leq L\textstyle\int_{x_0}^{x} \mid x - t \mid\mid y(t) - z(t) \mid \mathrm{d}t \\
&< \epsilon + L\textstyle\int_{x_0}^{x} \mid x - t \mid\mid y(t) - z(t) \mid \mathrm{d}t \ \forall \epsilon > 0 \\
&\leq \epsilon \exp(|\textstyle\int_{x_0}^{x} L \mid x - t \mid \mathrm{d}t|).
\end{aligned}$$

The second inequality above is obtained from the assumption in Theorem 3.2 stating that f is Lipschitizian with Lipschitz constant L. The last inequality is obtained using Gronwall's lemma as the previous strict inequality lays the condition for making it favourable to use Gronwall's lemma.
By letting $\epsilon \to 0$, we get $y(x) = z(x)$, proving uniqueness.

Theorem 3.3. *Let the conditions of* Theorem 3.2 *be satisfied.*
Then for all $x \in J_h$ we have the following error estimate:

$$\mid y(x) - y_m(x) \mid\leq N \exp(\sqrt{L}h) \min\left\{1, \frac{(\sqrt{L}h)^{2m}}{(2m)!}\right\}, m = 0, 1, 2, \cdots, \tag{3.13}$$

where $\max\limits_{x \in J_h} \mid y_1(x) - y_0(x) \mid\leq N$.

Proof. For $n > m$, the triangle inequality gives,

$$\begin{aligned}
\mid y_n(x) - y_m(x) \mid &\leq \textstyle\sum_{k=m}^{n-1} \mid y_{k+1}(x) - y_k(x) \mid \\
&\leq \textstyle\sum_{k=m}^{n-1} NL^k \dfrac{\mid x - x_0 \mid^{2k}}{(2k)!} \\
&\leq \textstyle\sum_{k=m}^{n-1} NL^k \dfrac{h^{2k}}{(2k)!} \\
&= N\left[\frac{(\sqrt{L}h)^{2m}}{(2m)!} + \frac{(\sqrt{L}h)^{2m+2}}{(2m+2)!} + \frac{(\sqrt{L}h)^{2m+4}}{(2m+4)!} + \cdots \frac{(\sqrt{L}h)^{2n-2}}{(2n-2)!}\right] \\
&\leq N\frac{(\sqrt{L}h)^{2m}}{(2m)!}\left[1 + \frac{\sqrt{L}h}{(2m+1)!} + \frac{(\sqrt{L}h)^2}{(2m+2)!} + \frac{(\sqrt{L}h)^3}{(2m+3)!} + \cdots + \frac{(\sqrt{L}h)^{2n-2m-1}}{(2n-1)!}\right] \\
&\leq \frac{N(\sqrt{L}h)^{2m}}{(2m)!}\textstyle\sum_{k=0}^{\infty} \dfrac{(\sqrt{L}h)^k}{(k)!} \\
&= \frac{N(\sqrt{L}h)^{2m}}{(2m)!} \exp(\sqrt{L}h).
\end{aligned}$$

We point out that the second inequality in the chain above is obtained from equation (3.12), while the last inequality is due to the fact that every term in the bracket above it is positive. The other inequalities and equalities are mere simplifications.

Letting $n \to \infty$, and by noting that $\lim\limits_{n \to \infty} y_n(x) = y(x)$, we get

$$\mid y(x) - y_m(x) \mid\leq N\frac{(\sqrt{L}h)^{2m}}{(2m)!} \exp(\sqrt{L}h). \tag{3.14}$$

Further, we already have $\mid y_n(x) - y_m(x) \mid\leq N \exp(\sqrt{L}h)$.
Letting $n \to \infty$, and by noting that $\lim\limits_{n \to \infty} y_n(x) = y(x)$, we get

$$\mid y(x) - y_m(x) \mid\leq N \exp(\sqrt{L}h). \tag{3.15}$$

From equations (3.14) and (3.15), we obtain the desired error bound (3.13).

4. Some Supporting Examples

Let $\bar{S}$:$| x |\leq \frac{1}{3}, | y-1 |\leq 1$ be a closed rectangle.

Consider the *IVP*

$$y' = -y, \; y(0) = 1. \tag{4.1}$$

We know that the exact solution is $y(x) = e^{-x}$. The first three iterates by Picard's method are

$$y_1(x) = 1 - x,$$
$$y_2(x) = 1 - x + \frac{x^2}{2!},$$
$$y_3(x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}.$$

The error estimates considering the third iterate is given by

$$| y(x) - y_3(x) |\leq 0.000482 + R \quad \text{where} \quad R = \textstyle\sum_{k=8}^{\infty} \dfrac{(-1)^k}{3^k k!}. \tag{4.2}$$

For the *IVP*

$$y'' = -y, \; y(0) = 1, \; y'(0) = 1, \tag{4.3}$$

the exact solution is $y(x) = \cos x + \sin x$. The first three iterates by Picard's method are

$$y_1(x) = 1 + x - \frac{x^2}{2!},$$
$$y_2(x) = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!},$$
$$y_3(x) = 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \frac{x^6}{6!}.$$

The error estimates considering the third iterate is given by

$$| y(x) - y_3(x) |\leq 0.00000009 + R, \quad \text{where } R = \textstyle\sum_{k=8}^{\infty} \dfrac{1}{3^k k!}. \tag{4.4}$$

Comparing equations (4.2) and (4.4), we see that we get a tighter error bound and thus a better approximate for the solution of the second order *IVP*.

As a final illustration, consider another illustration given below: Let $\overline{S_1}$:$| x |\leq \frac{1}{2}, | y-1 |\leq 2$ be a closed rectangle. Consider the *IVP*

$$y' = e^x + y, \; y(0) = 1. \tag{4.5}$$

The exact solution is computed to be $y(x) = (x+1)e^x$. The first three iterates by Picard's method are

$$y_1(x) = x + e^x,$$
$$y_2(x) = \frac{x^2}{2} - 1 + 2e^x,$$
$$y_3(x) = \frac{x^3}{3!} - x - 2 + 3e^x.$$

The error estimates considering the third iterate is given by

$$| y(x) - y_3(x) |\leq 0.0060848. \tag{4.6}$$

For the *IVP*

$$y'' = e^x + y, \; y(0) = 1, \; y'(0) = 1, \tag{4.7}$$

the exact solution is found to be

$$y(x) = \frac{1}{4}(2x-1)e^x + \frac{1}{4}e^{-x} + e^x.$$

The first three iterates by Picard's method are

$$y_1(x) = \frac{x^2}{2!} + e^x,$$
$$y_2(x) = \frac{x^4}{4!} - x - 1 + 2e^x,$$
$$y_3(x) = \frac{x^6}{6!} - \frac{x^3}{3!} - \frac{x^2}{2!} - 2x - 2 + 3e^x.$$

The error estimates considering the third iterate is given by

$$| y(x) - y_3(x) |\leq 0.0000018. \tag{4.8}$$

Comparing equations (4.6) and (4.8), we see that we get a tighter error bound and thus a better approximate for the solution of the second order *IVP*.

5. Conclusion

We have developed Picard's method of successive approximations for a particular type of second order *IVP*. We have obtained a bound in a closed form for the difference between two successive solutions. Further, we have obtained a tighter error bound on the solution as compared to the existing bound for first order *IVPs*. We have illustrated our results with suitable examples.

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ON KATUGAMPOLA FRACTIONAL q-INTEGRAL AND q-DERIVATIVE

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Abstract

We first study Katugampola fractional q-integral and q-derivative in the space $L^{1}_{q,p}[a,b]$ and obtain some properties of these operators, which include the image of power function, semi group property and composition of these operators. Next, we look at the existence and uniqueness of solution to generalized fractional q-Cauchy type problems involving Katugampola fractional q-derivative. Also, we obtain their compact form solutions using Adomian decomposition method, in terms of q-Mittag-Leffler type functions.

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1. Introduction

The fractional calculus is a discipline of mathematics that explores the extension of integrals and derivatives to non integer orders. The subject got enormous attention of many researchers and mathematicians during the last and present century. Many different fractional integrals and derivatives have been introduced over the past few decades viz. Hadamard, Katugampola, Prabhakar fractional derivatives [1, 14, 20, 21]. These derivatives found plentiful applications in distinct areas of science and engineering.

In mathematics, defining an analytical expression by a quantity q that generalizes a given expression and reduces back to the original expression in the limit $q \to 1^{-}$ is known as quantum calculus. Euler first introduced the q-calculus on the tracks of Newtons work of infinite series, but when Jackson [12] first defined and studied the q-integral, the q-calculus became a link between mathematics and physics. The fractional q-calculus has been applied in the field of non-linear mathematical analysis [2, 3]. In recent times, many researchers have analysed the concepts of q-calculus and have brought new results which are available in literature [3, 4, 15]. Initiating from the q-extension of Cauchy's formula [5], Al-Salam initiated fixing the notion of fractional q-calculus. After this Al-Salam [6] and Agarwal [7] took it forward by studying certain fractional q-integrals and q-derivatives. The fractional Riemann-Liouvilli q-integral operator was introduced by Al-Salam [6], from that time a couple of q-analogues of Riemann operator were studied in [4, 22]. Recently, with the eruption of research in fractional calculus, new developments were made in the theory of fractional q-difference calculus, especially in q-analogue of the properties of fractional integrals and derivatives namely q-Katugampola fractional integral and derivative [19]. The books [10, 15] cover various basic definitions and properties of *Quantum Calculus*.

The structure of remaining paper is as follows. In Section 2, we provide definitions for use in the following sections. In Section 3, we give definitions of Katugampola fractional q-integral and q-derivative in the space $L^{1}_{q,p}[a,b]$. In Section 4, we obtain certain basic properties of the fractional q-integral and q-derivative under consideration in the spaces defined in Section 2. In Section 5, we look at the existence and uniqueness of a solution to q-Cauchy problems involving fractional Katugampola q-derivative. In Section 6, we get solution to fractional problems of q-Cauchy type involving fractional Katugampola q-derivative using Adomian decomposition method, in terms of q-Mittag-Leffler type functions.

2. Definitions and preliminaries

In order to be complete, the following definitions are mentioned.

Definition 2.1. *For $p \in \mathbb{R}$, $0 < |q| < 1$, $L^{1}_{q,p}[a,b]$ is the Banach space of all the functions defined on $[a,b]$, satisfying* [4]

$$\|f\| = \int_a^b t^{p-1}|f(t)|d_q t < \infty.$$

Definition 2.2. *$AC_{p,q}[a,b]$ denotes a class of $(p,q)_{-}$ absolutely continuous functions f defined on $[a,b]$ such that* [18]

$$f(x) = C + \int_a^x t^{p-1}\varphi(t)d_q t,\ (\varphi \in L^{1}_{q,p}[a,b]),$$

where $C = f(a)$ *and* $\varphi(x) = {}^{p}\delta_q(f(x))$, ${}^{p}\delta_q^n = (x^{1-p}\mathcal{D}_q)^n = \left(x^{1-p}\frac{d_q}{d_q x}\right)^n$.

$AC^n_{p,q}[a,b]$ *is the space of all the functions* f *for which,* f, ${}^{p}\delta_q(f), \ldots, ({}^{p}\delta_q)^{n-1}(f)$ *are q-regular at a and* $({}^{p}\delta_q)^{n-1}(f) \in AC_{p,q}[a,b]$.

In [18], $f \in AC^n_{p,q}[a,b]$ *iff there exists* $\varphi \in L^1_{q,p}[a,b]$ *and*

$$f(x) = \textstyle\sum_{k=0}^{n-1} C_k (x^p - a^p)^{(k)}_{q^p} + \frac{([p]_q)^{1-n}}{\Gamma_{q^p}(n)} \int_a^x t^{p-1}(x^p - (tq)^p)^{(n-1)}_{q^p} \varphi(t) d_q t, \tag{2.1}$$

where, $\varphi(x) = ({}^{p}\delta_q)^n f(x)$ *and* $C_k = \frac{([p]_q)^{-k}}{\Gamma_{q^p}(k+1)}(({}^{p}\delta_q)^k f)(a)$.

Definition 2.3. *For* $0 < a < b$, *the definite q-integral is given by* [15] :

$$(\mathcal{J}_{0,q}f)(x) = \int_0^b f(x) d_q x := b(1-q)\textstyle\sum_{k=0}^{\infty} f(bq^k)q^k,$$

and

$$(\mathcal{J}_{a,q}f)(x) = \int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \tag{2.2}$$

The q-integral value for a function f *defined over the* $[a,b]$ *is given by* [24]

$$(\mathcal{J}_{a,q}f)(x) = \int_a^x f(t) d_q t = (1-q)(x-a)\textstyle\sum_{k=0}^{\infty} f({}_a\Phi_{q^k}(x))q^k, \quad x \in [a,b], \tag{2.3}$$

where, ${}_a\Phi_q(x) = qx + (1-q)a$.

Jackson q-integral for $a > 0$ *and positive integer n, becomes* [22]

$$\int_{aq^n}^a f(x) d_q x = a(1-q)\textstyle\sum_{k=0}^{n-1} f(aq^k)q^k. \tag{2.4}$$

For $\alpha > 0$ *and* $\lambda > -1$, *the following Jackson integral holds true* [17]:

$$\int_a^x t^{p-1}(x^p - (qt)^p)^{(\alpha-1)}_{q^p}(t^p - a^p)^{(\lambda)}_{q^p} d_q t = \frac{1}{[p]_q}\left(\frac{\Gamma_{q^p}(\alpha)\Gamma_{q^p}(\lambda+1)}{\Gamma_{q^p}(\alpha+\lambda+1)}\right)\left[(x^p - a^p)^{(\alpha+\lambda)}_{q^p}\right]. \tag{2.5}$$

The q-derivatives of the generalized expression $(x^p - y^p)^{(\alpha)}_{q^p}$ *with respect to x and y are* [17]:

$$\begin{aligned} {}_x\mathcal{D}_q(x^p - y^p)^{(\alpha)}_{q^p} &= x^{p-1}[p\alpha]_q(x^p - y^p)^{(\alpha-1)}_{q^p}, \\ {}_y\mathcal{D}_q(x^p - y^p)^{(\alpha)}_{q^p} &= -y^{p-1}[p\alpha]_q(x^p - (yq)^p)^{(\alpha-1)}_{q^p}. \end{aligned} \tag{2.6}$$

For $0 < |q| < 1$ *and a function* f, *the q-derivative* $\mathcal{D}_q$ *with respect to x, we have* [25]

$$\mathcal{D}_q \int_a^x f(x,t) d_q t = \int_a^x \mathcal{D}_q f(x,t) d_q t + f(qx, x), \qquad x > a. \tag{2.7}$$

For $\alpha, \beta \in \mathbb{R}$ *and* $a, b \in \mathbb{C}$, *we have the following identities* [8, 23]:

$$(a \pm b)^{(\alpha+\beta)}_q = (a \pm b)^{(\alpha)}_q (a \pm bq^\alpha)^{(\beta)}_q \quad \text{and} \quad [ab]_q = [a]_{q^b}[b]_q.$$

A generalized form of result given in [22] *can be written as*:

$$(i) \quad (a - b(q^p)^k)^{(\alpha)}_{q^p} = a^\alpha\left(1 - \frac{b}{a}(q^p)^k\right)^{(\alpha)}_{q^p}, \qquad a, b, \alpha \in \mathbb{R}^+, k, n \in \mathbb{N}. \tag{2.8}$$

$$(ii) \quad ((q^p)^n - (q^p)^k)^{(\alpha)}_{q^p} = 0, \quad (k \le n), \qquad \alpha \in \mathbb{R}^+, k, n \in \mathbb{N}. \tag{2.9}$$

If $\mu, \alpha, \beta \in \mathbb{R}^+$, *then the following identity holds* [19]

$$\textstyle\sum_{n=0}^{\infty} \frac{(1 - \mu(q^p)^{1-n})^{(\alpha-1)}_{q^p}(1 - (q^p)^{1+n})^{(\beta-1)}_{q^p}}{(1-q^p)^{(\alpha-1)}_{q^p}(1-q^p)^{(\beta-1)}_{q^p}}(q^p)^{n\alpha} = \frac{(1 - \mu q^p)^{(\alpha+\beta-1)}_{q^p}}{(1-q^p)^{(\alpha+\beta-1)}_{q^p}}. \tag{2.10}$$

Banach Fixed Point Theorem. *If* (X, d) *is a nonempty complete metric space and* $T : X \to X$ *is the map with*

$$d(Tx_1, Tx_2) \le \lambda d(x_1, x_2), \quad \text{for all } x_1, x_2 \in X \text{ and } 0 \le \lambda < 1.$$

Then there exist a unique fixed point $x^* \in X$ *for the operator* T [14].

For $\Re(\alpha) > 0$, *the q-Mittag-Leffler function is defined by* [13]

$$ {}_qE_{\alpha,\beta}(\lambda, x-a) = \textstyle\sum_{k=0}^{\infty} \lambda^k \frac{(x-a)^{\alpha k}_q}{\Gamma_q(\alpha k + \beta)}. \tag{2.11}$$

3. Katugampola fractional q-integral and q-derivative

Definition 3.1. *For $\alpha > 0$, $x > 0$ and $f : [a,b] \to \mathbb{C}$, the Katugampola fractional q-integral is defined as:*

$$\begin{aligned}({}^{p}\mathcal{J}^{\alpha}_{a,q}f)(x) &= \frac{(1-q)^{\alpha-1}}{(1-q^{p})^{(\alpha-1)}_{q^{p}}}\int_{a}^{x} t^{p-1}(x^{p}-(tq)^{p})^{(\alpha-1)}_{q^{p}} f(t)d_{q}t.\\ &= \frac{([p]_{q})^{1-\alpha}}{\Gamma_{q^{p}}(\alpha)}\int_{a}^{x} t^{p-1}(x^{p}-(tq)^{p})^{(\alpha-1)}_{q^{p}} f(t)d_{q}t.\end{aligned} \tag{3.1}$$

Definition 3.2. *If $n-1 < \alpha \le n$ and $p > 0$, then the corresponding Katugampola fractional q-derivative is defined as:*

$$({}^{p}\mathcal{D}^{\alpha}_{a,q}f)(x) = (x^{1-p}\mathcal{D}_{q})^{n}({}^{p}\mathcal{J}^{n-\alpha}_{a,q})f(x) = \frac{([p]_{q})^{1-n+\alpha}}{\Gamma_{q^{p}}(n-\alpha)}(x^{1-p}\mathcal{D}_{q})^{n}\int_{a}^{x} t^{p-1}(x^{p}-(tq)^{p})^{(n-\alpha-1)}_{q^{p}} f(t)d_{q}t.$$

$$({}^{p}\mathcal{D}^{0}_{a,q}f)(x) = f(x).$$

or we can write this as:

$$({}^{p}\mathcal{D}^{\alpha}_{a,q}f)(x) = {}^{p}\delta^{n}_{q}({}^{p}\mathcal{J}^{n-\alpha}_{a,q}f)(x), \tag{3.2}$$

provided that $f \in L^{1}_{q,p}[a,b]$ and ${}^{p}\mathcal{J}^{n-\alpha}_{a,q}f \in AC^{n}_{p,q}[a,b]$.

Remark 3.1.

1. The operators ${}^{p}\mathcal{J}^{\alpha}_{a,q}f$ and ${}^{p}\mathcal{D}^{\alpha}_{a,q}f$ are q-extensions of classical generalized fractional integral and derivative operators introduced by Katugampola [16].
2. For $a = 0$, (3.1) and (3.2) gives generalized fractional q-integral and q-derivative introduced by Momenzadeh and Mahmudov [19].
3. For $p \to 1$, (3.1) and (3.2) reduce to Riemann-Liouville fractional q-integral and q-derivative respectively [4].
4. For $p \to 0^{+}$ and $q \to 1^{-}$, (3.1) and (3.2) become Hadamard fractional integral and derivative respectively [14].

Theorem 3.1. *For $f \in L^{1}_{q,p}[a,b]$ and $\mathcal{J}^{n-\alpha}_{a,q}f \in AC^{(n)}_{p,q}[a,b]$, where $\alpha > 0$, $n = \lfloor\alpha\rfloor + 1$, we have*

$${}^{p}\mathcal{D}^{-\alpha}_{a,q}f(x) = {}^{p}\mathcal{J}^{\alpha}_{a,q}f(x) = \frac{([p]_{q})^{1-\alpha}}{\Gamma_{q^{p}}(\alpha)}\int_{a}^{x} t^{p-1}(x^{p}-(tq)^{p})^{(\alpha-1)}_{q^{p}} f(t)d_{q}t, \quad for\ x \in (a,b]. \tag{3.3}$$

Proof. From (3.2) we have,

$$\begin{aligned}{}^{p}\mathcal{D}^{-\alpha}_{a,q}f(x) &= \Big(x^{1-p}\frac{d_{q}}{d_{q}x}\Big)^{n}({}^{p}\mathcal{J}^{n+\alpha}_{a,q})f(x).\\ &= \Big(x^{1-p}\frac{d_{q}}{d_{q}x}\Big)^{n} \times \left\{\frac{\Big([p]_{q}\Big)^{1-(n+\alpha)}}{\Gamma_{q^{p}}(n+\alpha)}\int_{a}^{x} t^{p-1}(x^{p}-(tq)^{p})^{((n+\alpha)-1)}_{q^{p}} f(t)d_{q}t\right\}.\\ &= \Big(x^{1-p}\frac{d_{q}}{d_{q}x}\Big)^{n-1} \times \left\{\frac{\Big([p]_{q}\Big)^{1-(n+\alpha)}}{\Gamma_{q^{p}}(n+\alpha)}\Big(x^{1-p}\frac{d_{q}}{d_{q}x}\Big)\int_{a}^{x} t^{p-1}(x^{p}-(tq)^{p})^{((n+\alpha)-1)}_{q^{p}} f(t)d_{q}t\right\}.\end{aligned}$$

By setting $\phi(x,t) = t^{p-1}(x^{p}-(tq)^{p})^{((n+\alpha)-1)}_{q^{p}} f(t)$, we get $\phi(qx,x) = 0$ for all $x \in (a,b]$. So, in view (2.7), we can apply the differentiation inside the integral and repeat the process n-times, we obtain (3.3). □

Remark 3.2. If $p = 1$, then from (3.3) we get the result with Riemann-Liouville fractional q-integral [4, p. 125].

Theorem 3.2. *If $\alpha \in \mathbb{R}^{+}$ and $\lambda \in (-1,\infty)$, then the images of power function $(x^{p}-a^{p})^{(\lambda)}_{q^{p}}$ under ${}^{p}\mathcal{J}^{\alpha}_{a,q}$ and ${}^{p}\mathcal{D}^{\alpha}_{a,q}$ are given by*

$${}^{p}\mathcal{J}^{\alpha}_{a,q}(x^{p}-a^{p})^{(\lambda)}_{q^{p}} = \frac{1}{([p]_{q})^{\alpha}}\left(\frac{\Gamma_{q^{p}}(\lambda+1)}{\Gamma_{q^{p}}(\alpha+\lambda+1)}\right)(x^{p}-a^{p})^{(\alpha+\lambda)}_{q^{p}}. \tag{3.4}$$

and

$${}^{p}\mathcal{D}^{\alpha}_{a,q}(x^{p}-a^{p})^{(\lambda)}_{q^{p}} = \frac{([p]_{q})^{\alpha}\,\Gamma_{q^{p}}(\lambda+1)}{\Gamma_{q^{p}}(\lambda-\alpha+1)}(x^{p}-a^{p})^{(\lambda-\alpha)}_{q^{p}}. \tag{3.5}$$

Proof. For $\lambda \neq 0$, we use (2.5) and Definition 3.1 to obtain (3.4). In view of repeated application of (2.6) (n-times), (3.2) and (3.4), we arrive at (3.5). □

Remark 3.3. In particular, for $\lambda = 0$, using (2.6) and q-integration by parts, we get

$$({}^{p}\mathcal{J}_{a,q}^{\alpha}1)(x) = \frac{([p]_q)^{1-\alpha}}{\Gamma_{q^p}(\alpha)}\int_a^x t^{p-1}(x^p-(tq)^p)_{q^p}^{(\alpha-1)}d_qt = \frac{([p]_q)^{1-\alpha}}{\Gamma_{q^p}(\alpha)}\int_a^x \frac{{}_t\mathcal{D}_q\Big((x^p-t^p)_{q^p}^{(\alpha)}\Big)}{-[p\alpha]_q}d_qt.$$
$$= \frac{-1}{([p]_q)^{\alpha}\Gamma_{q^p}(\alpha+1)}\int_a^x {}_t\mathcal{D}_q\Big((x^p-t^p)_{q^p}^{(\alpha)}\Big)d_qt = \frac{1}{([p]_q)^{\alpha}\Gamma_{q^p}(\alpha+1)}(x^p-a^p)_{q^p}^{(\alpha)}. \tag{3.6}$$

Now with the help of (3.6), we have

$$({}^{p}\mathcal{D}_{a,q}^{\alpha}1)(x) = \frac{([p]_q)^{\alpha}}{\Gamma_{q^p}(-\alpha+1)}(x^p-a^p)_{q^p}^{(-\alpha)}.$$

4. Certain properties of Katugampola fractional q-integral and q-derivative

Lemma 4.1. *If $\alpha,\beta \geq 0$ and $0 < a < x < b$, then we have*

$$\int_0^a (x^p-(qt)^p)_{q^p}^{(\beta-1)}({}^{p}\mathcal{J}_{a,q}^{\alpha}f)(t)d_qt = 0.$$

Proof. Starting with the Definition 3.1, for $x = aq^n$, we have

$$(\,{}^{p}\mathcal{J}_{a,q}^{\alpha}f)(aq^n) = \frac{([p]_q)^{1-\alpha}}{\Gamma_{q^p}(\alpha)}\int_a^{aq^n} t^{p-1}((aq^n)^p-(tq)^p)_{q^p}^{(\alpha-1)}f(t)d_qt.$$

Now, on using (2.4), (2.8) and then (2.9), we have

$$(\,{}^{p}\mathcal{J}_{a,q}^{\alpha}f)(aq^n) = \frac{([p]_q)^{1-\alpha}}{\Gamma_{q^p}(\alpha)}\Big\{-(a^p)^{\alpha}(1-q)\textstyle\sum_{k=0}^{n-1}((q^p)^n-(q^p)^{k+1})_{q^p}^{(\alpha-1)}f(aq^k)(q^p)^k\Big\} = 0.$$

Thus, by the Definition 2.3 of definite q-integral, we get the required result. □

Theorem 4.1. *For $\alpha,\ \beta \in \mathbb{R}^+$, if $f \in L^1_{q,p}[a,b]$, then the semi-group property for Katugampola fractional q-integral ${}^{p}\mathcal{J}_{a,q}^{\alpha}$ is given by*

$$\Big({}^{p}\mathcal{J}_{a,q}^{\beta}\,{}^{p}\mathcal{J}_{a,q}^{\alpha}f\Big)(x) = \Big({}^{p}\mathcal{J}_{a,q}^{\alpha+\beta}f\Big)(x), \quad (0 < a < x < b). \tag{4.1}$$

Proof. Using (2.2) with Lemma 4.1, we have

$$\Big({}^{p}\mathcal{J}_{a,q}^{\beta}\,{}^{p}\mathcal{J}_{a,q}^{\alpha}f\Big)(x) = \frac{([p]_q)^{1-\beta}}{\Gamma_{q^p}(\beta)}\int_0^x t^{p-1}(x^p-(tq)^p)_{q^p}^{(\beta-1)}({}^{p}\mathcal{J}_{a,q}^{\alpha}f)(t)d_qt.$$

Therefore,

$$\Big({}^{p}\mathcal{J}_{a,q}^{\beta}\,{}^{p}\mathcal{J}_{a,q}^{\alpha}f\Big)(x) = \frac{([p]_q)^{1-\beta}([p]_q)^{1-\alpha}}{\Gamma_{q^p}(\beta)\Gamma_{q^p}(\alpha)}\int_0^x t^{p-1}(x^p-(tq)^p)_{q^p}^{(\beta-1)}\Big\{\int_0^t u^{p-1}(t^p-(uq)^p)_{q^p}^{(\alpha-1)}f(u)d_qu\Big\}d_qt$$
$$-\frac{([p]_q)^{1-\beta}([p]_q)^{1-\alpha}}{\Gamma_{q^p}(\beta)\Gamma_{q^p}(\alpha)}\int_0^x t^{p-1}(x^p-(tq)^p)_{q^p}^{(\beta-1)}\Big\{\int_0^a u^{p-1}(t^p-(uq)^p)_{q^p}^{(\alpha-1)}f(u)d_qu\Big\}d_qt.$$

Now applying the Proposition 3.5 of [19], i.e; $\Big({}^{p}\mathcal{J}_{0,q}^{\beta}\,{}^{p}\mathcal{J}_{0,q}^{\alpha}f\Big)(x) = \Big({}^{p}\mathcal{J}_{0,q}^{\alpha+\beta}f\Big)(x)$, we obtain

$$\Big({}^{p}\mathcal{J}_{a,q}^{\beta}\,{}^{p}\mathcal{J}_{a,q}^{\alpha}f\Big)(x) = \Big({}^{p}\mathcal{J}_{0,q}^{\alpha+\beta}f\Big)(x)$$
$$-\frac{([p]_q)^{1-\alpha}([p]_q)^{1-\beta}}{\Gamma_{q^p}(\alpha)\Gamma_{q^p}(\beta)}\int_0^x t^{p-1}(x^p-(tq)^p)_{q^p}^{(\beta-1)}\Big\{\int_0^a u^{p-1}(t^p-(uq)^p)_{q^p}^{(\alpha-1)}f(u)d_qu\Big\}d_qt,$$

or

$$\Big({}^{p}\mathcal{J}_{a,q}^{\beta}\,{}^{p}\mathcal{J}_{a,q}^{\alpha}f\Big)(x) = ({}^{p}\mathcal{J}_{a,q}^{\alpha+\beta}f)(x) + \frac{([p]_q)^{1-(\alpha+\beta)}}{\Gamma_{q^p}(\alpha+\beta)}\int_0^a t^{p-1}(x^p-(tq)^p)_{q^p}^{(\alpha+\beta-1)}f(t)d_qt$$
$$-\frac{([p]_q)^{1-\alpha}([p]_q)^{1-\beta}}{\Gamma_{q^p}(\alpha)\Gamma_{q^p}(\beta)}\int_0^x t^{p-1}(x^p-(tq)^p)_{q^p}^{(\beta-1)}\Big\{\int_0^a u^{p-1}(t^p-(uq)^p)_{q^p}^{(\alpha-1)}f(u)d_qu\Big\}d_qt,$$

or

$$\Big({}^{p}\mathcal{J}_{a,q}^{\beta}\,{}^{p}\mathcal{J}_{a,q}^{\alpha}f\Big)(x) = \Big({}^{p}\mathcal{J}_{a,q}^{\alpha+\beta}f\Big)(x) + a^p(1-q)\textstyle\sum_{k=0}^{\infty}A_kf(aq^k)(q^p)^k,$$

where,

$$A_k = \frac{([p]_q)^{1-(\alpha+\beta)}}{\Gamma_{q^p}(\alpha+\beta)}(x^p-(aq^{k+1})^p)_{q^p}^{(\alpha+\beta-1)}$$

$$-\frac{([p]_q)^{1-\alpha}([p]_q)^{1-\beta}}{\Gamma_{q^p}(\alpha)\Gamma_{q^p}(\beta)}\ x^p(1-q)\sum_{n=0}^{\infty}(x^p-(xq^{n+1})^p)_{q^p}^{(\beta-1)}((xq^n)^p-(aq^{k+1})^p)_{q^p}^{(\alpha-1)}(q^p)^n.$$

Now using (2.8), we have

$$A_k = ([p]_q)^{1-(\alpha+\beta)}((1-q^p)x^p)^{\alpha+\beta-1}$$
$$\times\left\{\frac{(1-(\frac{a}{x})^p(q^p)^{k+1})_{q^p}^{(\alpha+\beta-1)}}{(1-q^p)_{q^p}^{(\alpha+\beta-1)}}-\sum_{n=0}^{\infty}\frac{(1-(q^p)^{n+1})_{q^p}^{(\beta-1)}}{(1-q^p)_{q^p}^{(\beta-1)}}\frac{(1-(\frac{a}{x})^p(q^p)^{k+1-n})_{q^p}^{(\alpha-1)}}{(1-q^p)_{q^p}^{(\alpha-1)}}(q^p)^{n\alpha}\right\}.$$

Next, on using (2.10) with $\mu=(aq^k/x)^p$, we get $A_k=0$ for all $k\in\mathbb{N}$, and the Theorem4.1 is proved. □

Remark 4.1.

1. For $p\to 1$, we have the semi-group property corresponding to Theorem 4.1 with Reimann-Liouville q-fractional integral operator as studied by Rajkovic *et.al.* [22].
2. If $p\to 0^+$ and $q\to 1^-$, then we have the semi-group property with Hadamard fractional integral [14, p. 114].
3. Also if $q\to 1^-$, then we have semi-group property with Riemann-Liouville fractional integral [14, p. 73].

Theorem 4.2. *For $n-1<\alpha\le n,\ n\in\mathbb{N}$ and $0<|q|<1$, if $f\in L^1_{q,p}[a,b]$ and ${}^p\mathcal{J}^{n-\alpha}_{a,q}f\in AC^n_{p,q}[a,b]$, then*

$$({}^p\mathcal{D}^{\alpha}_{a,q}\,{}^p\mathcal{J}^{\alpha}_{a,q}f)(x)=f(x),\quad x\in(a,b]. \tag{4.2}$$

Proof. In view of semi-group property (4.1), we get

$$({}^p\mathcal{D}^{\alpha}_{a,q}\,{}^p\mathcal{J}^{\alpha}_{a,q}f)(x)=(x^{1-p}\mathcal{D}_q)^n({}^p\mathcal{J}^{n-\alpha}_{a,q})({}^p\mathcal{J}^{\alpha}_{a,q}f)(x).$$
$$=(x^{1-p}\mathcal{D}_q)^n({}^p\mathcal{J}^{n}_{a,q}f)(x).$$

Now, by repeated application of (2.7) $(n-1)$-times, we have

$$(x^{1-p}\mathcal{D}_q)^n({}^p\mathcal{J}^{n}_{a,q}f)(x)=(x^{1-p}\mathcal{D}_q)\times\frac{1}{\Gamma_{q^p}(1)}\int_a^x t^{p-1}f(t)d_qt.$$

Finally, again using (2.7), we reach at (4.2). □

Theorem 4.3. *For $n-1<\alpha\le n,\ n\in\mathbb{N}$ and $0<|q|<1$, if $f\in L^1_{q,p}[a,b]$ and ${}^p\mathcal{J}^{n-\alpha}_{a,q}f\in AC^n_{p,q}[a,b]$, then*

$$({}^p\mathcal{J}^{\alpha}_{a,q}\,{}^p\mathcal{D}^{\alpha}_{a,q}f)(x)=f(x)-\sum_{k=1}^{n}\frac{([p]_q)^{k-\alpha}({}^p\mathcal{D}^{(\alpha-k)}_{a,q}f)(a)}{\Gamma_{q^p}(\alpha-k+1)}(x^p-a^p)_{q^p}^{(\alpha-k)},\quad for\ x\in(a,b]. \tag{4.3}$$

In particular, we have

$$({}^p\mathcal{J}^{\alpha}_{a,q}\,{}^p\mathcal{D}^{\alpha}_{a,q}f)(x)=f(x)-\frac{([p]_q)^{1-\alpha}({}^p\mathcal{D}^{(\alpha-1)}_{a,q}f)(a)}{\Gamma_{q^p}(\alpha)}(x^p-a^p)_{q^p}^{(\alpha-1)},\quad for\ 0<\alpha\le 1.$$

Proof. According to Definition 3.1 and Definition 3.2 of ${}^p\mathcal{J}^{\alpha}_{a,q}$ and ${}^p\mathcal{D}^{\alpha}_{a,q}$, we have

$$({}^p\mathcal{J}^{\alpha}_{a,q}\,{}^p\mathcal{D}^{\alpha}_{a,q}f)(x)=\frac{([p]_q)^{1-\alpha}}{\Gamma_{q^p}(\alpha)}\int_a^x(x^p-(tq)^p)_{q^p}^{(\alpha-1)}\left\{\frac{d_q}{d_qt}\Big({}^p\delta_q^{n-1}({}^p\mathcal{J}^{n-\alpha}_{a,q}f)\Big)(t)\right\}d_qt.$$

Now, q-integrating by parts repeatedly and using the fact $(x^p-x^p)_{q^p}^{(\alpha-n)}=0$, leads to

$$({}^p\mathcal{J}^{\alpha}_{a,q}\,{}^p\mathcal{D}^{\alpha}_{a,q}f)(x)=-\sum_{k=1}^{n}\frac{([p]_q)^{k-\alpha}\Big({}^p\delta_q^{n-k}({}^p\mathcal{J}^{n-\alpha}_{a,q}f)\Big)(a)}{\Gamma_{q^p}(\alpha-k+1)}(x^p-a^p)_{q^p}^{(\alpha-k)}$$
$$+\frac{([p]_q)^{1-(\alpha-n)}}{\Gamma_{q^p}(\alpha-n)}\int_a^x t^{p-1}(x^p-(tq)^p)_{q^p}^{((\alpha-n)-1)}\left\{\Big({}^p\delta_q^{n-n}({}^p\mathcal{J}^{n-\alpha}_{a,q}f)\Big)(t)\right\}d_qt.$$

Thus, on using the semi-group property (4.1), we get (4.3). □

Theorem 4.4. *For $\beta\ge\alpha\ge 0$, if $f\in L^1_{q,p}[a,b]$, then*

$${}^p\mathcal{D}^{\alpha}_{a,q}\,{}^p\mathcal{J}^{\beta}_{a,q}f(x)={}^p\mathcal{J}^{\beta-\alpha}_{a,q}f(x),\quad x\in(a,b]. \tag{4.4}$$

Moreover, if ${}^p\mathcal{D}^{\alpha-\beta}_{a,q}f(x)$ exists in $(a,b]$ and $\alpha>\beta\ge 0$, then

$${}^p\mathcal{D}^{\alpha}_{a,q}\,{}^p\mathcal{J}^{\beta}_{a,q}f(x)={}^p\mathcal{D}^{\alpha-\beta}_{a,q}f(x). \tag{4.5}$$

Proof. First suppose that $\beta \geq \alpha \geq 0$. Now, if $\alpha = n$, a positive integer then in view of (3.1), (3.2) and repeated application of (2.7) n-times, we have

$$^{p}\mathcal{D}^{n}_{a,q}\, {}^{p}\mathcal{J}^{\beta}_{a,q} f(x) = {}^{p}\mathcal{J}^{\beta-n}_{a,q} f(x).$$

If $n-1 < \alpha < n$, then for $\beta = \alpha + (\beta - \alpha)$, we have from (4.1) and (4.2) that

$$^{p}\mathcal{D}^{\alpha}_{a,q}\, {}^{p}\mathcal{J}^{\beta}_{a,q} f(x) = {}^{p}\mathcal{D}^{\alpha}_{a,q}\, {}^{p}\mathcal{J}^{\alpha}_{a,q}\, {}^{p}\mathcal{J}^{\beta-\alpha}_{a,q} f(x) = {}^{p}\mathcal{J}^{\beta-\alpha}_{a,q} f(x).$$

Now, for $\alpha > \beta$, let $m-1 < \alpha \leq m$ and $n-1 < (\alpha - \beta) \leq n$, then $n \leq m$. So, by applying (3.2) and (4.1), we have

$$\begin{aligned} {}^{p}\mathcal{D}^{\alpha}_{a,q}\, {}^{p}\mathcal{J}^{\beta}_{a,q} f(x) &= {}^{p}\delta^{m}_{q}\, {}^{p}\mathcal{J}^{m-\alpha}_{a,q}\, {}^{p}\mathcal{J}^{\beta}_{a,q} f(x) = {}^{p}\delta^{m}_{q}\, {}^{p}\mathcal{J}^{m-n}_{a,q}\, {}^{p}\mathcal{J}^{n-\alpha+\beta}_{a,q} f(x). \\ &= {}^{p}\mathcal{D}^{n}_{a,q}\, {}^{p}\mathcal{J}^{n-\alpha+\beta}_{a,q} f(x) = {}^{p}\delta^{n}_{q}\, {}^{p}\mathcal{J}^{n-\alpha+\beta}_{a,q} f(x) = {}^{p}\mathcal{D}^{\alpha-\beta}_{a,q} f(x). \end{aligned}$$

□

Theorem 4.5. *For $\beta > 0$ and $n-1 < \beta \leq n$, $n \in \mathbb{N}$, if $f \in L^{1}_{q,p}[a,b]$ and ${}^{p}\mathcal{J}^{n-\beta}_{a,q} f \in AC^{n}_{p,q}[a,b]$, then for any $\alpha \geq 0$*

$$({}^{p}\mathcal{J}^{\alpha}_{a,q}\, {}^{p}\mathcal{D}^{\beta}_{a,q} f)(x) = {}^{p}\mathcal{D}^{-\alpha+\beta}_{a,q} f(x) - \sum_{k=1}^{n} \frac{([p]_q)^{k-\alpha}\left({}^{p}\mathcal{D}^{(\beta-k)}_{a,q} f\right)(a)}{\Gamma_{q^p}(\alpha - k + 1)} (x^p - a^p)^{(\alpha-k)}_{q^p}, \ for\ x \in (a,b]. \tag{4.6}$$

Proof. First if $\alpha \geq \beta$, then from (3.4), (4.1) and (4.3), we get

$$\begin{aligned} ({}^{p}\mathcal{J}^{\alpha}_{a,q}\, {}^{p}\mathcal{D}^{\beta}_{a,q} f)(x) &= {}^{p}\mathcal{J}^{\alpha-\beta}_{a,q} ({}^{p}\mathcal{J}^{\beta}_{a,q}\, {}^{p}\mathcal{D}^{\beta}_{a,q} f)(x) \\ &= {}^{p}\mathcal{J}^{\alpha-\beta}_{a,q} f(x) - \sum_{k=1}^{n} \frac{([p]_q)^{k-\alpha}\left({}^{p}\mathcal{D}^{(\beta-k)}_{a,q} f\right)(a)}{\Gamma_{q^p}(\alpha - k + 1)} (x^p - a^p)^{(\alpha-k)}_{q^p}, \quad \text{for all } x \in (a,b]. \end{aligned}$$

Now when $\beta > \alpha$, then according to (3.5), (4.3) and (4.4), we obtain

$$\begin{aligned} ({}^{p}\mathcal{J}^{\alpha}_{a,q}\, {}^{p}\mathcal{D}^{\beta}_{a,q} f)(x) &= {}^{p}\mathcal{D}^{\beta-\alpha}_{a,q} ({}^{p}\mathcal{J}^{\beta}_{a,q}\, {}^{p}\mathcal{D}^{\beta}_{a,q} f)(x) \\ &= {}^{p}\mathcal{D}^{\beta-\alpha}_{a,q} f(x) - \sum_{k=1}^{n} \frac{([p]_q)^{k-\alpha}\left({}^{p}\mathcal{D}^{(\beta-k)}_{a,q} f\right)(a)}{\Gamma_{q^p}(\alpha - k + 1)} (x^p - a^p)^{(\alpha-k)}_{q^p}, \quad \text{for all } x \in (a,b]. \end{aligned}$$

Finally, we reach at (4.6), in view of the Theorem 3.1. □

Theorem 4.6. *For $n-1 < \beta \leq n$, $n \in \mathbb{N}$, if $f \in L^{1}_{q,p}[a,b]$ and ${}^{p}\mathcal{J}^{n-\beta}_{a,q} f \in AC^{n}_{p,q}[a,b]$, then*

$$({}^{p}\mathcal{D}^{\alpha}_{a,q}\, {}^{p}\mathcal{D}^{\beta}_{a,q} f)(x) = {}^{p}\mathcal{D}^{\alpha+\beta}_{a,q} f(x) - \sum_{k=1}^{n} \frac{([p]_q)^{k-\alpha}\left({}^{p}\mathcal{D}^{(\beta-k)}_{a,q} f\right)(a)}{\Gamma_{q^p}(-\alpha - k + 1)} (x^p - a^p)^{(-\alpha-k)}_{q^p}, \ for\ x \in (a,b], \tag{4.7}$$

provided that ${}^{p}\mathcal{D}^{\alpha+\beta}_{a,q} f(x)$ exists for any $\alpha > 0$.

Proof. Using the relation $({}^{p}\mathcal{D}^{\alpha}_{a,q}\, {}^{p}\mathcal{D}^{\beta}_{a,q} f)(x) = {}^{p}\mathcal{D}^{\alpha+\beta}_{a,q} ({}^{p}\mathcal{J}^{\beta}_{a,q}\, {}^{p}\mathcal{D}^{\beta}_{a,q} f)(x)$ and also using (3.5), (4.3) and (4.5), we arrive at (4.7). □

Theorem 4.7. *For $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, if $f \in L^{1}_{q,p}[a,b]$, ${}^{p}\mathcal{J}^{n-\alpha}_{a,q} f \in AC^{n}_{p,q}[a,b]$, then*

$$({}^{p}\mathcal{D}^{\beta}_{a,q}\, {}^{p}\mathcal{D}^{\alpha}_{a,q} f)(x) = {}^{p}\mathcal{D}^{\beta+\alpha}_{a,q} f(x) - \sum_{j=1}^{n} \frac{([p]_q)^{j+\beta}\left({}^{p}\mathcal{D}^{(\alpha-j)}_{a,q} f\right)(a)}{\Gamma_{q^p}(-\beta - j + 1)} (x^p - a^p)^{(-\beta-j)}_{q^p}, \ for\ x \in (a,b].$$

provided that ${}^{p}\mathcal{D}^{\alpha+\beta}_{a,q} f(x)$ exists for any $\beta > 0$.

Proof. Same as Theorem 4.6. □

Remark 4.2. Taking $p \to 1$ in Theorem 4.2 to Theorem 4.7, the corresponding results with Riemann-Liouville fractional q-integral and q-derivative can be obtained [4].

Theorem 4.8. *For $\alpha > 0$, $n = \lfloor \alpha \rfloor + 1$ and $\mathcal{D}_{p,q} = (x^{1-p}\mathcal{D}_q)$. If $f \in AC^{n}_{p,q}[a,b]$, then*

$$\begin{aligned} \mathcal{D}^{\alpha}_{p,q} f(x) = \sum_{k=0}^{n-1} &\frac{([p]_q)^{-k+\alpha}}{\Gamma_{q^p}(k - \alpha + 1)} \mathcal{D}^{k}_{p,q} f(a) (x^p - a^p)^{(k-\alpha)}_{q^p} \\ &+ \frac{([p]_q)^{1-n+\alpha}}{\Gamma_{q^p}(n - \alpha)} \int_a^x t^{p-1} (x^p - (tq)^p)^{(n-\alpha-1)}_{q^p} \mathcal{D}^{n}_{p,q} f(t) d_q t, \end{aligned} \tag{4.8}$$

for all $x \in (a,b]$.

Also,

$$\mathcal{D}^{\alpha}_{p,q} f(a) = 0 \Longleftrightarrow \mathcal{D}^{k}_{p,q} f(a) = 0, \ \ (k = 0, 1 \ldots, n-1). \tag{4.9}$$

Proof. For $f \in AC^n_{p,q}[a,b]$, (2.1) gives

$$f(x) = \sum_{k=0}^{n-1} \frac{([p]_q)^{-k}}{\Gamma_{q^p}(k+1)} \mathcal{D}^k_{p,q} f(a)(x^p - a^p)^{(k)}_{q^p} + {}^p\mathcal{J}^n_{a,q}\mathcal{D}^n_{p,q} f(x). \tag{4.10}$$

Now, Apply $\mathcal{D}^\alpha_{p,q}$ to both sides of (4.10) and then using (4.4), we arrive at (4.8) and (4.9) follows from (4.8). □

Remark 4.3. For $a = 0$, ${}^p\mathcal{J}^\alpha_{a,q}$ and ${}^p\mathcal{D}^\alpha_{a,q}$ reduce to generalized fractional q-integral and q-derivative by Momenzadeh and Mahmudov [19]. All the properties in this section will also hold for ${}^p\mathcal{J}^\alpha_{0,q}$ and ${}^p\mathcal{D}^\alpha_{0,q}$, few of them are proved in [19].

5. Existence and uniqueness of solution to generalized fractional q-Cauchy type problem

This section is devoted to the study of existence and uniqueness of solutions to the generalized fractional q-Cauchy type problem involving Katugampola fractional q-Derivative ${}^p\mathcal{D}^\alpha_{a,q}$, given by

$$({}^p\mathcal{D}^\alpha_{a,q} y)(x) = f(x, y(x)), \quad 0 < a < x < b,\ \alpha > 0, \tag{5.1}$$

with initial conditions

$$({}^p\mathcal{D}^{\alpha-k}_{a,q} y)(a) = b_k \ (b_k \in \mathbb{R}; k = 1, 2, \dots n;\ n = -[-\alpha]). \tag{5.2}$$

Lemma 5.1. *For $\alpha > 0$ and $f : (a,b] \to \mathbb{C}$, if $f \in L^1_{q,p}[a,b]$ then ${}^p\mathcal{J}^\alpha_{a,q} f \in L^1_{q,p}[a,b]$ and*

$$\left\| {}^p\mathcal{J}^\alpha_{a,q} f \right\| \le M \left\| f \right\|. \tag{5.3}$$

Proof. For the function f defined in the interval $(a,b]$ and $f \in L^1_{q,p}[a,b]$, using Definition 3.1 and definition of q-integral given by (2.3) for $x \in (qa, a]$, gives

$$\begin{aligned}\left\| {}^p\mathcal{J}^\alpha_{a,q} f \right\| &\le \frac{([p]_q)^{1-\alpha}}{\Gamma_{q^p}(\alpha)}(1-q)\int_a^x t^{p-1}(t-a) \\ &\times \sum_{i=0}^\infty q^i (a + (t-a)q^i)^{p-1}\left(t^p - ((a+(t-a)q^i)q)^p\right)^{(\alpha-1)}_{q^p} \left| f(a + (t-a)q^i)\right| d_q t. \\ &= \frac{([p]_q)^{1-\alpha}}{\Gamma_{q^p}(\alpha)}(1-q)^2(x-a)^2 \sum_{j=0}^\infty\sum_{i=0}^\infty q^{i+j}(a+(x-a)q^j)^{p-1} q^j (a+(x-a)q^{i+j})^{p-1} \\ &\times \left((a+(x-a)q^j)^p - ((a+(x-a)q^{i+j})q)^p\right)^{(\alpha-1)}_{q^p} \left| f(a+(x-a)q^{i+j})\right|.\end{aligned}$$

Using series manipulation by taking i→ i-j, we have

$$\begin{aligned}\left\| {}^p\mathcal{J}^\alpha_{a,q} f \right\| &\le \frac{([p]_q)^{1-\alpha}}{\Gamma_{q^p}(\alpha)}(1-q)^2(x-a)^2 \sum_{i=0}^\infty q^i (a+(x-a)q^i)^{p-1}\left| f(a+(x-a)q^i)\right| \\ &\times \sum_{j=0}^i (a+(x-a)q^j)^{p-1} q^j \left((a+(x-a)q^j)^p - ((a+(x-a)q^i)q)^p\right)^{(\alpha-1)}_{q^p}.\end{aligned}$$

Now, let

$$\frac{([p]_q)^{1-\alpha}}{\Gamma_{q^p}(\alpha)}(1-q)(x-a)\sum_{j=0}^i (a+(x-a)q^j)^{p-1} q^j \left((a+(x-a)q^j)^p - ((a+(x-a)q^i)q)^p\right)^{(\alpha-1)}_{q^p} \le M.$$

Then,

$$\left\| {}^p\mathcal{J}^\alpha_{a,q} f \right\| \le (1-q)(x-a)M\sum_{i=0}^\infty q^i (a+(x-a)q^i)^{p-1}\left| f(a+(x-a)q^i)\right|.$$

In view of (2.3), we arrive at the desired result (5.3). □

Remark 5.1. In particular for $p = 1$, $a = 0$, we arrive at the same result for Riemann Liouville fractional q-integral proved in [4].

Theorem 5.1. *For $\alpha > 0$, $n - 1 < \alpha \le n$, if G is an open set in $\mathbb{C}$ and $f : (a,b] \times G \to \mathbb{R}$ with $f(x,y) \in L^1_{q,p}[a,b]$ for any $y \in G$. If $y \in L^1_{q,p}[a,b]$, then $y(x)$ is solution to the generalized fractional q-Cauchy type problem* (5.1)-(5.2) *if and only if $y(x)$ is a solution to q-integral equation*

$$y(x) = \sum_{k=1}^n \frac{b_k([p]_q)^{k-\alpha}}{\Gamma_{q^p}(\alpha-k+1)}(x^p - a^p)^{(\alpha-k)}_{q^p} + \frac{([p]_q)^{1-\alpha}}{\Gamma_{q^p}(\alpha)}\int_a^x t^{p-1}(x^p - (tq)^p)^{(\alpha-1)}_{q^p} f(x, y(t)) d_q t. \tag{5.4}$$

Proof. First, let $y(x)$ satisfies generalized fractional q-Cauchy type problem. Then ${}^{p}\mathcal{D}^{\alpha}_{a,q}y \in L^{1}_{q,p}[a,b]$ and therefore Theorem 4.3 gives

$$ {}^{p}\mathcal{J}^{\alpha}_{a,q}\,{}^{p}\mathcal{D}^{\alpha}_{a,q}y(x) = y(x) - \sum_{k=1}^{n}\frac{b_k([p]_q)^{k-\alpha}}{\Gamma_{q^p}(\alpha-k+1)}(x^p-a^p)^{(\alpha-k)}_{q^p}, \quad for\ x\in(a,b]. \tag{5.5}$$

Also

$$ {}^{p}\mathcal{J}^{\alpha}_{a,q}\,{}^{p}\mathcal{D}^{\alpha}_{a,q}y(x) = {}^{p}\mathcal{J}^{\alpha}_{a,q}f(x,y(x)) = \frac{([p]_q)^{1-\alpha}}{\Gamma_{q^p}(\alpha)}\int_a^x t^{p-1}(x^p-(tq)^p)^{(\alpha-1)}_{q^p}f(x,y(t))d_qt, \text{ for } x\in(a,b]. \tag{5.6}$$

Hence the necessity condition is proved on equating (5.5) and (5.6).

In sufficient part $y(x)$ satisfies (5.4) for all $x\in(a,b]$, therefore Lemma 5.1 gives, $y(x)\in L^{1}_{q,p}[a,b]$. Here,

$$ {}^{p}\mathcal{J}^{n-\alpha}_{a,q}y(x) = \sum_{k=1}^{n}\frac{b_k([p]_q)^{k-n}}{\Gamma_{q^p}(n-k+1)}(x^p-a^p)^{(n-k)}_{q^p} + \frac{([p]_q)^{1-n}}{\Gamma_{q^p}(n)}\int_a^x t^{p-1}(x^p-(tq)^p)^{(n-1)}_{q^p}f(x,y(t))d_qt, \tag{5.7}$$

for all $x\in(a,b]$. Hence from (2.1), ${}^{p}\mathcal{J}^{n-\alpha}_{a,q}y(x)\in AC^{n}_{p,q}[a,b]$. Therefore ${}^{p}\mathcal{D}^{\alpha}_{a,q}y(x)$ exits for all $x\in(a,b]$. Now

$$ {}^{p}\mathcal{D}^{\alpha}_{a,q}\frac{([p]_q)^{k-\alpha}}{\Gamma_{q^p}(\alpha-k+1)}(x^p-a^p)^{(\alpha-k)}_{q^p} = {}^{p}\delta^{n}_{q}\,{}^{p}\mathcal{J}^{n-\alpha}_{a,q}\frac{([p]_q)^{k-\alpha}}{\Gamma_{q^p}(\alpha-k+1)}(x^p-a^p)^{(\alpha-k)}_{q^p}.$$

$$ = {}^{p}\delta^{n}_{q}\frac{([p]_q)^{k-n}}{\Gamma_{q^p}(n-k+1)}(x^p-a^p)^{(n-k)}_{q^p} = 0,\ k=1,2,\ldots,n. \tag{5.8}$$

Applying ${}^{p}\mathcal{D}^{\alpha}_{a,q}$ on both the sides of (5.4), we have

$$ {}^{p}\mathcal{D}^{\alpha}_{a,q}y(x) = ({}^{p}\mathcal{D}^{\alpha}_{a,q}\,{}^{p}\mathcal{J}^{\alpha}_{a,q})f(x,y(x)) = f(x,y(x)), \quad a<x<b.$$

Now, using Theorem 4.7, gives

$$ ({}^{p}\mathcal{D}^{\alpha}_{a,q}y)(a) = \lim_{j\to\infty}({}^{p}\mathcal{D}^{\alpha}_{a,q}y)(a+xq^j).$$

$$ y(x) = b_k + \lim_{j\to\infty}{}^{p}\mathcal{J}^{\alpha}_{a,q}f(a+xq^j, y(a+xq^j)). \tag{5.9}$$

$$ y(x) = b_k + \lim_{j\to\infty}\frac{([p]_q)^{1-k}}{\Gamma_{q^p}(k)}\int_a^{a+xq^j}t^{p-1}(x^p-(tq)^p)^{(k-1)}_{q^p}f(x,y(t))d_qt. \tag{5.10}$$

For $f(x,y)\in L^{1}_{q,p}[a,b]$ the limit in (5.10) becomes zero. □

Theorem 5.2. *For $\alpha>0$, $n=\alpha p$ and G an open set in $\mathbb{C}$, unique solution to generalized fractional q-Cauchy problem* (5.1)-(5.2) *in $L^{1}_{q,p}[a,a+h]$ exist, if function $f:(a,b]\times G\to\mathbb{R}$ satisfy:*

1. *$f(x,y)\in L^{1}_{q,p}[a,b]$, for any $y\in G$.*
2. *For all $x\in(a,b]$ and $y_1,\ y_2\in G$, there exists a constant $A>0$ such that*

$$ \left|f(x,y_1(x))-f(x,y_2(x))\right| \le A\left|y_1(x)-y_2(x)\right|.$$

and $0<h\le b-a$ satisfies

$$ A\frac{([p]_q)^{1-\alpha}(1-q)h}{\Gamma_{q^p}(\alpha)}\sum_{j=0}^{i}(a+hq^j)^{p-1}q^j\Big((a+hq^j)^p-((a+hq^i)q)^p\Big)^{(\alpha-1)}_{q^p} < 1.$$

Proof. A solution of the generalized fractional q-Cauchy problem (5.1)-(5.2) in $L^{1}_{q,p}[a,a+h]$ is a fixed point of $T: L^{1}_{q,p}[a,a+h]\to L^{1}_{q,p}[a,a+h]$, as by Theorem 5.1, is equivalent to the Volterra q-integral equation (5.4). The operator T is given by

$$ Ty(x) = \sum_{k=1}^{n}\frac{b_k([p]_q)^{k-\alpha}}{\Gamma_{q^p}(\alpha-k+1)}(x^p-a^p)^{(\alpha-k)}_{q^p} + \frac{([p]_q)^{1-\alpha}}{\Gamma_{q^p}(\alpha)}\int_a^x t^{p-1}(x^p-(tq)^p)^{(\alpha-1)}_{q^p}f(x,y(t))d_qt.$$

Therefore, we will prove the theorem by proving, for $y\in L^{1}_{q,p}[a,a+h]$, $Ty\in L^{1}_{q,p}[a,a+h]$ and T as contraction mapping.

From Lemma 5.1, if $y\in L^{1}_{q,p}[a,a+h]$, then $Ty\in L^{1}_{q,p}[a,a+h]$.

Also, (5.3) gives

$$ \left\|Ty_1-Ty_2\right\| \le A\left\|{}^{p}\mathcal{J}^{\alpha}_{a,q}(y_1-y_2)\right\| \le w\left\|y_1-y_2\right\|,$$

where, $w = A\frac{([p]_q)^{1-\alpha}(1-q)h}{\Gamma_{q^p}(\alpha)}\sum_{j=0}^{i}(a+hq^j)^{p-1}q^j\Big((a+hq^j)^p-((a+hq^i)q)^p\Big)^{(\alpha-1)}_{q^p} < 1.$

Hence a unique $y^*\in L^{1}_{q,p}[a,a+h]$ such that $Ty^*=y^*$ exists, from the Banach fixed point theorem. On applying the sufficient part of Theorem 5.1, the result is obtained . □

6. Solution of generalized fractional q-Cauchy type problems involving Katugampola fractional q-derivative using Adomian decomposition method

Now in this section, we solve fractional q-Cauchy type problems by using the Adomian decomposition method (*ADM*) [9, 11] to understand the working. Let us first remember pivotal idea of Adomian decomposition method. *ADM* is a type of algorithm, gleaned from a method of decomposition, to formulate the approximate and even explicit solution for linear and non-linear operator equations with apt intial data. It merges the approach of ordinary or partial differentail equations or systems of such equations, into novel basic method that is applicable to both of them viz intial and boundary value problems.

Problem 6.1. *Consider the following fractional q-Cauchy type problem:*

$$\left({}^{p}\mathcal{D}_{a,q}^{\alpha}y\right)(x) - \lambda y(x) = f(x)\ (a < x \le b;\ n-1 < \alpha \le n;\ n \in \mathbb{N};\ \lambda \in \mathbb{R}), \tag{6.1}$$

with the initial condition

$$\left({}^{p}\mathcal{D}_{a,q}^{\alpha-j}y\right)(a) = b_j\ (b_j \in \mathbb{R};\ j = 1, 2, \ldots, n;\ n = -[-\alpha]). \tag{6.2}$$

Solution. Applying ${}^{p}\mathcal{J}_{a,q}^{\alpha}$ on both sides of (6.1) and then using Theorem 4.3 and initial condition, we get

$$y(x) = \sum_{j=1}^{n} \frac{b_j([p]_q)^{j-\alpha}}{\Gamma_{q^p}(\alpha - j + 1)}(x^p - a^p)_{q^p}^{(\alpha-j)} + \lambda\, {}^{p}\mathcal{J}_{a,q}^{\alpha}y(x) + {}^{p}\mathcal{J}_{a,q}^{\alpha}f(x).$$

Decomposition of $y(x)$ in the form of the sum of an infinite number of components can be written as

$$y(x) = \sum_{k=0}^{\infty} y_k(x). \tag{6.3}$$

We apply Adomian decomposition method to obtain components recursively as

$$y_0 = \sum_{j=1}^{n} \frac{b_j([p]_q)^{j-\alpha}}{\Gamma_{q^p}(\alpha - j + 1)}(x^p - a^p)_{q^p}^{(\alpha-j)} + {}^{p}\mathcal{J}_{a,q}^{\alpha}f(x), \tag{6.4}$$

and

$$y_{k+1}(x) = \lambda\, {}^{p}\mathcal{J}_{a,q}^{\alpha}y_k(x). \tag{6.5}$$

We obtain the components, by using (3.4) in recursive formulae (6.4) and (6.5)

$$y_k(x) = \sum_{j=1}^{n} b_j \frac{\lambda^k([p]_q)^{j-k\alpha-\alpha}}{\Gamma_{q^p}(k\alpha + \alpha - j + 1)}(x^p - a^p)_{q^p}^{(k\alpha+\alpha-j)}$$
$$+ \int_a^x t^{p-1} \frac{\lambda^k\left([p]_q\right)^{1-k\alpha-\alpha}}{\Gamma_{q^p}(k\alpha + \alpha)} (x^p - (tq)^p)_{q^p}^{(k\alpha+\alpha-1)} f(t) d_q t. \tag{6.6}$$

We have the solution of the Cauchy problem (6.1), by using (6.6) in (6.3) as

$$y(x) = \sum_{j=1}^{n} b_j \sum_{k=0}^{\infty} \frac{\lambda^k([p]_q)^{j-k\alpha-\alpha}}{\Gamma_{q^p}(k\alpha + \alpha - j + 1)}(x^p - a^p)_{q^p}^{(k\alpha+\alpha-j)}$$
$$+ \int_a^x t^{p-1} f(t) \sum_{k=0}^{\infty} \frac{\lambda^k([p]_q)^{1-k\alpha-\alpha}}{\Gamma_{q^p}(k\alpha + \alpha)}(x^p - (tq)^p)_{q^p}^{(k\alpha+\alpha-1)} d_q t. \tag{6.7}$$

In view of the definition of q-Mittag-Leffler function (2.11), with $q \to q^p$, we have the solution

$$y(x) = \sum_{j=1}^{n} b_j \frac{(x^p - a^p)_{q^p}^{(\alpha-j)}}{([p]_q)^{\alpha-j}}\, {}_{q^p}E_{\alpha,\alpha-j+1}\left[\frac{\lambda}{([p]_q)^{\alpha}}, (x^p - a^p q^{p(\alpha-j)})\right]$$
$$+ \int_a^x \frac{(x^p - (tq)^p)_{q^p}^{(\alpha-1)}}{([p]_q)^{\alpha-1}}\, {}_{q^p}E_{\alpha,\alpha}\left[\frac{\lambda}{([p]_q)^{\alpha}}, (x^p - (tq^{\alpha})^p)\right]\frac{f(t)}{t^{1-p}} d_q t. \tag{6.8}$$

In particular, the unique solution $y(x)$ of homogeneous generalized fractional q-Cauchy type problem

$$\left({}^{p}\mathcal{D}_{a,q}^{\alpha}y\right)(x) - \lambda y(x) = 0\ (a < x \le b; \alpha > 0; \lambda \in \mathbb{R}), \tag{6.9}$$

with the initial conditions (6.2), in the space $L_{q,p}^{1}[a, a+h]$ is given by

$$y(x) = \sum_{j=1}^{n} b_j \frac{(x^p - a^p)_{q^p}^{(\alpha-j)}}{([p]_q)^{\alpha-j}}\, {}_{q^p}E_{\alpha,\alpha-j+1}\left[\frac{\lambda}{([p]_q)^{\alpha}}, (x^p - a^p q^{p(\alpha-j)})\right].$$

□

Following are the two examples based on this problem.

Example 6.1. The fractional q-Cauchy type problem

$$\left({}^{p}\mathcal{D}^{\alpha}_{a,q}y\right)(x) - \lambda y(x) = f(x), \quad \left({}^{p}\mathcal{D}^{(\alpha-1)}_{a,q}y\right)(a) = b \ (b \in \mathbb{R}), \tag{6.10}$$

with $0 < \alpha < 1$ and $\lambda \in \mathbb{R}$ has the solution

$$y(x) = b\frac{(x^p - a^p)^{(\alpha-1)}_{q^p}}{([p]_q)^{\alpha-1}}\ {}_{q^p}E_{\alpha,\alpha}\left[\frac{\lambda}{([p]_q)^{\alpha}}, (x^p - a^p q^{p(\alpha-1)})\right] + \int_a^x \frac{(x^p - (tq)^p)^{(\alpha-1)}_{q^p}}{([p]_q)^{\alpha-1}}\ {}_{q^p}E_{\alpha,\alpha}\left[\frac{\lambda}{([p]_q)^{\alpha}}, (x^p - (tq^{\alpha})^p)\right]\frac{f(t)}{t^{1-p}}d_q t.$$

While the solution to the problem

$$\left({}^{p}\mathcal{D}^{\alpha}_{a,q}y\right)(x) - \lambda y(x) = 0, \ \left({}^{p}\mathcal{D}^{(\alpha-1)}_{a,q}y\right)(a) = b \ (b \in \mathbb{R}), \tag{6.11}$$

takes the form

$$y(x) = b\frac{(x^p - a^p)^{(\alpha-1)}_{q^p}}{([p]_q)^{\alpha-1}}\ {}_{q^p}E_{\alpha,\alpha}\left[\frac{\lambda}{([p]_q)^{\alpha}}, (x^p - a^p q^{p(\alpha-1)})\right].$$

In particular, the fractional q-Cauchy type problem

$$\left({}^{p}\mathcal{D}^{1/2}_{a,q}y\right)(x) - \lambda y(x) = f(x), \ \left({}^{p}\mathcal{D}^{(-1/2)}_{a,q}y\right)(a) = b \ (b \in \mathbb{R}), \tag{6.12}$$

has the solution given by

$$y(x) = b\frac{([p]_q)^{\frac{1}{2}}}{(x^p - a^p)^{(\frac{1}{2})}_{q^p}}\ {}_{q^p}E_{\frac{1}{2},\frac{1}{2}}\left[\frac{\lambda}{([p]_q)^{\frac{1}{2}}}, (x^p - a^p q^{p(\frac{1}{2})})\right] + \int_a^x \frac{([p]_q)^{\frac{1}{2}}}{(x^p - (tq)^p)^{(\frac{1}{2})}_{q^p}}\ {}_{q^p}E_{\frac{1}{2},\frac{1}{2}}\left[\frac{\lambda}{([p]_q)^{\frac{1}{2}}}, (x^p - (tq^{\frac{1}{2}})^p)\right]\frac{f(t)}{t^{1-p}}d_q t.$$

The solution to the problem

$$\left({}^{p}\mathcal{D}^{1/2}_{a,q}y\right)(x) - \lambda y(x) = 0, \ \left({}^{p}\mathcal{D}^{(-1/2)}_{a,q}y\right)(a) = b \ (b \in \mathbb{R}), \tag{6.13}$$

is given by

$$y(x) = b\frac{([p]_q)^{\frac{1}{2}}}{(x^p - a^p)^{(\frac{1}{2})}_{q^p}}\ {}_{q^p}E_{\frac{1}{2},\frac{1}{2}}\left[\frac{\lambda}{([p]_q)^{\frac{1}{2}}}, (x^p - a^p q^{p(\frac{1}{2})})\right].$$

Example 6.2. Let $b, d \in \mathbb{R}$. The fractional q -Cauchy type problem

$$\left({}^{p}\mathcal{D}^{\alpha}_{a,q}y\right)(x) - \lambda y(x) = f(x), \quad \left({}^{p}\mathcal{D}^{(\alpha-1)}_{a,q}y\right)(a) = b, \ \left({}^{p}\mathcal{D}^{(\alpha-2)}_{a,q}y\right)(a) = d, \tag{6.14}$$

with $1 < \alpha < 2$ and $\lambda \in \mathbb{R}$ has the solution

$$y(x) = b\frac{(x^p - a^p)^{(\alpha-1)}_{q^p}}{([p]_q)^{\alpha-1}}\ {}_{q^p}E_{\alpha,\alpha}\left[\frac{\lambda}{([p]_q)^{\alpha}}, (x^p - a^p q^{p(\alpha-1)})\right] + d\frac{(x^p - a^p)^{(\alpha-2)}_{q^p}}{([p]_q)^{\alpha-2}}\ {}_{q^p}E_{\alpha,\alpha-1}\left[\frac{\lambda}{([p]_q)^{\alpha}}, (x^p - a^p q^{p(\alpha-2)})\right] + \int_a^x \frac{(x^p - (tq)^p)^{(\alpha-1)}_{q^p}}{([p]_q)^{\alpha-1}}\ {}_{q^p}E_{\alpha,\alpha}\left[\frac{\lambda}{([p]_q)^{\alpha}}, (x^p - (tq^{\alpha})^p)\right]\frac{f(t)}{t^{1-p}}d_q t.$$

In particular, the solution to the problem

$$\left({}^{p}\mathcal{D}^{\alpha}_{a,q}y\right)(x) - \lambda y(x) = 0, \ \left({}^{p}\mathcal{D}^{(\alpha-1)}_{a,q}y\right)(a) = b, \ \left({}^{p}\mathcal{D}^{(\alpha-2)}_{a,q}y\right)(a) = d, \tag{6.15}$$

with $1 < \alpha < 2$ and $\lambda \in \mathbb{R}$ has the form

$$y(x) = b\frac{(x^p - a^p)^{(\alpha-1)}_{q^p}}{([p]_q)^{\alpha-1}}\ {}_{q^p}E_{\alpha,\alpha}\left[\frac{\lambda}{([p]_q)^{\alpha}}, (x^p - a^p q^{p(\alpha-1)})\right] + d\frac{(x^p - a^p)^{(\alpha-2)}_{q^p}}{([p]_q)^{\alpha-2}}\ {}_{q^p}E_{\alpha,\alpha-1}\left[\frac{\lambda}{([p]_q)^{\alpha}}, (x^p - a^p q^{p(\alpha-2)})\right].$$

7. Conclusion

In this paper, we have studied Katugampola fractional q-integral and q-derivative in the space $L^1_{q,p}[a,b]$ which are q-extensions of Katugampola fractional q-integral and q-derivative defined by [17]. Then we derived existence and uniqueness of solution to generalized fractional q-Cauchy type problems involving Katugampola fractional q-derivative and obtained their solutions using Adomian decomposition method.

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AN ECONOMIC PRODUCTION QUANTITY MODEL WITH SELLING PRICE DEPENDENT DEMAND UNDER INFLATION AND VARIABLE PRODUCTION RATE FOR DETERIORATING ITEMS

By

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Abstract

In this paper we have considered an economic production quantity model in which demand is assumed as a function of selling price with unsteady deterioration rate. We have considered variable production rate and the shortages are not allowed. The effect of inflation rate is also taken. Under this condition a profit function is formulated and suitable numerical examples or sensitivity analysis also provided by changing some parameters of the system.

2020 Mathematical Sciences Classification: 90B05, 90B10, 90B15, 90B30.

Keywords and Phrases: *EPQ* model, demand, selling price, inflation, deteriorating items, variable production rate, optimal function.

1. Introduction

In the industrial area there is a misunderstanding that production and inventory control are different functions. Though inventory control refers to the process of ordering, storing, using them in the factory. With production management one can control costs, manpower, raw material, warehousing the products and other capacity restriction and production control helps to produce items in plant. On the other hand inventory control decides the future demand, maximizing the total profit amount from minimizing the total investment with customer satisfaction. But in reality production and inventory control are dependent factors as production is upheld by inventories or are themselves the result of inventory. But in a particular case when items are purchased and resold then both of the terms have different meanings. Demand and deterioration rate are the main assumptions in the basic economic production model and in the *EOQ* model.

During the past few years the deterioration rate with different demand in the inventory management has received much attention from authors/researchers. For smooth and sufficient running of business affairs demand plays an important role in every part of an inventory control system. Demand may depend on selling price as in the case of such products like vegetables, fruits, and other food items which we use on a daily basis deteriorate and spoil during a short period so deterioration cannot be ignored for adequate inventory to fulfill the demands. Selling price and demand are dependent on each other as increment in selling price reduces demand and lower selling price has the reverse effect.

Patel[4] developed an economic production model for variable production rate and demand as an exponential function of time. Yadav and Aggarwal[16] created an inventory model with selling price dependent demand under inflation without shortage. An inventory model for deteriorating goods with time dependent quadratic demand and time varying holding cost with partial backlogging studied by Sharma and Yadav [12]. Shaikh and Patel [14] constructed an economic production model with demand is stock and price dependent and shortage are not allowed. Sharma and Sharma [13] analyzed a model by taking demand is a function of selling price and deterioration is taken as a quadratic function of time. Saha and Chakrabarti [8] studied an economic production model for variable production rate and probabilistic demand is assumed. Ghare and Schrader[2] represented an inventory model for exponentially decaying inventory with constant production. Kumar, Kundu and Goswami [3] produced an inventory model by taking finite production rate, fuzzy demand and deterioration rate. Samantha and Roy[10] worked on an inventory model for deteriorating items by assuming two types of production rate with shortage. Sana and chaudhary [11] established a production model in an inexact process of production in a volume flexible inventory model. Saha and Chakrabati [9] constructed a Supply Chain Inventory Model for Deteriorating Items with Price Dependent Demand and Shortage under Fuzzy Environment .Teng and Chang [15]developed an economic production model with stock and selling price dependent demand of deteriorating items. Patra and Mondal [6]represented a production model by taking fuzzy demand and variable production rate and time dependent selling price. Patel and Patel[5] considered demand dependent production rate and varying holding cost for deteriorating items in *EPQ* model. Cohen[1] explained an inventory model with known demand for joint pricing and ordering policy for exponentially decaying inventory. Preeti and Sharma [7] developed an inventory model for deteriorating items with power pattern demand and partial backlogging with time dependent holding cost.

In this paper we develop a production model for deteriorating items with unsteady deterioration rate with demand is an exponential function of selling price under the effect of inflation rate in the particular time period. Numerical examples are also provided to illustrate the model.

2. Notations and Assumptions

The model is considered under the following assumptions and notations

1. Customer demand rate which is an exponentially function of time depend on selling price of the item as $D(p(t)) = a - b(p(t))$ where a is fixed demand; $a, b > 0$ and $a >> b$.
2. $P(t)$ represents the selling price of the item at per unit time t and consider as $P(t) = pe^{rt}$ where p is the selling price at time $t = 0$.
3. r is the inflation rate which is taken as constant during the cycle.
4. θt = Deterioration rate per unit time.
5. There is no replacement and repair of deteriorated items throughout the total time period of the inventory.
6. $\pi(t)$ = initial level of inventory at any instant time $'t'$.
7. Q_1 = Inventory measure at timet_1.
8. Q_2 = Inventory measure at timet_2 .
9. Shortages are not allowed.
10. T = total cycle time length of an inventory.
11. Z =total profit amount.
12. K is a variable production rate consider during cycle as

$$K = \begin{Bmatrix} k_0 0 \leq t \leq t_1 \\ k_0 e^{-\mu(t-t_1)} t_1 \leq t \leq T \end{Bmatrix} \text{ where } \mu \text{ is constant } (0 < \mu < 1). \tag{2.1}$$

3. The Mathematical Model

The differential equations governing the system over the period $(0, T)$ are given below. Here the production rate k_0 is constant in the interval $(0, t_1)$ after that it shown increment during the interval (t_1, t_2).

$$\frac{d(\pi_1)}{d(t)} = k_0 - (a - bpe^{rt}) 0 \leq t \leq t_1, \tag{3.1}$$

$$\frac{d(\pi_2)}{d(t)} + \theta t.\pi_2 = k_0 e^{-\mu(t-t_1)} - (a - bpe^{rt}) t_1 \leq t \leq t_2, \tag{3.2}$$

$$\frac{d(\pi_3)}{d(t)} + \theta t.\pi_3 = -(a - bpe^{rt}) t_2 \leq t \leq T, \tag{3.3}$$

with the initial conditions

$$\pi(0) = 0, \pi(t_1) = Q_1, \pi(t_2) = Q_2, \pi(T) = 0. \tag{3.4}$$

Now solutions of the differential equation by adjusting the constant of integration and initial conditions

$$\pi_1(t) = (k_0 - a)t + bp/r(e^{rt} - 1), \tag{3.5}$$

$$\begin{aligned} \pi_2(t) = {} & k_0 t - at + bpt - k_0 t_1 + at_1 - bpt_1 + \mu k_0 t_1 t + Q_1 + \frac{t^2}{2}(-\mu k_0 + bpr - Q_1\theta) \\ & + \frac{t^3}{3}(-k_0\theta - bp\theta + a\theta) + \frac{t^4}{8}(+\mu k_0\theta - bpr\theta) + \frac{t_1^2}{2}(Q_1\theta - (6)\mu k_0 - bpr) + \frac{t_1^3}{6}(-k_0\theta + a\theta - bp\theta) \\ & + \frac{t_1^4}{8}(\frac{-\mu k_0\theta}{3} - bpr\theta) + \frac{k_0\theta t_1 t^2}{2} - \frac{a\theta t_1 t^2}{2} + \frac{bp\theta t_1 t^2}{2} - \frac{Q_1\theta^2 t_1^2 t^2}{4} + \frac{\mu k_0 \theta t_1^2 t^2}{4} + \frac{bpr\theta t_1^2 t}{4} - \frac{\mu k_0 \theta t_1 t^3}{3}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \pi_3(t) = {} & -a(t - \frac{\theta t^3}{3}) + aT(1 + \frac{\theta T^2}{6}) + bp(t + \frac{rt^2}{2} - \frac{\theta t^3}{3} - \frac{r\theta t^4}{8}) \\ & - bp(T + \frac{rT^2}{2} + \frac{\theta T^3}{6} + \frac{r\theta T^4}{8}) - \frac{a\theta T t^2}{2} + \frac{bp\theta T t^2}{2} + \frac{bpr\theta T^2 t^2}{4}. \end{aligned} \tag{3.7}$$

Using $\pi(t_1) = Q_1, \pi(t_2) = Q_2$ in equations (3.5), (3.6), we obtain

$$Q_1 = (k_0 - a)t_1 + \frac{bpe^{rt}}{r} - \frac{bp}{r}, \tag{3.8}$$

$$Q_2 = -a(t - \frac{\theta t_2^3}{3}) + aT(1 + \frac{\theta T^2}{6}) + bp(t + \frac{rt_2^2}{2} - \frac{\theta t_2^3}{3} - \frac{r\theta t_2^4}{8})$$

$$- bp(T + \frac{rT^2}{2} + \frac{\theta T^3}{6} + \frac{r\theta T^4}{8}) - \frac{a\theta T t_2^2}{2} + \frac{bp\theta T t_2^2}{2} + \frac{bpr\theta T^2 t_2^2}{4}. \quad (3.9)$$

Now again Putting $t = t_2$ in the solution of differential equation (3.6) and (3.7), we get

$$\pi_2(t_2) = k_0 t_2 - a t_2 + bp t_2 - k_0 t_1 + a t_1 - bp t_1 + \mu k_0 t_1 t_2 + Q_1 + \frac{t_2^2}{2}(-\mu k_0 + bpr - Q_1\theta)$$
$$+ \frac{t_2^3}{3}(-k_0\theta - bp\theta + a\theta) + \frac{t_2^4}{8}(+\mu k_0\theta - bpr\theta) + \frac{t_1^2}{2}(Q_1\theta - \mu k_0 - bpr) + \frac{t_1^3}{6}(-k_0\theta + a\theta - bp\theta)$$
$$+ \frac{t_1^4}{8}(\frac{-\mu k_0\theta}{3} - bpr\theta) + \frac{k_0\theta t_1 t_2^2}{2} - \frac{a\theta t_1 t_2^2}{2} + \frac{bp\theta t_1 t_2^2}{2} - \frac{Q_1\theta^2 t_1^2 t_2^2}{4} + \frac{\mu k_0\theta t_1^2 t_2^2}{4} + \frac{bpr\theta t_1^2 t_2}{4} - \frac{\mu k_0\theta t_1 t_2^3}{3}, \quad (3.10)$$

$$\pi_3(t_2) = -a(t_2 - \frac{\theta t_2^3}{3}) + aT(1 + \frac{\theta T^2}{6}) + bp(t_2 + \frac{rt_2^2}{2} - \frac{\theta t_2^3}{3} - \frac{r\theta t_2^4}{8}) - bp(T + \frac{rT^2}{2} + \frac{\theta T^3}{6} + \frac{r\theta T^4}{8}) - \frac{a\theta T t_2^2}{2} + \frac{bp\theta T t_2^2}{2} + \frac{bpr\theta T^2 t_2^2}{4}. \quad (3.11)$$

So by comparing equation (3.9) and (3.10), we have

$$t_2 = \frac{(a - bp)T}{k_0(1 + \mu t_1)}, \quad (3.12)$$

$$\pi_{2.1}(t_2) = \frac{1}{8}(+\mu k_0 - bpr)\theta t^4 + \frac{1}{3}(-k_0\theta - bp\theta + a\theta)t^3 + \frac{1}{2}(-\mu k_0 + bpr)t^2 + \frac{t^2 t_1^2}{4}(-\mu k_0\theta + bpr\theta)$$
$$+ (k_0 - a + bp + \mu k_0 t_1)t + \frac{t_1^4}{8}(\frac{-\mu k_0\theta}{3} - bpr\theta) + \frac{t_1^3}{3}(k_0\theta - a\theta + bp\theta) + \frac{t_1^2}{2}(\mu k_0 - bpr), \quad (3.13)$$

(we are not considering higher power of θ here).

The different costs of the economic production model are as

$$\text{Set up cost } = A, \quad (3.14)$$

$$\text{Holding cost } = C_h[\int_0^{t_1}\pi_1(t)dt + \int_{t_1}^{t_2}\pi_{2.1}(t)dt + \int_0^{T}\pi_3(t)dt], \quad (3.15)$$

$$\text{Deterioration rate} = C_d[\int_{t_1}^{t_2}\theta t\pi_2(t)dt + \int_{t_2}^{T}\theta t\pi_3(t)dt], \quad (3.16)$$

$$SR = C_s[\int_0^{T}(a - bpe^{rt})dt], \quad (3.17)$$

$$\text{Development cost } = \alpha k_0. \quad (3.18)$$

Now the total profit $'Z'$ during a cycle as following

$$Z = 1/T(SR - Sec - HC - DC - \alpha k_0). \quad (3.19)$$

Now substituting the value from equation (3.13), (3.14), (3.15), (3.16), (3.17) in equation (3.18), we get the total profit per unit in the term of t_1 , t_2, T. By putting the value $t_1 = vT$ and $t_2 = \frac{(a-bp)T}{k_0(1+vt_1)}$ in equation (3.18), we will get the total profit expression in terms of T.

For optimal solution we differentiate the equation with respect to T and then equate it to zero as expressed below:

$$\frac{\partial Z(T)}{\partial T} = 0, \quad (3.20)$$

which will satisfy the condition for optimal solution

$$\frac{\partial^2 Z(T)}{\partial T} \geq 0. \quad (3.21)$$

4. Numerical Example

We have considered the values given below of the variable to illustrate the proposed model: $a = 50$, $r = 0.01$, $b = 1.9$, $p = 4$, $k_0 = 100$, $\mu = 0.04$, $v = 0.10$, $\theta = 0.07$, $C_h = 7.5$, $C_d = 30$, $C_s = 50$, $A = 100$, $\alpha = 0.1$ by substituting these values in equation (3.20) we get the optimal cycle length $T = 0.97265$ and total profit $(Z) = 1906.600$ .the graphical representation between profit Z and T is also given which show the concavity of the profit function.

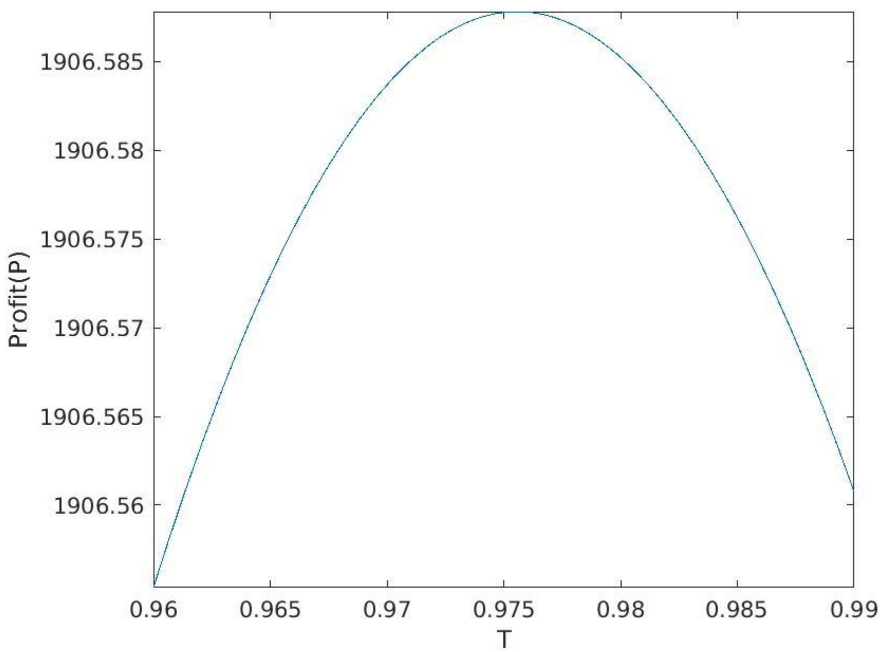

Figure 4.1

5. Sensitivity Analysis

We have studied the sensitivity analysis by changing one parameter as ±20% and ±10% and keep the rest parameter fixed on the basis of above mentioned values in the deterministic model.

Table 5.1: Sensitivity analysis associated with the model for various parameters

Parameter	% change	T	Profit (Z)
a	+20%	T=0.9609375	2404.100
	+10%	T=0.966796875	2154.300
	-10%	T=0.9921875	1661.000
	-20%	T=1.028125	1417.800
θ	+20%	T=0.953125	1904.200
	+10%	T=0.96484375	1905.400
	-10%	T=0.986328125	1907.800
	-20%	T=0.99609375	1909.800
C_h	+20%	T=0.921875	1889.400
	+10%	T=0.9453125	1897.800
	-10%	T=1.009375	1915.700
	-20%	T=1.05625	1925.100
C_s	+20%	T=0.974609375	2330.600
	+10%	T=0.97265625	2118.600
	-10%	T=0.9765625	1694.600
	-20%	T=0.97632815	1482.600
k_0	+20%	T=0.9296875	1892.800
	+10%	T=0.9453125	1899.100
	-10%	T=1.00625	1915.800
	-20%	T=1.03125	1927.600

From the above analysis, the following observation can be made as when parameter a and C_s increase or decrease the profit show increment or decrement respectively. From the above table we can see that when we change k_0 decrease to increase the total profit decrease and the other factors are less effective.

6. Conclusion

In the present paper we have developed an economic production model for deteriorating items considering demand as an exponential function of selling price with effect of inflation rate. The variable production rate is assumed with unsteady deterioration rate. The sensitivity analysis with respect to parameters has been provided which shows the increment/decrement in the value of profit. The proposed model can be extended for different demand rates and deterioration.

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ON APPLICATION OF SAIGO'S FRACTIONAL q-INTEGRAL OPERATORS TO BASIC ANALOGUE OF FOX'S H-FUNCTIONS

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Abstract

This paper deals with the derivation of the Saigo's fractional q-integral operator of the basic analogue of the Fox's H-function defined by Saxena, Modi and Kalla [6]. In the present paper, an application of the Saigo's fractional q-integral operator to various q-integral of Fox's H-function have been investigated. Some special cases have also been deduced.

2020 Mathematical Sciences Classification: 26A33, 33C60, 33D05, 05A30.
Keywords and Phrases: Saigo's fractional q-integral operator, Kober fractional q-integral, Weyl fractional q-integral, basic analogue of Fox's H-function.

1. Introduction

Saxena, Modi and Kalla [6] introduced a basic analogue of the H-function in terms of the Mellin-Barnes type basic contour integral in the following form:

$$H^{m,n}_{M,N}\left[\begin{array}{c|c}(a,A) & \\ (b,B) & x;q\end{array}\right] = H^{m,n}_{M,N}\left[\begin{array}{c|c}(a_1,A_1),...,(a_M,A_M) & \\ (b_1,B_1),...,(b_N,B_N) & x;q\end{array}\right], \tag{1.1}$$

$$= \frac{1}{2\pi i}\int_\Omega \frac{\prod_{j=1}^m G(q^{b_j-B_js})\prod_{j=1}^n G(q^{1-a_j+A_js})}{\prod_{j=m+1}^N G(q^{1-b_j+B_js})\prod_{j=n+1}^M G(q^{a_j-A_js})G(q^{1-s})}\frac{\pi x^s}{\sin\pi s}d_qs, \tag{1.2}$$

where

$$G(q^\lambda) = \prod_{n=0}^\infty \frac{1}{(1-q^{\lambda+n})} = \frac{1}{(q^\lambda;q)_\infty}, \tag{1.3}$$

and $0\le m\le N;\ \ 0\le n\le M;\ \ A_j$ and B_j are all positive integers. The contour of integration Ω is a line parallel to $Re(ws)=0$, in such a way that all of the poles of $G(q^{b_j-B_js})$; $1\le j\le m$ are to the right and those of $G(q^{1-a_j+A_js})$; $1\le j\le n$ to the left of Ω. The integral converges if $Re\{s\log(x)-\log\sin\pi s\}<0$ large values of $|s|$ on the contour, that is if $\left|\left\{\arg(x)-w_2w_1^{-1}-\log|x|\right\}\right|<\pi$ where $0<|q|<1$, $\log q=-w=-(w_1+iw_2)$, w_1,w_2 being real.

For $A_j=1(j=1,...,M)$ and $B_j=1(j=1,...,N)$ the definition (1.1) reduces to the q-analogue of the Meijer's G-function due to Saxena et.al [6].

$$G^{m,n}_{M,N}\left[\begin{array}{c|c}a_1,...,a_M & \\ b_1,...,b_N & x;q\end{array}\right] = \frac{1}{2\pi i}\int_\Omega \frac{\prod_{j=1}^m G(q^{b_j-s})\prod_{j=1}^n G(q^{1-a_j+s})}{\prod_{j=m+1}^N G(q^{1-b_j+s})\prod_{j=n+1}^M G(q^{a_j-s})G(q^{1-s})}\frac{\pi x^s}{\sin\pi s}d_qs, \tag{1.4}$$

where $0\le m\le N$; $0\le n\le M$ and $Re\{s\log-\log\sin\pi s\}<0$.

Saxena and Kumar [7] introduced the basic analogue of $J_\mu(x)$, $Y_\mu(x)$, $K_\mu(x)$, $H_\mu(x)$ in terms of $H_q(.)-$ function as follows:

$$J_\mu(x;q) = \{G(q)\}^2\, H^{1,0}_{0,3}\left[\begin{array}{c|c}- & \\ (\frac{\mu}{2},1),(\frac{-\mu}{2},1),(1,1) & \frac{x^2(1-q)^2}{4};q\end{array}\right], \tag{1.5}$$

where $J_\mu(x;q)$ denotes the q-analogue of Bessel function of first kind $J_\mu(x)$.

$$Y_\mu(x;q) = \{G(q)\}^2\, H^{2,0}_{1,4}\left[\begin{array}{c|c}(\frac{-\mu-1}{2},1) & \\ (\frac{\mu}{2},1),(\frac{-\mu}{2},1),(\frac{-\mu-1}{2},1),(1,1) & \frac{x^2(1-q)^2}{4};q\end{array}\right], \tag{1.6}$$

where $Y_\mu(x;q)$ denotes the q-analogue of Bessel function $Y_\mu(x)$.

$$K_\mu(x;q) = (1-q)\, H^{2,0}_{0,3}\left[\begin{array}{c|c}- & \\ (\frac{\mu}{2},1),(\frac{-\mu}{2},1),(1,1) & \frac{x^2(1-q)^2}{4};q\end{array}\right], \tag{1.7}$$

where $K_\mu(x;q)$ denotes the q-analogue of Bessel function of third kind $K_\mu(x)$.

$$H_\mu(x;q) = \left(\frac{1-q}{2}\right)^{1-\alpha} H^{3,1}_{1,4}\left[\begin{array}{c|c} (\frac{\alpha+1}{2},1) & \\ (\frac{\mu}{2},1),(\frac{-\mu}{2},1),(\frac{\alpha+1}{2},1),(1,1) & \end{array}\ \frac{x^2(1-q)^2}{4};q\right], \tag{1.8}$$

where $H_\mu(x;q)$ denotes the q-analogue of Struve function $H_\mu(x)$.

The object of the present paper is to evalute Saigo's fractional q-integral operator involving the q-analogue of Meijer's G-function or Fox's H-function. Some special cases have also been derived as the applications of the main results.

2. Definitions

To explore our work, we use following definitions:

Definition 2.1. *For $\lambda \in \mathbb{C}$ and $0 < |q| < 1$, the q-shifted factorial is defined as*

$$(\lambda;q)_n = \prod_{k=0}^{n-1}(1-\lambda q^k) = \frac{(\lambda;q)_\infty}{(\lambda q^n;q)_\infty}; n\in\mathbb{N} \quad and \quad (\lambda;q)_0 = 1, \tag{2.1}$$

in term of q-gamma function

$$(\lambda;q)_n = \frac{\Gamma_q(\lambda+n)}{\Gamma_q(\lambda)}(1-q)^n; \quad n>0. \tag{2.2}$$

Definition 2.2. *The q-analogue of the power function is defined and denoted as*

$$(a-b)_n = a^n\left(\frac{b}{a};q\right)_n = a^n \prod_{k=0}^{\infty}\left[\frac{1-(\frac{b}{a})q^k}{1-(\frac{b}{a})q^{k+n}}\right]; \quad (a\neq 0). \tag{2.3}$$

Definition 2.3. *The q-gamma function is defined as*

$$\Gamma_q(\lambda) = \frac{G(q^\lambda)}{G(q)}(1-q)^{1-\lambda} = (1-q)_{\lambda-1}(1-q)^{1-\lambda};\ \lambda\neq 0,-1,-2,..., \tag{2.4}$$

where $G(q^\lambda) = \dfrac{1}{(q^\lambda;q)_\infty}$.

Definition 2.4. *The q-integral of a function is defined* [1] *as*

$$\int_0^x f(t)d_qt = x(1-q)\sum_{k=0}^{\infty}q^k f(xq^k), \tag{2.5}$$

$$\int_x^\infty f(t)d_qt = x(1-q)\sum_{k=1}^{\infty}q^{-k} f(xq^{-k}). \tag{2.6}$$

Definition 2.5. *The q-bionomial series is defined* [1] *as*

$${}_1\Phi_0\left[\begin{array}{c}\lambda\\ -\end{array};\quad x;q\right] = \sum_{n=0}^{\infty}\frac{(\lambda;q)_n}{(q;q)_n}x^n = \frac{(\lambda x;q)_\infty}{(x;q)_\infty}. \tag{2.7}$$

Definition 2.6. *Heine's q-analogue of Gauss summation theorem is given by Gasper and Rahman* [1] *is defined as*

$${}_2\Phi_1\left[\begin{array}{c}q^a,q^b\\ q^c\end{array};\quad q^{c-a-b};q\right] = \sum_{n=0}^{\infty}\frac{(q^a;q)_n(q^b;q)_n}{(q^c;q)_n(q;q)_n}\left(q^{c-a-b}\right)^n = \frac{\Gamma_q(c)\Gamma_q(c-a-b)}{\Gamma_q(c-a)\Gamma_q(c-b)}, \tag{2.8}$$

Definition 2.7. *The q-analogue of Saigo's fractional integral operator, given by Garg and Chanchlani* [2] *is defined as for $Re(\alpha)>0$ and $\beta,\gamma\in\mathbb{C}$*

$$I_q^{\alpha,\beta,\gamma}f(x) = \frac{x^{-\beta-1}}{\Gamma_q(\alpha)}\int_0^x\left(\frac{tq}{x};q\right)_{\alpha-1}\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\gamma};q)_m}{(q^\alpha;q)_m(q;q)_m}q^{(\gamma-\beta)m}(-1)^m q^{-\binom{m}{2}}\left(\frac{t}{x}-1\right)_m f(t)d_qt, \tag{2.9}$$

and

$$K_q^{\alpha,\beta,\gamma}f(x) = \frac{q^{-\alpha(\alpha+1)/2-\beta}}{\Gamma_q(\alpha)}\int_x^\infty\left(\frac{x}{t};q\right)_{\alpha-1}t^{-\beta-1}\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\gamma};q)_m}{(q^\alpha;q)_m(q;q)_m}q^{(\gamma-\beta)m}(-1)^m q^{-\binom{m}{2}}\left(\frac{x}{qt}-1\right)_m f\left(tq^{1-\alpha}\right)d_qt. \tag{2.10}$$

3. Main Results

In this section we shall evalute the Saigo's fractional q-integrals involving the basic analogue of Fox's H-functions. Further the applications of Saigo's fractional q-integrals have been deduced as special cases of the main results. The main results are presented in the following theorems.

Theorem 3.1. *If the equation* (1.1) *satisfied then for* $Re(\alpha) > 0$ *and* $\beta, \gamma \in \mathbb{C}$*, the Saigo's fractional q-integral of Fox's* $H_q(.)-$ *function is given by the following formula:*

$$I_q^{\alpha,\beta,\gamma}\; H_{M,N}^{m,n}\left[\begin{array}{c|c} (a,A) & \\ (b,B) & wx^{\lambda};q \end{array}\right] = x^{-\beta}\,(1-q)^{\alpha}\; H_{M+1,N+2}^{m,n+2}\left[\begin{array}{c|c} (a,A),(0,\lambda),(\beta-\gamma,\lambda) & \\ (b,B),(\beta,\lambda),(-\alpha-\gamma,\lambda) & wx^{\lambda};q \end{array}\right]. \tag{3.1}$$

Proof. Using the eq.(2.5), the Saigo's fractional q-integral operator given by Garg and Chanchlani [2] reduces to

$$I_q^{\alpha,\beta,\gamma} f(x) = x^{-\beta}\,(1-q)^{\alpha}\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\gamma};q)_m}{(q;q)_m}q^{(\gamma-\beta+1)m}\sum_{k=0}^{\infty}q^k\frac{(q^{\alpha+m};q)_k}{(q;q)_k}f(xq^{k+m}), \tag{3.2}$$

$$\Rightarrow I_q^{\alpha,\beta,\gamma} H_{M,N}^{m,n}\left[\begin{array}{c|c} (a,A) & \\ (b,B) & wx^{\lambda};q \end{array}\right]$$

$$= x^{-\beta}\,(1-q)^{\alpha}\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\gamma};q)_m}{(q;q)_m}q^{(\gamma-\beta+1)m}\sum_{k=0}^{\infty}q^k\frac{(q^{\alpha+m};q)_k}{(q;q)_k}H_{M,N}^{m,n}\left[\begin{array}{c|c} (a,A) & \\ (b,B) & wx^{\lambda}q^{(k+m)\lambda};q \end{array}\right], \tag{3.3}$$

$$= x^{-\beta}\,(1-q)^{\alpha}\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\gamma};q)_m}{(q;q)_m}q^{(\gamma-\beta+1)m}\sum_{k=0}^{\infty}q^k\frac{(q^{\alpha+m};q)_k}{(q;q)_k}$$

$$\times\frac{1}{2\pi i}\int_{\Omega}\frac{\prod_{j=1}^{m}G(q^{b_j-B_js})\prod_{j=1}^{n}G(q^{1-a_j+A_js})}{\prod_{j=m+1}^{N}G(q^{1-b_j+B_js})\prod_{j=n+1}^{M}G(q^{a_j-A_js})G(q^{1-s})}\frac{\pi\left(wx^{\lambda}q^{\lambda k+\lambda m}\right)^s}{\sin\pi s}d_qs. \tag{3.4}$$

On interchaging the order of summation and integration, then by using eq.(2.7),

R.H.S. of (3.4) $= x^{-\beta}\,(1-q)^{\alpha}\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\gamma};q)_m}{(q;q)_m}q^{(\gamma-\beta+1)m}$

$$\times\frac{1}{2\pi i}\int_{\Omega}\frac{\prod_{j=1}^{m}G(q^{b_j-B_js})\prod_{j=1}^{n}G(q^{1-a_j+A_js})}{\prod_{j=m+1}^{N}G(q^{1-b_j+B_js})\prod_{j=n+1}^{M}G(q^{a_j-A_js})G(q^{1-s})}\frac{\pi\left(wx^{\lambda}q^{\lambda m}\right)^s}{(q^{\lambda s+1};q)_{\alpha+m}\sin\pi s}d_qs. \tag{3.5}$$

Again interchanging the order of summation and integration, we get

R.H.S. of (3.4) $= \frac{x^{-\beta}(1-q)^{\alpha}}{2\pi i}\int_{\Omega}\frac{\prod_{j=1}^{m}G(q^{b_j-B_js})\prod_{j=1}^{n}G(q^{1-a_j+A_js})}{\prod_{j=m+1}^{N}G(q^{1-b_j+B_js})\prod_{j=n+1}^{M}G(q^{a_j-A_js})G(q^{1-s})}\frac{\pi\left(wx^{\lambda}\right)^s}{\sin\pi s}$

$$\times\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\gamma};q)_m}{(q;q)_m\,(q^{\lambda s+1};q)_{\alpha+m}}q^{(\gamma+\lambda s-\beta+1)m}d_qs. \tag{3.6}$$

Also, we have result due to Garg and Chanchlani [2, eq.(2.11) and eq.(2.14), p.173],

$$\left\{(1-q)^{\alpha}\sum_{m=0}^{\infty}\frac{(q^{\alpha+\beta};q)_m(q^{-\eta};q)_m}{(q;q)_m\,(q^{1+\mu};q)_{\alpha+m}}q^{(\eta+\mu-\beta+1)m} = \frac{\Gamma_q(\mu+1)\Gamma_q(\mu-\beta+\eta+1)}{\Gamma_q(\mu-\beta+1)\Gamma_q(\mu+\alpha+\eta+1)}\right\}. \tag{3.7}$$

Now making an appeal to (2.4) and (3.6), we derive

$$\frac{x^{-\beta}}{2\pi i}\int_{\Omega}\frac{\prod_{j=1}^{m}G(q^{b_j-B_js})\prod_{j=1}^{n}G(q^{1-a_j+A_js})}{\prod_{j=m+1}^{N}G(q^{1-b_j+B_js})\prod_{j=n+1}^{M}G(q^{a_j-A_js})G(q^{1-s})}\frac{\pi\left(wx^{\lambda}\right)^s}{\sin\pi s}\frac{G(q^{\lambda s+1})G(q^{\lambda s-\beta+\gamma+1})}{G(q^{\lambda s-\beta+1})G(q^{\lambda s+\alpha+\gamma+1})}(1-q)^{\alpha}\,d_qs, \tag{3.8}$$

$$= x^{-\beta}\,(1-q)^{\alpha}\times\left\{H_{M+1,N+2}^{m,n+2}\left[\begin{array}{c|c} (a,A),(0,\lambda),(\beta-\gamma,\lambda) & \\ (b,B),(\beta,\lambda),(-\alpha-\gamma,\lambda) & wx^{\lambda};q \end{array}\right]\right\}, \tag{3.9}$$

which completes the proof of the Theorem 3.1 . □

Special Case 3.1. On taking $\beta = 0$ in the Theorem 3.1 and using the result $I_q^{\alpha,0,\gamma}f(x) = I_q^{\gamma,\alpha}f(x)$ due to Garg and Chanchlani [2, eq.(2.9), p.173]

$$\Rightarrow I_q^{\alpha,0,\gamma}\; H_{M,N}^{m,n}\left[\begin{array}{c|c} (a,A) & \\ (b,B) & wx^{\lambda};q \end{array}\right] = I_q^{\gamma,\alpha}\; H_{M,N}^{m,n}\left[\begin{array}{c|c} (a,A) & \\ (b,B) & wx^{\lambda};q \end{array}\right], \tag{3.10}$$

Putting $\beta = 0$ in eq. (3.8) and replacing γ by u , α by v

$$I_q^{u,v}\; H_{M,N}^{m,n}\left[\begin{array}{c|c} (a,A) & \\ (b,B) & wx^{\lambda};q \end{array}\right] = (1-q)^{v}\; H_{M+1,N+1}^{m,n+1}\left[\begin{array}{c|c} (a,A),(-u,\lambda) & \\ (b,B),(-v-u,\lambda) & wx^{\lambda};q \end{array}\right], \tag{3.11}$$

which is the Kober fractional q-integral of the $H_q(.)-$ function, obtained by Saxena, Yadav, Purohit and Kalla, [8, eq.(24), (2005), p.4]

Special Case 3.2. On taking $\beta = -\alpha$ in the Theorem 3.1 and using the result $I_q^{\alpha,-\alpha,\gamma} f(x) = I_q^{\alpha} f(x)$ due to Garg and Chanchlani [2, eq.(2.7), p.173], we derive

$$I_q^{\alpha,-\alpha,\gamma}\, H_{M,N}^{m,n}\left[\begin{array}{c|c} (a,A) & \\ (b,B) & \end{array}\; wx^{\lambda}; q\right] = I_q^{\alpha}\, H_{M,N}^{m,n}\left[\begin{array}{c|c} (a,A) & \\ (b,B) & \end{array}\; wx^{\lambda}; q\right], \tag{3.12}$$

Putting $\beta = -\alpha$ in eq. (3.8) and replacing α by u, we obtain

$$I_q^{u}\, H_{M,N}^{m,n}\left[\begin{array}{c|c} (a,A) & \\ (b,B) & \end{array}\; wx^{\lambda}; q\right] = x^{u}\,(1-q)^{u}\, H_{M+1,N+1}^{m,n+1}\left[\begin{array}{c|c} (a,A),(0,\lambda) & \\ (b,B),(-u,\lambda) & \end{array}\; wx^{\lambda}; q\right], \tag{3.13}$$

which is the Riemann-Liouville fractional q-integral of the $H_q(.)-$ function, obtained by Kalla, Yadav and Purohit [3, eq.(2.1), (2005), p.317].

Theorem 3.2. *If the equation* (1.1) *satisfied then for $Re(\alpha) > 0$ and $\beta, \gamma \in \mathbb{C}$, the Saigo's fractional q-integral of Fox's $H_q(.)-$ function is given by the following formula:*

$$K_q^{\alpha,\beta,\gamma}\, H_{M,N}^{m,n}\left[\begin{array}{c|c} (a,A) & \\ (b,B) & \end{array}\; wx^{\lambda}; q\right] = x^{-\beta} q^{-\alpha(\alpha+1)/2}\,(1-q)^{\alpha}$$
$$\times H_{M+2,N+1}^{m+2,n}\left[\begin{array}{c|c} (a,A),(\beta,\lambda),(\gamma,\lambda) & \\ (b,B),(0,\lambda),(\alpha+\beta+\gamma,\lambda) & \end{array}\; wx^{\lambda} q^{-\alpha\lambda}; q\right], \tag{3.14}$$

Proof. Using the eq.(2.6), the Saigo's fractional q-integral operator given by Garg and Chanchlani [2] reduces to

$$K_q^{\alpha,\beta,\gamma} f(x) = x^{-\beta} q^{-\alpha(\alpha+1)/2}\,(1-q)^{\alpha} \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m (q^{-\gamma};q)_m}{(q;q)_m} q^{\gamma m} \sum_{k=0}^{\infty} q^{\beta k} \frac{(q^{\alpha+m};q)_k}{(q;q)_k} f(xq^{-\alpha-k-m}) \tag{3.15}$$

$$\Rightarrow K_q^{\alpha,\beta,\gamma} H_{M,N}^{m,n}\left[\begin{array}{c|c} (a,A) & \\ (b,B) & \end{array}\; wx^{\lambda}; q\right] = x^{-\beta} q^{-\alpha(\alpha+1)/2}\,(1-q)^{\alpha} \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m (q^{-\gamma};q)_m}{(q;q)_m} q^{\gamma m}$$
$$\times \sum_{k=0}^{\infty} q^{\beta k} \frac{(q^{\alpha+m};q)_k}{(q;q)_k} H_{M,N}^{m,n}\left[\begin{array}{c|c} (a,A) & \\ (b,B) & \end{array}\; wx^{\lambda} q^{(-\alpha-k-m)\lambda}; q\right] \tag{3.16}$$

$$= x^{-\beta} q^{-\alpha(\alpha+1)/2}\,(1-q)^{\alpha} \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m (q^{-\gamma};q)_m}{(q;q)_m} q^{\gamma m} \sum_{k=0}^{\infty} q^{\beta k} \frac{(q^{\alpha+m};q)_k}{(q;q)_k}$$
$$\times \frac{1}{2\pi i} \int_{\Omega} \frac{\prod_{j=1}^{m} G(q^{b_j - B_j s}) \prod_{j=1}^{n} G(q^{1-a_j+A_j s})}{\prod_{j=m+1}^{N} G(q^{1-b_j+B_j s}) \prod_{j=n+1}^{M} G(q^{a_j - A_j s}) G(q^{1-s})} \frac{\pi \left(wx^{\lambda} q^{(-\alpha-k-m)\lambda}\right)^{s}}{\sin \pi s} d_q s. \tag{3.17}$$

On interchaging the order of summation and integration, then by using (2.7), the above expression (3.17) reduces to

$$x^{-\beta} q^{-\alpha(\alpha+1)/2}\,(1-q)^{\alpha} \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m (q^{-\gamma};q)_m}{(q;q)_m} q^{\gamma m}$$
$$\times \frac{1}{2\pi i} \int_{\Omega} \frac{\prod_{j=1}^{m} G(q^{b_j - B_j s}) \prod_{j=1}^{n} G(q^{1-a_j+A_j s})}{\prod_{j=m+1}^{N} G(q^{1-b_j+B_j s}) \prod_{j=n+1}^{M} G(q^{a_j - A_j s}) G(q^{1-s})} \frac{\pi \left(wx^{\lambda}\right)^{s} q^{(-\alpha\lambda - m\lambda)s}}{(q^{\beta-\lambda s};q)_{\alpha+m} \sin \pi s} d_q s. \tag{3.18}$$

Again interchanging the order of summation and integration, (3.18) gives

$$\frac{x^{-\beta} q^{-\alpha(\alpha+1)/2}\,(1-q)^{\alpha}}{2\pi i} \int_{\Omega} \frac{\prod_{j=1}^{m} G(q^{b_j - B_j s}) \prod_{j=1}^{n} G(q^{1-a_j+A_j s})}{\prod_{j=m+1}^{N} G(q^{1-b_j+B_j s}) \prod_{j=n+1}^{M} G(q^{a_j - A_j s}) G(q^{1-s})}$$
$$\times \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m (q^{-\gamma};q)_m}{(q;q)_m\,(q^{\beta-\lambda s};q)_{\alpha+m}} q^{(\gamma-\lambda s)m} \frac{\pi \left(wx^{\lambda}\right)^{s} q^{-\alpha\lambda s}}{\sin \pi s} d_q s, \tag{3.19}$$

Also, we have result due to Garg and Chanchlani [2, eq.(2.11) and eq.(2.14), p.173],

$$\left\{(1-q)^{\alpha} \sum_{m=0}^{\infty} \frac{(q^{\alpha+\beta};q)_m (q^{-\eta};q)_m}{(q;q)_m\,(q^{1+\mu};q)_{\alpha+m}} q^{(\eta+\mu-\beta+1)m} = \frac{\Gamma_q(\mu+1)\Gamma_q(\mu-\beta+\eta+1)}{\Gamma_q(\mu-\beta+1)\Gamma_q(\mu+\alpha+\eta+1)}\right\}. \tag{3.20}$$

Now using (2.4) and above result (3.19), we get

$$\frac{x^{-\beta}q^{-\alpha(\alpha+1)/2}}{2\pi i}\int_{\Omega}\frac{\prod_{j=1}^{m}G(q^{b_j-B_js})\prod_{j=1}^{n}G(q^{1-a_j+A_js})}{\prod_{j=m+1}^{N}G(q^{1-b_j+B_js})\prod_{j=n+1}^{M}G(q^{a_j-A_js})G(q^{1-s})}$$

$$\times\frac{G(q^{\beta-\lambda s})G(q^{\gamma-\lambda s})\,(1-q)^{\alpha}}{G(q^{-\lambda s})G(q^{\alpha+\beta+\gamma-\lambda s})}\,\frac{\pi\left(wx^{\lambda}\right)^{s}q^{-\alpha\lambda s}}{\sin\pi s}d_qs, \tag{3.21}$$

$$= x^{-\beta}q^{-\alpha(\alpha+1)/2}\,(1-q)^{\alpha}\,H^{m+2,n}_{M+2,N+1}\left[\begin{array}{c|c}(a,A),(\beta,\lambda),(\gamma,\lambda) & \\ (b,B),(0,\lambda),(\alpha+\beta+\gamma,\lambda) & wx^{\lambda}q^{-\alpha\lambda};q\end{array}\right], \tag{3.22}$$

which completes the proof of the Theorem 3.2 . □

Special Case 3.3. On taking $\beta = 0$ in the Theorem 3.2 and using the result $K_q^{\alpha,0,\gamma}f(x) = q^{-\alpha(\alpha+1)/2}K_q^{\gamma,\alpha}f(x)$ due to Garg and Chanchlani [2, eq.(2.10), p.173], we derive

$$K_q^{\alpha,0,\gamma}\,H^{m,n}_{M,N}\left[\begin{array}{c|c}(a,A) & \\ (b,B) & wx^{\lambda};q\end{array}\right] = q^{-\alpha(\alpha+1)/2}K_q^{\gamma,\alpha}\,H^{m,n}_{M,N}\left[\begin{array}{c|c}(a,A) & \\ (b,B) & wx^{\lambda};q\end{array}\right]. \tag{3.23}$$

Putting $\beta = 0$ in eq. (3.21) and replacing γ by u and α by v , we obtain

$$K_q^{u,v}\,H^{m,n}_{M,N}\left[\begin{array}{c|c}(a,A) & \\ (b,B) & wx^{\lambda};q\end{array}\right] = (1-q)^{v}\,H^{m+1,n}_{M+1,N+1}\left[\begin{array}{c|c}(a,A),(u,\lambda) & \\ (b,B),(v+u,\lambda) & wx^{\lambda}q^{-v\lambda};q\end{array}\right], \tag{3.24}$$

which is the Weyl fractional q-integral of the $H_q(.)-$ function, obtained by Yadav, Purohit and Kalla [10, eq.(2.2), (2008), p.136]

Special Case 3.4. On taking $\beta = -\alpha$ in the Theorem 3.2 and using the result $K_q^{\alpha,-\alpha,\gamma}f(x) = K_q^{\alpha}f(x)$ due to Garg and Chanchlani [2, eq.(2.8), p.173], we get

$$K_q^{\alpha,-\alpha,\gamma}\,H^{m,n}_{M,N}\left[\begin{array}{c|c}(a,A) & \\ (b,B) & wx^{\lambda};q\end{array}\right] = K_q^{\alpha}\,H^{m,n}_{M,N}\left[\begin{array}{c|c}(a,A) & \\ (b,B) & wx^{\lambda};q\end{array}\right]. \tag{3.25}$$

Putting $\beta = -\alpha$ in eq. (3.21) and replacing α by u with $w = \lambda = 1$, we obtain

$$K_q^{u}\,H^{m,n}_{M,N}\left[\begin{array}{c|c}(a,A) & \\ (b,B) & x;q\end{array}\right] = x^{u}q^{-u(u+1)/2}\,(1-q)^{u}\,H^{m+1,n}_{M+1,N+1}\left[\begin{array}{c|c}(a,A),(-u,1) & \\ (b,B),(0,1) & xq^{-u};q\end{array}\right], \tag{3.26}$$

which is the Weyl fractional q-integral of the $H_q(.)-$ function, obtained by Yadav and Purohit [9, eq.(25), (2006), p.239].

4. Conclusion

The Theorems 3.1 and 3.2, in this paper are believed to be new contribution to the theory of fractional q-calculus and generalized special functions. The certain basic q-integrals of the $H_q(.)$-function, obtained as the application of the main results.

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PAIR OF NON-SELF-MAPPINGS AND COMMON FIXED POINTS IN PARTIAL METRIC SPACES

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Abstract

Gajić and Rakočević proved a common fixed point theorem for non-self mappings on a Takahashi convex metric space. In this paper, we prove a common fixed point theorem on a pair of non-self mappings obeying specified conditions on a convex partial metric space. We also provide an illustrative example on the use of the theorem.

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1. Introduction and Preliminaries

Metric spaces were introduced by Fréchet [4] in 1906, and provide a procedure of measuring distances between points in a set. Partial metric spaces were originally developed by Mathews [14] in 1994 as part of the study of denotational semantics of dataflow networks. They are a generalization of metric spaces. While in metric spaces the distance between a point and itsef is always equal to zero, in partial metric spaces this distance can be a positive real number. Partial metric spaces have now found vast applications in computer, information and biological sciences [5].

Mathews [14] also proved that the Banach Fixed Point Theorem [2] for complete metric spaces extends to complete partial metric spaces. Since then, researchers have proved fixed point theorems for various types of contraction mappings that apply to partial metric spaces. Many of these involve self mappings. In this study, we propose a fixed point theorem for a pair of non-self mappings in a complete partial metric space. In so doing, we extend a theorem by Gajić and Rakočević [6].

In recent years, researchers extended Banach fixed point theorem [2] in different directions. Imdad and Kumar [8] proved some fixed point theorems for two pairs of non-self mappings by employing Boyd and Wong type contractive conditions and generalized some recent results due to Zaheer and Abdalla. Kumar [10] proved a fixed point theorem for a pair of compatible F-contraction maps in an ordered complete partial metric spaces and generalized several results in literature. Recently in 2021, Kumar and Sholastica [11] generalized some fixed point theorems for multivalued F-contractions in partial metric spaces. Several authors generalized and extended existing results in partial metric space and one can see for more details [12, 13, 15, 16, 17, 20] and the references therein.

We now introduce those results which will be of use in this paper.

Definition 1.1 ([14]). *A partial metric on a non-empty set X is a mapping $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$,*
P0: $0 \leq p(x, x) \leq p(x, y)$,
P1: $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,
P2: $p(x, y) = p(y, x)$ and
P3: $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.
A pair (X, p) is said to be a partial metric space.

From Definition 1.1, we deduce that for all x, y, z in a partial metric space (X, p), we have:

$$\text{(i)} \quad p(x, y) = 0 \text{ implies } x = y, \tag{1.1}$$

$$\text{(ii)} \quad p(x, y) \leq p(x, z) + p(z, y). \tag{1.2}$$

Proof. If $p(x, y) = 0$, then $p(x, x) = 0$ because $0 \leq p(x, x) \leq p(x, y)$ from P0. Similarly, $p(x, y) = 0$ implies $p(y, y) = 0$ because $0 \leq p(y, y) \leq p(x, y)$. Hence $p(x, y) = 0$ implies $p(x, x) = p(x, y) = p(y, y) = 0$. From P1 this means that $x = y$.

From P3, we infer that

$$p(x, y) \leq p(x, z) + p(z, y).$$

□

As an example, let $X = \mathbb{R}^+$ and let $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+,\ p(x, y) = \max\{x, y\}$. Then (X, p) is a partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X with a base being the family of open balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where
$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in a partial metric space (X, p) converges to $x \in X$ if and only if

$$p(x, x) = \lim_{n\to\infty} p(x, x_n).$$

Definition 1.2 ([14])**.** *(i) A sequence $\{x_n\}$ in a partial metric space (X, p) is called a Cauchy sequence if and only if*

$$\lim_{n,m\to\infty} p(x_n, x_m) < +\infty.$$

(ii) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that

$$p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

We define 0-complete partial metric spaces.

Definition 1.3 ([14])**.**

(i) A sequence $\{x_n\}$ in a partial metric space (X, p) is called 0-Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$.

(ii) A partial metric space (X, p) is said to be 0-complete if every 0-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$.

Lemma 1.1 ([14])**.** *If p is a partial metric on X, then the mapping*
$p^s : X \times X \to [0, +\infty)$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a metric.

In this paper we will denote p^s as the metric derived from the partial metric p.

Lemma 1.2 ([14])**.**

(i) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).

(ii) (X, p) is complete if and only if (X, p^s) is complete. Furthermore $\lim_{n\to\infty} p(x_n, x) = 0$ *if and only if* $p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

We define a convex partial metric space.

Definition 1.4 ([18])**.** *Let (X, p) be a partial metric space and $I = [0, 1]$ be the closed unit interval. A mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for all $(x, y, t) \in X \times X \times I$,*

$$p\,(u, W(x, y, t)) \leq tp(u, x) + (1 - t)p(u, y)$$

for every $u \in X$. A partial metric space (X, p), together with the convex structure W, is called a convex partial metric space.

If (X, p) is a convex partial metric space, then for every $x, y \in X$, we term

$$\mathrm{seg}[x, y] := \{W(x, y, t) : t \in [0, 1]\}. \tag{1.3}$$

In this study, we will use the following properties for a convex partial metric space with convex structure W.

Lemma 1.3. *Let $x, y \in X$ where (X, p) is a convex partial metric space with convex structure W. Let $w \in \mathrm{seg}[x, y]$. Then for all $u \in X$, we have*

(i) $p(u, w) \leq \max\{p(u, x), p(u, y)\}$,

(ii) $p(x, w) \leq p(x, y)$.

Proof. Suppose $\Gamma = \max\{p(u, x), p(u, y)\}$. Applying Definition 1.4, we have

$$\begin{aligned} p(u, w) &\leq tp(u, x) + (1 - t)p(u, y) \\ &\leq t\Gamma + (1 - t)\Gamma \\ &= \Gamma \\ &= \max\{p(u, x), p(u, y)\}. \end{aligned}$$

We have proved Lemma 1.3 (i).

Now let us set $x = u$ in Lemma 1.3 (i). We get

$$\begin{aligned} p(x, w) &\leq \max\{p(x, x), p(x, y)\} \\ &= p(x, y), \text{ from P0 of Definition 1.1.} \end{aligned}$$

□

Definition 1.5 ([3]).

(i) A subset B of a partial metric space (X, p) is said to be bounded *if there is a positive number M such that $p(x, y) \leq M$ for all $x, y \in B$.*

(ii) The diameter of a bounded set B is defined as

$$\operatorname{diam}(B) = \sup_{u,v \in B}\{p(u, v)\}.$$

Let $f : C \to X$ be a mapping, where $C \subseteq X$. We say that f is a *self mapping* if $C = X$, otherwise f is called a *non-self mapping*. If there is an element $x \in C$ such that $fx = x$, we say that x is a *fixed point* of f in X.

Suppose we have two mappings $f, g : C \to X$, with $C \subseteq X$. Let there be $x \in C$ such that $fx = gx = w$. We say that x is a *coincidence point* of f and g in X. If $x = w$, then we call x a *common fixed point* of f and g in X.

Suppose we have two mappings $f, g : C \to X$ with $C \subseteq X$. We say f and g are *coincidentally commuting* if for all $x \in C$, we have

$$fx = gx \Rightarrow fgx = gfx.$$

In this paper, we aim to extend the following theorem by Gajić and Rakočević [6] which proves the existence of a common fixed point for non-self mappings in context of metric spaces under specified conditions.

Theorem 1.1 ([6]). *Let (X, d) be a complete Takahashi convex metric space with convex structure W which is continuous in the third variable. Let C be a non-empty closed subset of X and ∂C be the boundary of C. Let $f, g : C \to X$ and suppose $\partial C \neq \emptyset$. Let us assume that f and g satisfy the following conditions:*

(i) For every $x, y \in C, d(gx, gy) \leq M_\omega(x, y)$ where $M_\omega(x, y) = \max\{\omega_1[d(fx, fy)], \omega_2[d(fx, gx)], \omega_3[d(fy, gy)], \omega_4[d(fx, gy)], \omega_5[d(gx, fy)]\}$, $\omega_i : [0, +\infty) \to [0, +\infty), i \in \{1, 2, 3, 4, 5\}$ is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < r$ for $r > 0$, and $\lim_{r\to\infty}[r - \omega_i(r)] = +\infty$,

(ii) $\partial C \subseteq f(C)$,

(iii) $g(C) \cap C \subset f(C)$,

(iv) $fx \in \partial C \Rightarrow gx \in C$ and

(v) $f(C)$ is closed in X.

Then there exists a coincidence point v in C. Moreover, if $\{f, g\}$ are coincidentally commuting, then v remains a unique common fixed point of f and g.

We now proceed to the main results.

2. Main Results

This paper seeks to extend Theorem 1.1 to partial metric spaces as follows:

Theorem 2.1. *Let (X, p) be a complete convex partial metric space with convex structure W which is continuous in the third variable. Let C be a non-empty subset of X with a non-empty boundary ∂C. Let $g, f : C \to X$ satisfy the following conditions:*

(i) For every $x, y \in C, p(gx, gy) \leq M_\omega(x, y)$ where $M_\omega(x, y) = \max\{\omega_1[p(fx, fy)], \omega_2[p(fx, gx)], \omega_3[p(fy, gy)], \omega_4[p(fx, gy)], \omega_5[p(gx, fy)]\}$ and $\omega_i : [0, +\infty) \to [0, +\infty), i \in \{1, 2, 3, 4, 5\}$ is a non-decreasing semi-continuous function from the right, such that $\omega_i(r) < r$ for $r > 0$,

(ii) $\partial C \subseteq f(C)$,

(iii) $g(C) \cap C \subset f(C)$,

(iv) $fx \in \partial C$ implies $gx \in C$ and

(v) $f(C)$ is 0-complete in (X, p).

Then there exists a coincidence point v of f and g in C. Moreover, if $\{f, g\}$ are coincidentally commuting, then v remains a unique common fixed point of f and g and $p(v, v) = 0$.

Proof. Commencing with an arbitrary point $z \in \partial C$, we construct a sequence $\{x_n\}$ of points in C as follows:

From assumption (ii), we can choose a point $x_0 \in C$ such that $fx_0 = z$. From (iv), we have $gx_0 \in C$. According to (iii), we choose $x_1 \in C$ such that $fx_1 = gx_0$. We locate gx_1.

We consider two scenarios. If $gx_1 \in C$, then, using (iii), we can choose $x_2 \in C$ such that $fx_2 = gx_1$. On the other end, if $gx_1 \notin C$, because W is continuous in the third variable, there is $\lambda_1 \in [0, 1]$ such that $W(gx_0, gx_1, \lambda_1) \in \operatorname{seg}[gx_0, gx_1] \cap \partial C$. As $W(gx_0, gx_1, \lambda_1) \in \partial C$, by (ii), we can choose $x_2 \in C$ such that $fx_2 = W(gx_0, gx_1, \lambda_1)$.

We proceed inductively as follows. If $gx_n \in C$, then by assumption (iii), we can choose $x_{n+1} \in C$ such that $fx_{n+1} = gx_n$.

If however $gx_n \notin C$ and $n \geq 1$, it means there is a $\lambda_n \in (0, 1)$ such that $W(gx_{n-1}, gx_n, \lambda_n) \in \operatorname{seg}[gx_{n-1}, gx_n] \cap \partial C$ and hence, by (ii), we can choose $x_{n+1} \in C$ such that

$$fx_{n+1} = W(g(x_{n-1}, gx_n, \lambda_n) \in \partial C.$$

We prove that, for $n \geq 1$,

$$gx_n \neq fx_{n+1} \Rightarrow gx_{n-1} = fx_n. \tag{2.1}$$

Suppose we have $gx_{n-1} \neq fx_n$. Then we have $fx_n \in \partial C$, which by assumption (iv) means $gx_n \in C$. This implies, by the construction of the sequence, that $gx_n = fx_{n+1}$, which is a contradiction. Thus we have proved (2.1).

We now prove that the sequences $\{fx_n\}$ and $\{gx_n\}$ are bounded, that is, we want to show that the set

$$A = \left(\bigcup_{i=0}^{\infty} fx_i\right) \cup \left(\bigcup_{i=0}^{\infty} gx_i\right)$$

is bounded.

For each $n \geq 1$, we set

$$A_n = \left(\bigcup_{i=0}^{n-1} fx_i\right) \cup \left(\bigcup_{i=0}^{n-1} gx_i\right).$$

Let $\alpha_n = \operatorname{diam}(A_n)$. We shall now prove that

$$\alpha_n \leq \max\{p(fx_0, gx_j), 0 \leq j \leq n-1\}. \tag{2.2}$$

Let us consider the case where $\alpha_n = 0, n \geq 1$. If this is the case, then we have for $0 \leq i, j \leq n-1$

$$\operatorname{diam}\left(\bigcup_{i=0}^{n-1} fx_i\right) \cup \left(\bigcup_{i=0}^{n-1} gx_i\right) = 0,$$

implying

$$\max\{p(fx_i, fx_j)\} = \max\{p(fx_i, gx_j)\} = \max\{p(gx_i, gx_j)\} = 0. \tag{2.3}$$

Employing Equation (1.1), the equation (2.3) implies

$$fx_i = gx_i = fx_0, \tag{2.4}$$

for all $0 \leq i \leq n-1$. In particular, we have

$$fx_0 = gx_0, \tag{2.5}$$

making x_0 a coincidence point of f and g. If f and g are also coincidentally commuting, it means that at the coincidence point x_0, we have

$$fgx_0 = gfx_0 = ggx_0. \tag{2.6}$$

Applying the assumption, we have for some $t \in \{1, 4, 5\}$,

$$\begin{aligned}
p(ggx_0, gx_0)) &\leq M_\omega(gx_0, x_0) \\
&= \max\{(\omega_1[p(fgx_0, fx_0)], \omega_2[p(fgx_0, ggx_0)], \\
&\qquad \omega_3[p(fx_0, gx_0)], \omega_4[p(fgx_0, gx_0)], \omega_5[p(ggx_0, fx_0)]\}) \\
&= \max\{(\omega_1[p(ggx_0, gx_0)], \omega_2[p(ggx_0, ggx_0)], \\
&\qquad \omega_3[p(gx_0, gx_0)], \omega_4[p(ggx_0, gx_0)], \omega_5[p(ggx_0, gx_0)]\}) \\
&= \omega_t[p(ggx_0, gx_0)] \\
&< p(ggx_0, gx_0), \text{ for } p(ggx_0, gx_0) > 0 \\
&\Rightarrow p(ggx_0, gx_0) = 0
\end{aligned}$$

$$\Rightarrow ggx_0 = gx_0 \text{ [from Equation (1.1)]},$$

making $v = gx_0$ a fixed point of g. From (2.5) and (2.6),
$v = gx_0 = fx_0$ is also a fixed point of f, making v a common fixed point of f and g.

We show that v is unique. Suppose v' is also a common fixed point. Then for some $s \in \{1, 4, 5\}$, we have

$$\begin{aligned}
p(v', v) = p(gu, gv) &\leq M_\omega(v', v) \\
&= \max\{(\omega_1[p(fv', fv)], \omega_2[p(fv', gv')], \omega_3[p(fv, gv)], \\
&\qquad \omega_4[p(fv', gv)], \omega_5[p(gv', fv)])\} \\
&= \max\{(\omega_1[p(v', v)], \omega_2[p(v', v')], \omega_3[p(v, v)], \\
&\qquad \omega_4[p(v', v)], \omega_5[p(v', v)])\} \\
&= \omega_s[p(v', v)] \\
&< p(u, v) \text{ for } p(v', v) > 0 \\
&\Rightarrow p(v', v) = 0 \\
&\Rightarrow v' = v.
\end{aligned}$$

Hence, when $\alpha_n = 0, v = fx_0$ is the unique common fixed point of f and g and $p(v, v) = 0$.

Let us consider the situation where $\alpha_n > 0$. To prove (2.2), we consider the following cases.

Case 1. Consider the case where $\alpha_n = p(fx_i, gx_j)$ for some $0 \leq i, j \leq n-1$.

(1.i) If $i \geq 1$ and $fx_i = gx_{i-1}$, we have for some $s \in \{1, 2, \ldots, 5\}$

$$\alpha_n = p(fx_i, gx_j) = p(gx_{i-1}, gx_j) \leq M_\omega(x_{i-1}, x_j) \leq \omega_s(\alpha_n) < \alpha_n$$

which is a contradiction. Hence, $i = 0$.

(1.ii) If however $i \geq 1$ and $fx_i \neq gx_{i-1}$, it implies $i \geq 2$
and $fx_i \in \operatorname{seg}[gx_{i-2}, gx_{i-1}]$. Hence, by Lemma 1.3 (i), we have

$$p(fx_i, gx_j) = p(gx_j, fx_i) \leq \max\{p(gx_{i-2}, gx_j), p(gx_{i-1}, gx_j)\}.$$

Hence

$$\begin{aligned}
\alpha_n &= p(fx_i, gx_j) \\
&\leq \max\{p(gx_{i-2}, gx_j), p(gx_{i-1}, gx_j)\} \\
&\leq \max\{M_\omega(x_{i-2}, x_j), M_\omega(x_{i-1}, x_j)\} \\
&\leq \omega_t(\alpha_n) \\
&< \alpha_n,
\end{aligned}$$

which is a contradiction. Therefore, $i = 0$.

Case 2. Consider $\alpha_n = p(fx_i, fx_j)$ for some $0 \leq i, j \leq n-1$.

(2.i) If $fx_j = gx_{j-1}$, we have $\alpha_n = p(fx_i, gx_{j-1})$ and this case reduces to Case (1.i) implying $i = 0$.

(2.ii) If $fx_j \neq gx_{j-1}$ then as in Case (1.ii), we have $j \geq 2$ and
$fx_j \in \operatorname{seg}[gx_{j-2}, gx_{j-1}]$. By Lemma 1.3 (i), we have

$$p(fx_i, fx_j) \leq \max\{p(gx_{j-2}, fx_i), p(gx_{j-1}, fx_i)\}. \tag{2.7}$$

This means $\alpha_n \leq p(gx_{j-1}, fx_i) = p(fx_i, gx_{j-1})$ for some $0 \leq i, j \leq n-1$, which leads to $i = 0$ according to the argument in Case 1.

Case 3. Consider $\alpha_n = p(gx_i, gx_j)$ for some $0 \leq i, j \leq n-1$. This case is not possible (see Case (1.i)).

Hence, we have proved (2.2).

Now for some $0 \leq j \leq n-1$ and some $t \in \{1, 2, \ldots, 5\}$, we have

$$\begin{aligned}
\alpha_n &\leq p(fx_0, gx_j) \\
&\leq p(fx_0, gx_0) + p(gx_0, gx_j), \text{ from (1.2)} \\
&\leq p(fx_0, gx_0) + M_\omega(x_0, x_j) \\
&\leq p(fx_0, gx_0) + \omega_t(\alpha_n)
\end{aligned}$$

$$\text{implying } \alpha_n - \omega_t(\alpha_n) \leq p(fx_0, gx_0). \tag{2.8}$$

By (i) in the assumption, there is $r_0 \geq 0$ such that for each $s \in \{1, 2, \ldots, 5\}$, we have
$r - \omega_s(r) > p(fx_0, gx_0)$ for $r > r_0$.

There is a subsequence $\{a_n\}$ of $\{\alpha_n\}$ and $s \in \{1, 2, \ldots, 5\}$ such that for each n, we have
$a_n - \omega_s(a_n) \leq p(fx_0, gx_0)$.
Thus, by (2.8), we have
$a_n \leq r_0$, $n = 1, 2, \ldots$
and evidently
$\alpha = \lim a_n = \mathrm{diam}(A) \leq r_0$.
This shows that the sequences $\{fx_i\}$ and $\{gx_i\}$ are bounded.
Now let us show that $\{fx_i\}$ and $\{gx_i\}$ are Cauchy sequences.
Let

$$B_n = \left(\bigcup_{i=n}^{\infty} fx_i\right) \cup \left(\bigcup_{i=n}^{\infty} gx_i\right), n \geq 2.$$

By (2.2), we have $\beta_n = \mathrm{diam}\{B_n\} \leq \sup_{j \geq n} p(fx_n, gx_j)$. If $fx_n = gx_{n-1}$ then by Case (1.i) for some $j \geq n$ and some $r \in \{1, 2, \ldots, 5\}$, we have

$$\beta_n = p(fx_n, gx_j) = p(gx_{n-1}, gx_j) \leq \omega_r(\beta_{n-1}). \tag{2.9}$$

If $fx_n \neq gx_{n-1}$, it means $fx_n \in \mathrm{seg}\{gx_{n-2}, gx_{n-1}\}$. Hence, as in Case (1.ii) for some $j \geq n$ and some $s \in \{1, 2, \ldots, 5\}$, we have

$$\begin{aligned}\beta_n &= p(fx_n, gx_j)\\ &= p(gx_j, fx_n)\\ &\leq \max\{p(gx_{n-2}, gx_j), p(gx_{n-1}, gx_j)\}\\ &\leq \omega_s(\beta_{n-2}).\end{aligned} \tag{2.10}$$

We note that $\beta_n \geq \beta_{n+1}$ for each n. Hence, by (2.9) and (2.10), for all $n \geq 2$ and $t \in \{1, 2, \ldots, 5\}$, we have

$$\beta_n \leq \omega_t(\beta_{n-2}). \tag{2.11}$$

Let us set

$$\lim_{n\to\infty} \beta_n = \beta. \tag{2.12}$$

We claim $\beta = 0$. If $\beta > 0$, taking $n \to \infty$ in (2.11), we get $\beta \leq \omega_t(\beta) < \beta$ which is a contradiction. Hence, $\beta = 0$.
Consider a subsequence $\{gx_{n_k}\}$ of $\{gx_n\}$, for which $gx_{n_k} = fx_{n_k+1}$. From (2.9), we have

$$\begin{aligned}\beta_n &= p(fx_n, gx_j) \text{ for some } j \geq n\\ \Rightarrow \lim_{n\to\infty} \beta_n = \beta &= \lim_{n,j\to\infty} p(fx_n, gx_j),\ \text{ for } j \geq n\\ &= \lim_{n,k\to\infty} p(fx_n, gx_{n_k}),\ \text{ for } n_k \geq n\\ &= \lim_{n,k\to\infty} p(fx_n, fx_{n_k+1}),\ \text{ for } n_k \geq n\\ &\Rightarrow \lim_{n,m\to\infty} p(fx_n, fx_m) = 0.\end{aligned}$$

Thus we have proved that $\{fx_n\} \subset C$ is a Cauchy sequence in (X, p), according to Lemma 1.2. From (iv) in the assumption, $f(C)$ is 0-complete. According to Definition 1.3, this means there is $v \in fC$ such that

$$\lim_{n,m\to\infty} p(fx_m, fx_n) = \lim_{n\to\infty} p(v, fx_n) = p(v, v) = 0. \tag{2.13}$$

We have shown that $fx_n \to v$. Consider the subsequence $\{g_{n_k}\}$ again. We have

$$\begin{aligned}&gx_{n_k} = fx_{n_k+1}\\ &\Rightarrow \lim_{k\to\infty} gx_{n_k+1} = \lim_{k\to\infty} fx_{n_k+1} = v\\ &\Rightarrow gx_n \to v.\end{aligned}$$

Hence, we can write

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = v. \tag{2.14}$$

As $v \in f(C)$, we can find $w \in C$ such that $fw = v$. We show that w is a coincidence point of f and g.
Consider the following:

$$p(gw, gx_n) \leq \max\{\omega_1[p(fw, fx_n)], \omega_2[p(fw, gw)], \omega_3[p(fx_n, gx_n)],$$

$$\omega_4[p(fw, gx_n)], \omega_5[p(gw, fx_n)]\}.$$

Taking $n \to \infty$ and noting that $fw = v$, we get for some $s \in \{1, 2, 3, 4, 5\}$

$$\begin{aligned} p(gw, v) &\leq \max\{\omega_1[p(v, v)], \omega_2[p(v, gw)], \omega_3[p(v, v)], \omega_4[p(v, v)], \\ &\qquad \omega_5[p(gw, v)]\} \\ &\leq \omega_s p(gw, v) \\ &< p(gw, v) \text{ for } p(gw, v) > 0, \\ &\Rightarrow p(gw, v) = 0 \\ &\Rightarrow gw = v. \end{aligned}$$

As $fw = gw = v$, w is a coincidence point of f and g. If f and g are coincidentally commuting, it means $f(gw) = g(fw)$ implying $fv = gv$.

We consider the following:

$$\begin{aligned} p(gv, v) &= p(gv, gw) \\ &\leq \max\{\omega_1[p(fv, fw)], \omega_2[p(fv, gv)], \omega_3[p(fw, gw)], \\ &\qquad \omega_4[p(fv, gw)], \omega_5[p(gv, fw)]\} \\ &= \max\{\omega_1[p(gv, v)], \omega_2[p(gv, gv)], \omega_3[p(v, v)], \\ &\qquad \omega_4[p(gv, v)], \omega_5[p(gv, v)]\} \\ &\leq \omega_t[p(gv, v)] \text{ for some } t \in \{1, 2, 3, 4, 5\}, \\ &< p(gv, v) \text{ for } p(gv, v) > 0, \\ &\Rightarrow p(gv, v) = 0 \\ &\Rightarrow gv = v = fv, \end{aligned}$$

making v a common fixed point of f and g.

We show that v is unique. Suppose v' is also a common fixed point of f and g. Then we have

$$\begin{aligned} p(gv, gv') &= p(v, v') \\ &\leq \max\{\omega_1[p(fv, fv')], \omega_2[p(fv, gv)], \omega_3[p(fv', gv')], \\ &\qquad \omega_4[p(fv, gv')], \omega_5[p(gv, fv')]\} \\ &= \max\{\omega_1[p(v, v')], \omega_2[p(v, v)], \omega_3[p(v', v')], \\ &\qquad \omega_4[p(v, v')], \omega_5[p(v, v')]\} \\ &\leq \omega_t[(p(v, v')] \text{ for some } t \in \{1, 2, 3, 4, 5\}, \\ &< p(v, v') \text{ for } p(v, v') > 0, \\ &\Rightarrow p(v, v') = 0 \\ &\Rightarrow v = v'. \end{aligned}$$

This shows v is unique and $p(v, v) = 0$. □

Remark 2.1. If (X, d) is a metric space with convex structure W, we get Theorem 1.1 by Gajić and Rakočević [6].

If we set $f = I$, the identity mapping, in Theorem 2.1, we get the following corollary.

Corollary 2.1. *Let (X, p) be a complete convex partial metric space with convex structure W which is continuous in the third variable. Let C be a non-empty subset of X with a non-empty boundary ∂C. Let $g : C \to X$ satisfy the following conditions:*
(i) For every $x, y \in C$, $p(gx, gy) \leq M_\omega(x, y)$ where $M_\omega(x, y)$
$= \max\{\omega_1[p(x, y)], \omega_2[p(x, gx)], \omega_3[p(y, gy)], \omega_4[p(x, gy)], \omega_5[p(gx, y)]\}$
and $\omega_i : [0, +\infty) \to [0, +\infty)$, $i \in \{1, 2, 3, 4, 5\}$ is a non-decreasing semicontinuous function from the right, such that $\omega_i(r) < r$ for $r > 0$,
(ii) $x \in \partial C \Rightarrow gx \in C$.

Then there exists a point $v \in C$ which is a fixed point of g and $p(v, v) = 0$.

Finally, if in Theorem 2.1 we set $\omega_i(r) = hr$ for $i \in \{1, 2, 3, 4, 5\}$ with $0 \leq h < 1$, then we get the following corollary:

Corollary 2.2. *Let (X, p) be a complete convex partial metric space with convex structure W which is continuous in the third variable. Let C be a non-empty subset of X with a non-empty boundary ∂C. Let $g, f : C \to X$ satisfy the following conditions:*
(i) For every $x, y \in C$ let there be $h \in (0, 1)$ such that
$p(gx, gy) \leq h \max\{p(fx, fy), p(fx, gx), p(fy, gy), p(fx, gy), p(gx, fy)\}$,
(ii) $\partial C \subseteq f(C)$,
(iii) $g(C) \cap C \subset f(C)$,
(iv) $fx \in \partial C \Rightarrow gx \in C$ and
(v) $f(C)$ be 0-complete in (X, p).

Then there exists a coincidence point v in C. Moreover, if $\{f, g\}$ are coincidentally commuting, then v remains a unique common fixed point of f and g and $p(v, v) = 0$.

Theorem 2.1 is also valid if we alter the first sentence as in the following corollary.

Corollary 2.3. *Let (X, p) be a complete convex partial metric space with convex structure W which is continuous in the third variable. Let C be a non-empty closed subset of X, the closure being with respect to (X, p^s). Let ∂C, the boundary of C with respect to (X, p^s), be non-empty. Let $g, f : C \to X$ satisfy the following conditions:*

*(i) For every $x, y \in C$, $p(gx, gy) \leq M_\omega(x, y)$ where $M_\omega(x, y) =$
$\max\{\omega_1[p(fx, fy)], \omega_2[p(fx, gx)], \omega_3[p(fy, gy)], \omega_4[p(fx, gy)], \omega_5[p(gx, fy)]\}$
and $\omega_i : [0, +\infty) \to [0, +\infty), i \in \{1, 2, 3, 4, 5\}$ is a non-decreasing semi-continuous function from the right, such that $\omega_i(r) < r$ for $r > 0$,*

(ii) $\partial C \subseteq f(C)$,

(iii) $g(C) \cap C \subset f(C)$,

(iv) $fx \in \partial C$ implies $gx \in C$ and

(v) $f(C)$ is 0-complete in (X, p).

Then there exists a coincidence point v of f and g in C. Moreover, if $\{f, g\}$ are coincidentally commuting, then v remains a unique common fixed point of f and g and $p(v, v) = 0$.

We now provide an example for the use of our result. The example is based on the Corollary 2.3, as it is better suited for the partial metric we are using.

Example 2.1. Let (X, p) be a partial metric space with $X = [0, +\infty)$ and $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. We note that (X, p) is a convex partial metric space with convex structure $W(x, y, t) = tx + (1 - t)y$. Let $C = [0, 2]$. Let us define $f, g : C \to X$ as

$$gx = \begin{cases} 4^x - 1, & x \in [0, 1] \\ 2, & x \in (1, 2] \end{cases} \qquad fx = \begin{cases} 16^x - 1, & x \in [0, 1] \\ 4, & x \in (1, 2] \end{cases}$$

From the given information, $f(C) = [0, 15]$ is closed with respect to (X, p^s). This makes $f(C)$ is complete in (X, p^s) and hence complete in (X, p). Because $f(C)$ is complete it is also 0-complete.

We have $\partial C = \{0, 2\} \subset [0, 15] = f(C)$. We also have $g(C) = [0, 3]$ and
$g(C) \cap C = [0, 2] \subset [0, 15] = f(C)$.

For $fx \in \partial C = \{0, 2\}$, we have
$x \in \{0, \log 3/\log 16\} = \{0, 0.3962\} \Rightarrow gx \in \{0, 0.7321\} \subset [0, 2] = C$.

We note that the mappings f and g are not continuous at $x = 1$ and the pair $\{f, g\}$ is coincidentally commuting as $fg(0) = gf(0) = 0$.

Consider the function

$$h(x) = \frac{16^x - 1}{4^x - 1}.$$

Using the L'Hopital rule, we have

$$\lim_{x \to 0} h(x) = \lim_{x \to 0} \frac{16^x \ln 16}{4^x \ln 4} = \frac{2 \ln 4}{\ln 4} = 2.$$

We note that

$$h(x) = \frac{16^x - 1}{4^x - 1} = \frac{(4^x + 1)(4^x - 1)}{4^x - 1} = 4^x + 1.$$

Hence, we have $h'(x) = 4^x \ln 4 > 0$ implying that $h(x)$ is an increasing function. Hence for $x > 0$, we have

$$h(x) = \frac{16^x - 1}{4^x - 1} > 2$$

$$\Rightarrow (4^x - 1) < \frac{1}{2}(16^x - 1) < \frac{3}{4}(16^x - 1). \tag{2.15}$$

Without loss of generality, suppose $y \geq x$.

Let $x, y \in (1, 2]$. Then

$$p(gx, gy) = 2 < 3 = 0.75 \max\{4, 4\} = \frac{3}{4}\, p(fx, fy).$$

Let $x \in [0, 1], y \in (1, 2]$. Note that if $x \in [0.5, 0.5804]$, then $4^x - 1 \leq 1.2358 < \frac{1}{2} \times 4$. Hence,

$$\begin{aligned} p(gx, gy) &= \max\{4^x - 1, 1\} \\ &= \begin{cases} 1, & x \in [0, 0.5] \\ 4^x - 1, & x \in (0.5, 1] \end{cases} \\ &< \frac{1}{2} \begin{cases} 4, & x \in [0, 0.5804] \\ 16^x - 1, & x \in (0.5804, 1] \end{cases} \\ &= \frac{1}{2} \max\{fx, fy\} \\ &= \frac{1}{2} p(fx, fy). \end{aligned}$$

When $x = 0, y \in (0, 1]$, using (2.15), we have

$$\begin{aligned} p(gx, gy) &= \max\{0, 4^y - 1\} = 4^y - 1 \\ &< \frac{3}{4}(16^y - 1) = \frac{3}{4} \max\{0, 16^y - 1\} = \frac{3}{4} p(fx, fy). \end{aligned}$$

If $x, y \in (0, 1]$ using (2.15), we have

$$\begin{aligned} p(gx, gy) &= \max\{4^x - 1, 4^y - 1\} = 4^y - 1 \\ &< \frac{3}{4}(16^y - 1) = \frac{3}{4} \max\{16^x - 1, 16^y - 1\} = \frac{3}{4} p(fx, fy). \end{aligned}$$

Finally if $(x, y) = (0, 0)$ we have

$$\begin{aligned} p(gx, gy) &= \max\{0, 0\} = 0 \\ &\leq \frac{3}{4} \max\{0, 0\} = \frac{3}{4} p(fx, fy). \end{aligned}$$

Hence for all $x, y \in C$, we have,

$$\begin{aligned} p(gx, gy) \leq \max\{&\omega_1[p(fx, fx)], \omega_2[p(fx, fx)], \omega_3[p(fx, fx)], \\ &\omega_4[p(fx, fx)], \omega_5[p(fx, fx)]\}, \end{aligned}$$

where $\omega_s(r) = \dfrac{3}{4} r$ for $s \in \{1, 2, 3, 4, 5\}$. Thus all conditions of Theorem 2.1 are satisfied and 0 is the unique fixed point of f and g. Furthermore $p(0, 0) = 0$.

3. Conclusion

We proved a common fixed point theorem on a pair of non-self mappings obeying specified conditions on a convex partial metric space. This theorem extend and generalizes the results due to Gajić and Rakočević [6] and many others in literature.

Conflict of Interests. The authors declare that there is no conflict of interests regarding the publication of this paper.

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SPECIAL NORMAL AND NEO-NORMAL PROJECTIVE RECURRENT, BI-RECURRENT, FINSLER SPACES ADMITTING AFFINE MOTION

By

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Abstract

This paper deals with the study of the recurrent and bi-recurrent, Neo-normal / normal and special normal projective Finsler spaces admitting an affine motion. The relation between two Ricci tensors has been established in a normal projective Finsler space and in a special normal projective Finsler space, the recurrence tensor of a bi-recurrent vector field generating an affine motion can not be independent of the directional arguments and is always non-symmetric. Also, some special types of affine motion generated by a vector field whose covariant derivative is recurrent have been discussed in this paper.

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Keywords and Phrases: Finsler spaces, Neo-normal, Recurrent, Bi-current, Affine motion.

1. Introduction

Normal projective recurrent Finsler spaces have been studied by Yano [7]. He introduced the covariant derivative of any vector field with respect to $T^i(x, y)$ with respect to x^j as

$$\underset{j}{\nabla} T^i = \partial_j T^i - (\partial_m T^i)\Pi^m_{hj} y^h + T^m \Pi^i_{mj}, \tag{1.1}$$

where

$$\Pi^i_{jk}(x, y) = G^i_{jk} - \frac{1}{n+1}\dot{\partial}_r G^r_{jk} y^i \tag{1.2}$$

is normal projective connection coefficient, homogeneous of degree zero and satisfies the following relations:

$$\begin{aligned} \dot{\partial}_r \Pi^i_{jk} y^r &= 0, \\ \Pi^i_{jk} &= \Pi^i_{kj}. \end{aligned} \tag{1.3}$$

The commutation formula involving the process of covariant differentiation given by Eq. (1.1) gives rise to projective normal curvature tensor field $N^i_{jkh}(x, y)$ which satisfies the following identities and contractions:

$$\begin{aligned} N^i_{jkh} + N^i_{khj} + N^i_{hjk} &= 0, \\ N^i_{ikh} &= N_{kh}, \\ \dot{\partial}_r N^i_{jkh} y^r &= 0, \\ N^i_{jkh} &= -N^i_{kjh}. \end{aligned} \tag{1.4}$$

Mishra and Mehar [1] considered a space equipped with normal projective connection coefficients Π^i_{kh} whose curvature tensor N^i_{jkh} is recurrent with respect to normal projective connection coefficients Π^i_{kh} and called it as an RNP-Finsler space. They obtained several results concerning projective motion in such a space. Mishra et al. [2] studied the projective motion in a Finsler space with the vanishing covariant derivative of the curvature tensor N^i_{jkh} with respect to normal projective connection coefficient Π^i_{kh} , called them as SNP-Finsler space. Pande [3] observed that the recurrent Finsler spaces and symmetric Finsler spaces characterised by the recurrence of Berwald curvature tensor H^i_{jkh} and the vanishing of the covariant derivative of Berwald curvature tensor H^i_{jkh} with respect to Berwalds connection coefficients G^i_{kh} exactly coincide with RNP and SNP- Finsler space. Subsequently, Pande and Dwivedi [4] studied an RNP-Finsler space and obtained many identities in an RNP-Finsler space, most of these identities are also true in a recurrent Finsler space with respect to Berwalds connection coefficients. The different types of affine and projective motions generated in a general Finsler space along with many identities for these motions has been discussed by Qasem and Saleem [5]. Singh [6] has discussed the affine motion in birecurrent Finsler space and obtained various results. This paper deals with the recurrent and bi-recurrence, Neo-normal / normal and special normal projective Finsler spaces admitting an affine motion.

2. Normal, Neo- normal and Special Normal Projective Finsler Space

Yano [7] defined the normal projective curvature tensor N^i_{jkh} as follows

$$N^i_{jkh} = \partial_h \Pi^i_{kj} + \Pi^i_{ljh}\Pi^l_{kp}y^p + \Pi^i_{lh}\Pi^l_{kj} - k/h, \tag{2.1}$$

where

$$\Pi^i_{jkh} = \dot{\partial}_j \Pi^i_{kh} = G^i_{jkh} - \frac{1}{n+1}(\delta^i_j G^r_{khr} + y^j G^r_{jkhr}) \tag{2.2}$$

and

$$G^i_{ljkh} = \dot{\partial}_l G^i_{jkh}. \tag{2.3}$$

Π^i_{jkh} constitute the components of a tensor and Yano [7] denoted this tensor by U^i_{jkh}. Thus

$$U^i_{jkh} = G^i_{jkh} - \frac{1}{n+1}(\partial^i_j G^r_{khr} + y^j G^r_{jkhr}). \tag{2.4}$$

The tensor U^i_{jkh} satisfies the following identities and contractions:

$$\begin{aligned} U^i_{jkh} &= U^i_{jhk}, \\ U^i_{jki} &= G^i_{jki}, \\ U^i_{jkh}y^j &= 0, \\ U^i_{jkh}y^h &= \frac{1}{n+1}y^i G^h_{jkh}, \\ U^i_{ikh} &= \frac{2}{n+1}G^i_{ikh}. \end{aligned} \tag{2.5}$$

We now give the following definitions which shall be used in the later discussions:

Definition 2.1. *The Finsler manifold F_n of non-zero curvature is said to be of Normal projective recurrent curvature if the normal projective curvature tensor satisfies*

$$\nabla_m N^i_{jkh} = \lambda_m N^i_{jkh}, \tag{2.6}$$

for some non-null covariant vector field λ_m, the vector field λ_m is called the recurrence vector of the manifold.

Definition 2.2. *The point transformation $\bar{x}^i = x^i + v^i(x)dt$ considered at each point in the normal projective recurrent space is called special projective affine motion iff*

$$£_v \Pi^i_{jk} = 0, \tag{2.7}$$

where $£_v$ denotes the operator of Lie-derivative.

Definition 2.3. *The Finsler manifold F_n admits a Neo-normal concurrent affine motion if it admits the infinitesimal transformation of the type*

$$\begin{aligned} \bar{x}^i &= x^i + \in v^i, \\ \nabla_j v^i &= \rho(x,y)\delta^i_j, \end{aligned} \tag{2.8}$$

along with (2.7).

Definition 2.4. *The Finsler manifold F_n shall be called special normal projective if it satisfies Eq* (2.6) *and Eq.*(2.7) *together, whereas the manifold shall be called Neo-normal projective if it satisfies Eq.* (2.6) *and Eq.*(2.8).

The relation in between the normal projective curvature tensor and the Berwalds curvature tensor is given by Pande [3] *as*

$$N^i_{jkh} = H^i_{jkh} - \frac{1}{n+1}y^i \dot{\partial}_j H^r_{rkh}. \tag{2.9}$$

We now allow a contraction in Eq. (2.9) *with respect to indices i and j , and get*

$$N^r_{rkh} = H^r_{rkh}, \tag{2.10}$$

where we have taken into account the fact that the tensor H^r_{rkh} is positively homogeneous of degree zero in y^i.

Transvecting (2.9) *by y^j, we get*

$$N^i_{jkh} y^j = H^i_{kh}. \tag{2.11}$$

The normal projective curvature tensor N^i_{jkh} and the projective curvature tensor W^i_{jkh} are connected by

$$W^i_{jkh} = N^i_{jkh} + (\dot{\partial}_k M_{hj} - M_{kh}\delta^i_j \; k/h), \tag{2.12}$$

where

$$M_{kh} = -\frac{1}{n^2-1}(n.N_{kh} + N_{hk}), \tag{2.13}$$

and

$$N_{jh} = N^r_{jhr}. \tag{2.14}$$

Contracting Eq. (2.9) *with respect to the indices i and h, we get*

$$N_{jk} = H_{jk} - \frac{1}{n+1}\left\{\dot{\partial}_j (H^r_{rki} y^i) - H^r_{rkj}\right\}. \tag{2.15}$$

We now simplify this equation to get

$$N_{jk} = H_{jk} - \frac{1}{n+1}\left[\dot{\partial}_j \left\{(H_{ik} - H_{ki}) y^i\right\} - (H_{jk} - H_{kj})\right]. \tag{2.16}$$

Equation (2.16) *can further be simplified in the following form*

$$N_{jk} = \frac{n}{n+1} H_{jk} - \frac{1}{n+1} H_{kj} + \frac{n-1}{n+1}\dot{\partial}_j \dot{\partial}_k .H \tag{2.17}$$

Thus, we can state:

Theorem 2.1. *In a normal projective Finsler space the relation in between the two Ricci tensors N_{jk} and H_{jk} is given by* (2.17).

3. An N^h- Recurrent Space

We consider a normal projective Finsler space whose N-curvature tensor N^i_{jkh} satisfies Eq. (2.6). If we assume that the metric tensor g_{ij} is N-covariant constant, then Eq.(2.6) immediately gives

$$\underset{m}{\triangle} N_{jikh} = \lambda_m N_{jikh}. \tag{3.1}$$

Contracting Eq.(3.1) with respect to the indices i and j and thereafter using Eq.(1.4b), we get

$$\underset{m}{\triangle} N_{kh} = \lambda_m N_{kh}. \tag{3.2}$$

The equation (3.2) implies that the Ricci-tensor N_{jk} of an N^h-recurrent space is also recurrent. The Finsler space characterised by Eq.(3.2) shall be called Ricci- recurrent space. These observations enable us to state that every N^h-recurrent space is Ricci recurrent but not conversely because we cannot get back Eq. (2.6) from Eq.(3.2). However, if the space is assumed to be of dimension 3, then its curvature tensor N_{ijkh} may be assumed to be of the form

$$N_{ijkh} = g_{ik} L_{jh} + g_{ih} L_{ik} - k/h, \tag{3.3}$$

where

$$L_{ik} = \frac{1}{n-2}(N_{ik} - \frac{r}{2} g_{ik}),$$

$$r = \frac{1}{n-1} N^i_i. \tag{3.4}$$

Now, we consider a Finsler space $F_n (n > 3)$ whose associative curvature tensor satisfies Eq.(3.3). Such spaces have been introduced by Matsumoto who calls them as N3-like Finsler space. If an N3-like Finsler space is Ricci recurrent which is characterised by Eq.(3.2) then obviously it is N^h- recurrent space. Thus we can state

Theorem 3.1. *An N^h-recurrent space is always a Ricci-recurrent space.*

Transvecting Eq. (2.6) *by y^j, we get*

$$\underset{m}{\triangle} N^i_{jkh} y^i = \lambda_m N^i_{jkh} y^j. \tag{3.5}$$

Further, transvecting Eq. (3.5) *by y^k, we get*

$$\underset{m}{\triangle} N^i_{jkh} y^j y^k = \lambda_m N^i_{jkh} y^j y^k. \tag{3.6}$$

The contraction of Eq. (3.5) *with respect to indices i and j gives*

$$\underset{m}{\triangle} N^i_{jkh} y^j = \lambda_m N^i_{ikh} y^j. \tag{3.7}$$

Thus, we can state:

Theorem 3.2. *The tensors N^i_{kh}, N^i_h of an N^h- recurrent Finsler space are h-recurrent provided we be in a position to write $N^i_{jkh} y^i = N^i_{kh}$ and $N^i_{jkh} y^j y^k = N^i_h$.*

Following Yano [7] *and using Eq.* (2.6)*, we have the following result in a projective normal Finsler space*

$$\lambda_j N^i_{khl} y^j + \lambda_k N^i_{hjl} + \lambda_h N^l_{jkl} + y^s \left\{ (\dot{\partial}_r \Pi^i_{lh}) N^r_{jks} + (\dot{\partial}_r \Pi^l_{lk}) N^r_{hks} + (\dot{\partial}_r \Pi^i_{lj}) N^r_{khs} \right\} = 0. \tag{3.8}$$

4. Special Normal Projective Finsler Space Admitting An Affine Motion

Let us consider an infinitesimal transformation generated by a birecurrent vector characterized by

$$\underset{j}{\triangle} \underset{k}{\triangle} v^i = u_{jk} v^i. \tag{4.1}$$

The Lie-derivative of Π^i_{jk} is given by

$$\pounds_v \Pi^i_{jk}(x, y) = \underset{j}{\triangle} \underset{k}{\triangle} v^i + N^i_{jkh} v^h + (\dot{\partial}_h \Pi^i_{lk})(\underset{r}{\triangle} v^h) y^r. \tag{4.2}$$

Using Eq. (4.1) and Eq. (2.2) in Eq.(4.2), we get

$$\pounds_v \Pi^i_{jk} = u_{jk} v^i + N^i_{jkh} v^h + \Pi^i_{hJk} (\underset{r}{\triangle} v^h) y^r.$$

If the vector field v^i generates an affine motion, then the Lie-derivative of Π^i_{jk} vanishes and hence

$$u_{jk} v^i + N^i_{jkh} v^h + \Pi^i_{hjk} (\underset{r}{\triangle} v^h) y^r = 0. \tag{4.3}$$

Transvecting Eq.(4.3) by y^k and using Eq.(2.2) thereafter, we get

$$u_{jk} v^i y^k + N^i_{jkh} v^h v^k = 0. \tag{4.4}$$

Transvecting Eq. (4.4) by y_i, we get

$$u_{jk} v^i y^k y_i + N^i_{jkh} v^h v^k y_i = 0. \tag{4.5}$$

From Eq. (4.5), we have atleast one of the two conditions

$$\begin{aligned} u_{jk} y^k &= 0, \\ y_i v^i &= 0. \end{aligned} \tag{4.6}$$

Under the following assumption only

$$N^i_{jkh} y_i = 0. \tag{4.7}$$

Equation (4.6b) can not be true, for partial differentiation of Eq.(4.6b) with respect to y^j automatically gives $v^i = 0$ which is not possible because this vector has always been assumed to be a non-zero vector. Therefore, we finally come to the conclusion that condition (4.6) necessary holds. Thus, we can state:

Theorem 4.1. *If the birecurrent vector field v^i characterised by Eq.* (4.1) *generates an affine motion in a special normal projective Finsler space, then condition* (4.6) *is not necessary whereas it becomes necessary under the condition* (4.7).

Conversely, let us suppose that the recurrence tensor u_{jk} of a birecurrent vector field v^i characterised by Eq. (4.1) *in a special normal projective Finsler space satisfies Eq.* (4.6) *and further this vector field generates an infinitesimal transformation. The Lie derivative of the connection coefficient Π^i_{jk} with respect to such a transformation is given by Eq.* (4.3)*. Transvecting Eq.* (4.3) *by y^k, we get*

$$(\pounds_v \Pi^i_{jk}) y^k = N^i_{jkh} v^h y^k. \tag{4.8}$$

Differentiating Eq. (4.6) *partially with respect to* y^m, *we get*

$$y^k(\dot{\partial}_m u_{jk}) + u_{jk} = 0. \tag{4.9}$$

Eq. (4.1) *partially with respect to* y^m, *we get*

$$\dot{\partial}_m(\nabla_j\nabla_k v^i) = (\dot{\partial}_m u_{jk})v^i. \tag{4.10}$$

Further, using the commutation formula (4.8), *we get*

$$\dot{\partial}_j(\Pi^i_{mkr}v^r) + \Pi^i_{mjr}\nabla_k v^r - \Pi^r_{mjk}\nabla_r v^i = (\dot{\partial}_m u_{jk})v^i. \tag{4.11}$$

We now transvect Eq. (4.9) *with* y^k *and get*

$$\Pi^i_{mjr}\nabla_k v^r + (\dot{\partial}_m \Pi^i_{mkr})v^r = y^k(\dot{\partial}_m u_{jk})v^i. \tag{4.12}$$

Using Eq. (4.10), *Eq.* (4.9) *reduces to*

$$\Pi^r_{mjk}\nabla_r v^i + (\dot{\partial}_j \Pi^i_{mkr})v^r = y^k(\dot{\partial}_m u_{jk})v^i. \tag{4.13}$$

Here, although the tensor Π^r_{mjk} *is symmetric in its last two lower indices yet the recurrence* u_{jk} *appearing in* (4.13) *is non-symmetric. Thus, we can state*

Theorem 4.2. *In a special normal projective Finsler space the recurrence tensor* u_{jk} *of a birecurrent vector field* v^i *characterised by* (4.1) *is always non-symmetric.*

Taking the skew symmetric part of (4.1) *and then using commutation formula, we get*

$$-(\dot{\partial}_r v^i)N^r_{jkh}y^h + v^h N^i_{jkh} = (u_{jk} - u_{kj})v^i. \tag{4.14}$$

Transvecting the Bianchi identity by v^h, *we get*

$$N^i_{jkh}v^h + N^i_{khj}v^h + N^i_{hjk}v^h = 0. \tag{4.15}$$

Using Eq. (4.15) *in Eq.* (4.14), *we get*

$$-(\dot{\partial}_r v^i)N^r_{jkh}y^h = \{N^i_{hjk} + (u_{jk} - u_{kj})\}v^h. \tag{4.16}$$

If the birecurrent vector field characterised by Eq. (4.1) *be assumed this stage to be independent of directional arguments then from Eq.* (4.16), *we get*

$$H^i_{khj} + N^i_{hjk} = u_{kj} - u_{jk}. \tag{4.17}$$

Therefore, we can state.

Theorem 4.3. *In a special normal projective Finsler space, the recurrence tensor* u_{jk} *of a birecurrent vector field* v^i *always satisfies Eq.*(4.17) *if the vector field* v^i *be assumed to be independent of directional arguments.*

If the recurrence tensor u_{jk} *of a birecurrent vector field* v^i *generating an affine motion be assumed to be independent of directional arguments then from Eq.*(4.9) *we immediately get* $u_{jk} = 0$ *and this observation leads to a contradiction. Hence, we can state:*

Theorem 4.4. *In a special normal projective Finsler space, the recurrence tensor of a birecurrent vector field generating an affine motion can not be independent of the directional arguments.*

We have already seen in the above discussion that whenever the recurrence tensor u_{jk} *satisfies Eq.*(4.6), *we get Eq.* (4.12), *which shows that the recurrence tensor is necessarily non-symmetric. Thus we conclude:*

Theorem 4.5. *In a special normal projective Finsler space if a birecurrent vector field generates an affine motion then its recurrence tensor is always non-symmetric.*

5. Special Type of Affine Motion

In this section, we will propose an affine motion generated by a vector field v^i whose covariant derivative is recurrent, i.e.

$$\nabla_j\nabla_k v^i = \lambda_j\nabla_k v^i, \tag{5.1}$$

where λ_j is a non-null covariant vector field. In this connection first of all we establish the following:

Theorem 5.1. *A vector field* $v^i(x^j)$ *satisfying first two of the following conditions must satisfy the third.*

(A) $v^m N^i_{jkm} = \mu_j\nabla_k v^i$,

(B) $\nabla_j\nabla_k v^i = \lambda_j\nabla_k v^k$,

(C) $\Pi^i_{jkr}(\nabla_s v^r)y^s = 0$,

where λ_j *and* μ_j *are non-zero covariant vector fields.*

Proof. Let us suppose that the vector field $v^i(x^j)$ satisfies (A) and (B) both. The Lie-derivative of Π^i_{jk} with respect to infinitesimal transformation generated by a vector field $v^i(x^j)$ is given by Eq.(4.2) which in view of (A) and (B) may be written as

$$\pounds_v \Pi^i_{jk} = (\lambda_j + \mu_j)\nabla_k v^i + \Pi^i_{hjk}(\nabla_r v^h)y^r. \tag{5.2}$$

Since Π^i_{jk} and Π^i_{jkr} are symmetric with respect to the indices j and k in Eq. (5.2), $(\lambda_j + \mu_j)\nabla_k v^i$ should also be symmetric in these two indices, i.e.

$$(\lambda_j + \mu_j)\nabla_k v^i = (\lambda_k + \mu_k)\nabla_j v^i. \tag{5.3}$$

Equation (5.3) automatically implies at least one of the following conditions:

$$\begin{aligned} \nabla_k v^i &= 0, \\ \lambda_j + \mu_j &= 0i, \\ \nabla_k v^i &= (\lambda_k + \mu_k)v^i. \end{aligned} \tag{5.4}$$

for some non-zero vector field v^i. If (5.4a) holds, Equation (5.2) reduces to (C). Thus the condition(C) holds good. Now let us suppose that a vector field $v^i(x^j)$ satisfies the condition (B). Differentiating (B) partially with respect to y^h and using the relevant commutation formula, we get

$$\nabla_j \dot\partial_h(\nabla_k v^i) + v^r\Pi^i_{khr} = (\dot\partial_h\lambda_j)(\nabla_k v^i) + \lambda_j\left\{\nabla_k(\dot\partial_h v^i) + v^r\Pi^i_{khr}\right\}. \tag{5.5}$$

Since λ_j and v^i both are supposed to be independent of directional arguments, hence from Eq.(5.5)

$$\nabla_j \dot\partial_h(\nabla_k v^i) + v^r\Pi^i_{khr} = \lambda_j v^r\Pi^i_{khr}. \tag{5.6}$$

We now further consider the case when Eq. (5.5) is assumed to satisfy any one of the following two conditions:

$$\begin{aligned} \dot\partial_h(\nabla_k v^i) &= 0, \\ (\dot\partial_h\lambda_j) &= 0. \end{aligned} \tag{5.7}$$

If Eq. (5.7) holds then Eq.(5.5) assumes the form

$$\nabla_j \dot\partial_h(\nabla_k v^i) = \lambda_j\left\{\nabla_k(\dot\partial_h v^i) + v^r\Pi^i_{khr}\right\}. \tag{5.8}$$

We now differentiate Eq. (5.7) partially with respect to y^h and use the relevant commutation formula to get $\nabla_h v^i = 0$, it is a trivial case. Hence in order to consider non-trivial case, we consider Eq. (5.7). In such a case Eq.(5.5) becomes

$$v^r\Pi^i_{khr} = (\dot\partial_h\lambda_j)(\nabla_k v^i) + \lambda_j\left\{\nabla_k(\dot\partial_h v^i) + v^r\Pi^i_{khr}\right\}. \tag{5.9}$$

If λ_j be used to be unity then from Eq. (5.9), we get

$$(\dot\partial_h)(\nabla_k v^i) + \nabla_k(\dot\partial_h v^i) = 0. \tag{5.10}$$

If we also assume the existence of Eq. (5.7) then from Eq.(5.10), we get

$$\nabla_k(\dot\partial_h v^i) = 0. \tag{5.11}$$

All the possibilities and assumptions lead us to the following statements. □

Theorem 5.2. *If a vector field $v^i(x^j)$ satisfies Eq.* (5.7) *then the recurrence vector is independent of directional arguments and also in this case the vector v^i is also independent of the directional arguments.*

Theorem 5.3. *The condition (A) is necessary for a vector field $v^i(x^j)$ satisfying the condition (B) where v^i is independent of the directional arguments, to generate an affine motion.*

Let us consider a vector field $v^i(x^j)$ satisfying

$$\alpha N^i_{jkh} = \beta_{kh}\nabla_j v^i, \tag{5.12}$$

in a Finsler space of recurrent curvature characterised by Eq. (2.6).

Dividing Eq. (5.12) *by α and putting $\frac{\beta_{kh}}{\alpha} = \bar\gamma_{kh}$, we get*

$$N^i_{jkh} = \bar\gamma_{kh}\nabla_j v^i. \tag{5.13}$$

Differentiating Eq. (5.13) *normal projective covariantly with respect to x^m, we get*

$$\nabla_m N^i_{jkh} = (\nabla_h\bar\gamma_{kh})\nabla_j v^i + \bar\gamma_{kh}\nabla_m\nabla_j v^i. \tag{5.14}$$

Using Eqs. (2.6) *and* (5.13) *in Eq.*(5.14)*, we get*

$$(\lambda_m \bar{\gamma}_{kh} - \nabla_m \bar{\gamma}_{kh}) \nabla_j v^i = \bar{\gamma}_{kh} \nabla_m \nabla_j v^i. \tag{5.15}$$

Since the tensor field $\bar{\gamma}_{kh}$ *is non vanishing, we may therefore, choose a tensor field satisfying* $\bar{\gamma}_{kh} f^{kh} = 1$. *Transvecting Eq.* (5.15) *by* f^{kh}*, we shall have the condition (B), where*

$$v^m = (\lambda_m \bar{\gamma}_{kh} - \nabla_m \bar{\gamma}_{kh}) f^{kh}. \tag{5.16}$$

Transvecting Eq. (5.13) *by* v^h*, we get*

$$N^i_{jkh} v^h = \bar{\gamma}_{kh} v^h \nabla_j v^i. \tag{5.17}$$

If we write $\bar{\gamma}_{kh} v^h = \alpha_k$ *then equation* (5.17) *can be seen to be identical with condition (A). Therefore, we can state:*

Theorem 5.4. *A vector field* v^i *generates an affine motion in a symmetric space characterised by* $\nabla_m N^i_{jkh} = 0$ *if there exists a non-zero scalar field* α *and a non-zero vector field* β_{kh} *satisfying Eq.* (5.12).

Theorem 5.5. *If a Finsler space of recurrent curvature admits a vector field* v^i *satisfying Eq.*(5.12) *then the vector field* v^i *is orthogonal to the recurrence vector if there exists a non-zero scalar field* α *and a non-zero tensor field* A_{kh} *satisfying Eq.*(5.12) *where* $A_{kh} = \nabla_h \alpha_k - \nabla_k \alpha_h$

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COEFFICIENT BOUNDS FOR CERTAIN SUBCLASSES OF CLOSE-TO-CONVEX AND QUASI-CONVEX FUNCTIONS WITH FIXED POINT

By

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Abstract

In this paper, we apply the concept of subordination to introduce certain subclasses of close-to-convex and quasi-convex functions with fixed point in the unit disc $E = \{z :| z |< 1\}$. We establish the upper bounds of the first four coefficients for these classes. This work will motivate the other researchers of this field to study some more relevant classes.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f in the unit disc $E = \{z :| z |< 1\}$ and which are of the form $f(z) = z+\sum_{k=2}^{\infty} a_k z^k$. By $\mathcal{S}$, we denote the class of functions $f \in \mathcal{A}$, which are univalent in E. Let $\mathcal{U}$ be the class of Schwarzian functions of the form $u(z) = \sum_{k=1}^{\infty} c_k z^k$, which are analytic in the unit disc E and satisfying the conditions $u(0) = 0$ and $| u(z) |< 1$.

Before defining our main classes, firstly we discuss the following standardized classes of univalent functions:

$\mathcal{S}^* = \left\{f(z) \in \mathcal{A} : Re\left(\dfrac{zf'(z)}{f(z)}\right) > 0,\ z \in E\right\}$, the class of starlike functions.

$\mathcal{K} = \left\{f(z) \in \mathcal{A} : Re\left(\dfrac{(zf'(z))'}{f'(z)}\right) > 0,\ z \in E\right\}$, the class of convex functions.

The classes $\mathcal{S}^*$ and $\mathcal{K}$ are related by the Alexander relation [3] as $f \in \mathcal{K}$ if and only if $zf' \in \mathcal{S}^*$.

Kaplan [7] introduced the concept of close-to-convex functions. A function $f \in \mathcal{A}$ is said to be close-to-convex if there exists a convex function h such that $Re\left(\dfrac{f'(z)}{h'(z)}\right) > 0$ or equivalently there exists a starlike function g such that $Re\left(\dfrac{zf'(z)}{g(z)}\right) > 0$. The class of close-to-convex functions is denoted by C. Further, Noor [11] established the class C^* of quasi-convex functions as

$$C^* = \left\{f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{h'(z)}\right) > 0, h \in \mathcal{K}, z \in E\right\}.$$

Every quasi-convex function is convex and close-to-convex and so is univalent. Also $f \in C^*$ if and only if $zf' \in C$.

Let f and g be two analytic functions in E. Then f is said to be subordinate to g (symbolically $f \prec g$) if there exists a Schwarzian function $u(z) \in \mathcal{U}$ such that $f(z) = g(u(z))$.

The class $\mathcal{P}[C, D]$ consists of the functions p analytic in E with $p(0) = 1$ and subordinate to $\dfrac{1 + Cz}{1 + Dz}$, $(-1 \leq D < C \leq 1)$. This class was established by Janowski [5] and so the functions in the class $\mathcal{P}[C, D]$ are known as Janowski-type functions. Kanas and Ronning [6] introduced an interesting class $\mathcal{A}(w)$ of analytic functions of the form

$$f(z) = (z - w) + \sum_{k=2}^{\infty} a_k (z - w)^k$$

and normalized by the conditions $f(w) = 0, f'(w) = 1$, where w is a fixed point in E.

Also the classes of w-starlike functions and w-convex functions were defined in [6] as follows:

$\mathcal{S}^*(w) = \left\{f \in \mathcal{A}(w) : Re\left(\dfrac{(z - w)f'(z)}{f(z)}\right) > 0, z \in E\right\}$,

and

$\mathcal{K}(w) = \left\{f \in \mathcal{A}(w) : 1 + Re\left(\dfrac{(z - w)f''(z)}{f'(z)}\right) > 0, z \in E\right\}$.

The class $\mathcal{S}^*(w)$ is defined by the geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$ and the corresponding class $\mathcal{K}(w)$ is defined by the property that the image of any circular arc centered at w is convex. For $w = 0$, the classes $\mathcal{S}^*(w)$ and $\mathcal{K}(w)$ agree with the well known classes of starlike and convex functions, respectively. Also it is obvious that $f \in \mathcal{K}(w)$ if and only if $(z-w)f' \in \mathcal{S}^*(w)$. Various authors such as Acu and Owa [1], Al-Hawary [2] and Olatunji and Oladipo [12] have worked on the classes of analytic functions with fixed point.

For $-1 \le B < A \le 1$, Singh and Singh [15] discussed the subclasses of $\mathcal{S}^*(w)$ and $\mathcal{K}(w)$ defined as

$$\mathcal{S}^*(w; A, B) = \left\{ f \in \mathcal{A}(w) : \frac{(z-w)f'(z)}{f(z)} \prec \frac{1 + A(z-w)}{1 + B(z-w)}, z \in E \right\},$$

and

$$\mathcal{K}(w; A, B) = \left\{ f \in \mathcal{A}(w) : \frac{((z-w)f'(z))'}{f'(z)} \prec \frac{1 + A(z-w)}{1 + B(z-w)}, z \in E \right\}.$$

For $A = 1, B = -1$, the classes $\mathcal{S}^*(w; A, B)$ and $\mathcal{K}(w; A, B)$ reduce to $\mathcal{S}^*(w)$ and $\mathcal{K}(w)$, respectively.

To avoid repetition throughout this paper, we assume that $-1 \le D < C \le 1$, $-1 \le B < A \le 1$, $z \in E$.

Motivated and stimulated by the above defined classes, we now introduce the following subclasses of w-close-to-convex and w-quasi-convex functions with subordination:

Definition 1.1. *A function $f(z) \in \mathcal{A}(w)$ is said to be in the class $C(w; A, B; C, D)$ if*

$$\frac{(z-w)f'(z)}{g(z)} \prec \frac{1 + C(z-w)}{1 + D(z-w)},$$

where $g(z) = (z-w) + \sum_{k=2}^{\infty} b_k (z-w)^k \in \mathcal{S}^*(w; A, B)$.

The following points are to be noted:

(i) $C(0; A, B; C, D) \equiv C(A, B; C, D)$, the subclass of close-to-convex functions studied by Singh and Mehrok [13].

(ii) $C(0; 1, -1; C, D) \equiv C(C, D)$, the subclass of close-to-convex functions studied by Mehrok [8].

(iii) $C(w; 1, -1; C, D) \equiv C(w; C, D)$, the subclass of w-close-to-convex functions.

(iv) $C(w; 1, -1; 1, -1) \equiv C(w)$, the class of w-close-to-convex functions.

(v) $C(0; 1, -1; 1, -1) \equiv C$, the class of close-to-convex functions.

Definition 1.2. *A function $f(z) \in \mathcal{A}(w)$ is said to be in the class $C_1(w; A, B; C, D)$ if*

$$\frac{(z-w)f'(z)}{h(z)} \prec \frac{1 + C(z-w)}{1 + D(z-w)},$$

where $h(z) = (z-w) + \sum_{k=2}^{\infty} d_k (z-w)^k \in \mathcal{K}(w; A, B)$.

The following observations are obvious:

(i) $C_1(0; A, B; C, D) \equiv C_1(A, B; C, D)$, the subclass of close-to-convex functions studied by Singh and Mehrok [13].

(ii) $C_1(0; 1, -1; C, D) \equiv C_1(C, D)$, the subclass of close-to-convex functions studied by Mehrok and Singh [9].

(iii) $C_1(w; 1, -1; C, D) \equiv C_1(w; C, D)$, the subclass of w-close-to-convex functions.

(iv) $C_1(w; 1, -1; 1, -1) \equiv C_1(w)$, the subclass of w-close-to-convex functions.

(v) $C_1(0; 1, -1; 1, -1) \equiv C_1$, the subclass of close-to-convex functions studied by Abdel-Gawad and Thomas [4].

Definition 1.3. *A function $f(z) \in \mathcal{A}(w)$ is said to be in the class $C^*(w; A, B; C, D)$ if*

$$\frac{((z-w)f'(z))'}{h'(z)} \prec \frac{1 + C(z-w)}{1 + D(z-w)},$$

where $h(z) = (z-w) + \sum_{k=2}^{\infty} d_k (z-w)^k \in \mathcal{K}(w; A, B)$.

We have the following observations:

(i) $C^*(0; A, B; C, D) \equiv C^*(A, B; C, D)$, the subclass of quasi-convex functions studied by Singh and Singh [14].

(ii) $C^*(0; 1, -1; C, D) \equiv C^*(C, D)$, the subclass of quasi-convex functions discussed by Singh and Singh [14].

(iii) $C^*(w; 1, -1; C, D) \equiv C^*(w; C, D)$, the subclass of w-quasi-convex functions.

(iv) $C^*(w; 1, -1; 1, -1) \equiv C^*(w)$, the class of w-quasi-convex functions.

(v) $C^*(0; 1, -1; 1, -1) \equiv C^*$, the class of quasi-convex functions.

Definition 1.4. *A function $f(z) \in \mathcal{A}(w)$ is said to be in the class $C_1^*(w; A, B; C, D)$ if*

$$\frac{((z-w)f'(z))}{g'(z)} \prec \frac{1 + C(z-w)}{1 + D(z-w)},$$

where $g(z) = (z-w) + \sum_{k=2}^{\infty} b_k (z-w)^k \in \mathcal{S}^*(w; A, B)$.

The following points are obvious:

(i) $C_1^*(0; A, B; C, D) \equiv C_1^*(A, B; C, D)$*, the subclass of quasi-convex functions studied by Singh and Singh* [14].
(ii) $C_1^*(0; 1, -1; C, D) \equiv C_1^*(C, D)$*, the subclass of quasi-convex functions discussed by Singh and Singh* [13].
(iii) $C_1^*(w; 1, -1; C, D) \equiv C_1^*(w; C, D)$*, the subclass of w-quasi-convex functions.*
(iv) $C_1^*(w; 1, -1; 1, -1) \equiv C_1^*(w)$*, the subclass of w-quasi-convex functions.*
(v) $C_1^*(0; 1, -1; 1, -1) \equiv C_1^*$*, the subclass of quasi-convex functions.*

In this paper, we seek upper bounds of the first four coefficients for the functions belonging to the classes $C(w; A, B; C, D)$, $C_1(w; A, B; C, D)$, $C^*(w; A, B; C, D)$ and $C_1^*(w; A, B; C, D)$. This paper will motivate the other researchers to investigate some more interesting classes.

2. Preliminary Results

Lemma 2.1 ([12]). *For* $u(z) = \sum_{k=1}^{\infty} c_k (z-w)^k$ *and* $p(z) = \dfrac{1 + Cu(z)}{1 + Du(z)} = 1 + \sum_{k=1}^{\infty} p_k (z-w)^k$*, we have,*

$$|p_n| \le \frac{(C-D)}{(1+d)(1-d)^n}, n \ge 1, |w| = d.$$

Lemma 2.2 ([15]). *If* $g(z) = (z-w) + \sum_{k=2}^{\infty} b_k (z-w)^k \in \mathcal{S}^*(w; A, B)$*, then*

$$|b_2| \le \frac{(A-B)}{1-d^2}, \tag{2.1}$$

$$|b_3| \le \frac{(A-B)}{2(1-d^2)^2}[(1+d) + (A-B)], \tag{2.2}$$

$$|b_4| \le \frac{(A-B)}{6(1-d^2)^3}[(A-B) + (1+d)][(A-B) + 2(1+d)], \tag{2.3}$$

and

$$|b_5| \le \frac{(A-B)}{24(1-d^2)^4}[(A-B) + (1+d)][(A-B) + 2(1+d)][(A-B) + 3(1+d)]. \tag{2.4}$$

Lemma 2.3 ([15]). *If* $h(z) = (z-w) + \sum_{k=2}^{\infty} d_k (z-w)^k \in \mathcal{K}(w; A, B)$*, then*

$$|d_2| \le \frac{(A-B)}{2(1-d^2)}, \tag{2.5}$$

$$|d_3| \le \frac{(A-B)}{6(1-d^2)^2}[(1+d) + (A-B)], \tag{2.6}$$

$$|d_4| \le \frac{(A-B)}{24(1-d^2)^3}[(A-B) + (1+d)][(A-B) + 2(1+d)], \tag{2.7}$$

and

$$|d_5| \le \frac{(A-B)}{120(1-d^2)^4}[(A-B) + (1+d)][(A-B) + 2(1+d)][(A-B) + 3(1+d)]. \tag{2.8}$$

3. Main Results

Theorem 3.1. *If* $f \in C(w; A, B; C, D)$*, then*

$$|a_2| \le \frac{(A-B) + (C-D)}{2(1-d^2)}, \tag{3.1}$$

$$|a_3| \le \frac{[(A-B) + (1+d)][(A-B) + 2(C-D)]}{6(1-d^2)^2}, \tag{3.2}$$

$$|a_4| \le \frac{[(A-B) + (1+d)][(A-B) + 2(1+d)][(A-B) + 3(C-D)]}{24(1-d^2)^3}, \tag{3.3}$$

and

$$|a_5| \le \frac{[(A-B) + (1+d)][(A-B) + 2(1+d)][(A-B) + 3(1+d)][(A-B) + 4(C-D)]}{120(1-d^2)^4}. \tag{3.4}$$

Proof. From Definition 1.1, by principle of subordination, we have

$$\frac{(z-w)f'(z)}{g(z)} = p(z) = \frac{1+Cu(z)}{1+Du(z)} = 1 + \textstyle\sum_{k=1}^{\infty} p_k(z-w)^k, \tag{3.5}$$

where $u(z) = \sum_{k=1}^{\infty} c_k(z-w)^k$.

Expansion of (3.5) leads to

$$1 + 2a_2(z-w) + 3a_3(z-w)^2 + 4a_4(z-w)^3 + 5a_5(z-w)^4 + ...$$
$$= [1+p_1(z-w)+p_2(z-w)^2+p_3(z-w)^3+p_4(z-w)^4+...][1+b_2(z-w)+b_3(z-w)^2+b_4(z-w)^3+b_5(z-w)^4+...]. \tag{3.6}$$

On equating the coefficients of $(z-w), (z-w)^2, (z-w)^3$ and $(z-w)^4$ in (3.6), it yields

$$2a_2 = p_1 + b_2, \tag{3.7}$$

$$3a_3 = p_2 + b_2p_1 + b_3, \tag{3.8}$$

$$4a_4 = p_3 + b_2p_2 + b_3p_1 + b_4, \tag{3.9}$$

and

$$5a_5 = p_4 + b_2p_3 + b_3p_2 + b_4p_1 + b_5. \tag{3.10}$$

By taking modulus and application of triangle inequality, the equations (3.7), (3.8), (3.9) and (3.10) transform to

$$2|a_2| \le |p_1| + |b_2|, \tag{3.11}$$

$$3|a_3| \le |p_2| + |b_2||p_1| + |b_3|, \tag{3.12}$$

$$4|a_4| \le |p_3| + |b_2||p_2| + |b_3||p_1| + |b_4|, \tag{3.13}$$

and

$$5|a_5| \le |p_4| + |b_2||p_3| + |b_3||p_2| + |b_4||p_1| + |b_5|. \tag{3.14}$$

Using Lemma 2.1 and inequality (2.1) in (3.11), the result (3.1) is obvious.
Again using Lemma 2.1 and inequalities (2.1) and (2.2) in (3.12), the simplification leads to the result (3.2).
Further using ineequalities (2.1), (2.2) and (2.3) and applying Lemma 2.1, the result (3.3) can be easily obtained from (3.13). □

On using inequalities (2.1), (2.2), (2.3) and (2.4) and application of Lemma 2.1 in (3.14), it leads to the result (3.4).

For $A = 1, B = -1$, Theorem 3.1 yields the following result:

Corollary 3.1. *If $f \in C(w; C, D)$, then*

$$|a_2| \le \frac{2+(C-D)}{2(1-d^2)},$$

$$|a_3| \le \frac{(3+d)[1+(C-D)]}{3(1-d^2)^2},$$

$$|a_4| \le \frac{(2+d)(3+d)[2+3(C-D)]}{12(1-d^2)^3},$$

and

$$|a_5| \le \frac{(2+d)(3+d)(5+3d)[1+2(C-D)]}{30(1-d^2)^4}.$$

On putting $A = 1, B = -1, C = 1, D = -1$, Theorem 3.1 agrees with the following result:

Corollary 3.2. *If $f \in C(w)$, then*

$$|a_2| \le \frac{2}{1-d^2},$$

$$|a_3| \le \frac{3+d}{(1-d^2)^2},$$

$$|a_4| \le \frac{2(2+d)(3+d)}{3(1-d^2)^3},$$

and

$$|a_5| \le \frac{(2+d)(3+d)(5+3d)}{6(1-d^2)^4}.$$

Theorem 3.2. *If $f \in C_1(w; A, B; C, D)$, then*

$$|a_2| \leq \frac{(A-B)+2(C-D)}{4(1-d^2)},$$

$$|a_3| \leq \frac{[(A-B)+(1+d)][(A-B)+3(C-D)]+3(C-D)(1+d)}{18(1-d^2)^2},$$

$$|a_4| \leq \frac{(A-B)[(A-B)+(1+d)][(A-B)+2(1+d)+4(C-D)]+12(C-D)(1+d)[(A-B)+2(1+d)]}{96(1-d^2)^3},$$

and

$$|a_5| \leq \frac{60(C-D)(1+d)^2[(A-B)+2(1+d)]}{600(1-d^2)^4}$$
$$+\frac{(A-B)[(A-B)+(1+d)]\{20(C-D)(1+d)+[(A-B)+2(1+d)][(A-B)+5(C-D)+3(1+d)]\}}{600(1-d^2)^4}.$$

Proof. With the application of principle of subordination in Definition 1.2 and using Lemma 2.1 and Lemma 2.3, the results of Theorem 3.2 can be easily obtained by following the procedure of Theorem 3.1. □

For $A = 1, B = -1$, the following result can be obtained from Theorem 3.2:

Corollary 3.3. *If $f \in C_1(w; C, D)$, then*

$$|a_2| \leq \frac{1+(C-D)}{2(1-d^2)},$$

$$|a_3| \leq \frac{(3+d)[2+3(C-D)]+3(C-D)(1+d)}{18(1-d^2)^2},$$

$$|a_4| \leq \frac{6(C-D)(1+d)(2+d)+(3+d)[2(C-D)+(2+d)]}{24(1-d^2)^3},$$

and

$$|a_5| \leq \frac{30(C-D)(2+d)(1+d)^2+(3+d)[10(C-D)(1+d)+(2+d)\{5(C-D)+3d+5\}]}{150(1-d^2)^4}.$$

For $A = 1, B = -1, C = 1, D = -1$, Theorem 3.2 yields the following result:

Corollary 3.4. *If $f \in C_1(w)$, then*

$$|a_2| \leq \frac{3}{2(1-d^2)},$$

$$|a_3| \leq \frac{15+7d}{9(1-d^2)^2},$$

$$|a_4| \leq \frac{13d^2+45d+42}{24(1-d^2)^3},$$

and

$$|a_5| \leq \frac{60(2+d)(1+d)^2+(3+d)[3d^2+41d+50]}{150(1-d^2)^4}.$$

Theorem 3.3. *If $f \in C^*(w; A, B; C, D)$, then*

$$|a_2| \leq \frac{(A-B)+(C-D)}{4(1-d^2)}, \tag{3.15}$$

$$|a_3| \leq \frac{[(A-B)+(1+d)][(A-B)+2(C-D)]}{18(1-d^2)^2}, \tag{3.16}$$

$$|a_4| \leq \frac{[(A-B)+(1+d)][(A-B)+2(1+d)][(A-B)+3(C-D)]}{96(1-d^2)^3}, \tag{3.17}$$

and

$$|a_5| \leq \frac{[(A-B)+(1+d)][(A-B)+2(1+d)][(A-B)+3(1+d)][(A-B)+4(C-D)]}{600(1-d^2)^4}. \tag{3.18}$$

Proof. From Definition 1.3, by principle of subordination, we have

$$\frac{((z-w)f'(z))'}{h'(z)} = p(z) = \frac{1+Cu(z)}{1+Du(z)} = 1 + \textstyle\sum_{k=1}^{\infty} p_k (z-w)^k, \tag{3.19}$$

where $u(z) = \sum_{k=1}^{\infty} c_k (z-w)^k$.
(3.19) can be expanded as
$1 + 4a_2(z-w) + 9a_3(z-w)^2 + 16a_4(z-w)^3 + 25a_5(z-w)^4 + ...$

$$= [1+p_1(z-w)+p_2(z-w)^2+p_3(z-w)^3+p_4(z-w)^4+...][1+2d_2(z-w)+3d_3(z-w)^2+4d_4(z-w)^3+5d_5(z-w)^4+...]. \tag{3.20}$$

On equating the coefficients of $(z-w), (z-w)^2, (z-w)^3$ and $(z-w)^4$ in (3.20), it yields

$$4a_2 = p_1 + 2d_2, \tag{3.21}$$

$$9a_3 = p_2 + 2d_2p_1 + 3d_3, \tag{3.22}$$

$$16a_4 = p_3 + 2d_2p_2 + 3d_3p_1 + 4d_4, \tag{3.23}$$

and

$$25a_5 = p_4 + 2d_2p_3 + 3d_3p_2 + 4d_4p_1 + 5d_5. \tag{3.24}$$

On applying Lemma 2.1, Lemma 2.3 and following the procedure of Theorem 3.1, the results (3.15), (3.16), (3.17) and (3.18) can be easily obtained. □

On putting $A = 1, B = -1$ in Theorem 3.3, the following result is obvious:

Corollary 3.5. *If $f \in C^*(w; C, D)$, then*

$$|a_2| \le \frac{2+(C-D)}{4(1-d^2)},$$

$$|a_3| \le \frac{(3+d)[1+(C-D)]}{9(1-d^2)^2},$$

$$|a_4| \le \frac{(2+d)(3+d)[2+3(C-D)]}{48(1-d^2)^3},$$

and

$$|a_5| \le \frac{(2+d)(3+d)(5+3d)[1+2(C-D)]}{150(1-d^2)^4}.$$

On putting $A = 1, B = -1, C = 1, D = -1$ in Theorem 3.3, the following result is obvious:

Corollary 3.6. *If $f \in C^*(w)$, then*

$$|a_2| \le \frac{1}{1-d^2},$$

$$|a_3| \le \frac{3+d}{3(1-d^2)^2},$$

$$|a_4| \le \frac{(2+d)(3+d)}{6(1-d^2)^3},$$

and

$$|a_5| \le \frac{(2+d)(3+d)(5+3d)}{30(1-d^2)^4}.$$

Theorem 3.4. *If $f \in C_1^*(w; A, B; C, D)$, then*

$$|a_2| \le \frac{2(A-B)+(C-D)}{4(1-d^2)},$$

$$|a_3| \le \frac{[(A-B)+(1+d)][3(A-B)+2(C-D)]+2(A-B)(C-D)}{18(1-d^2)^2},$$

$$|a_4| \le \frac{6(C-D)(1+d)[2(A-B)+(1+d)]+(A-B)[(A-B)+(1+d)][4(A-B)+8(1+d)+9(C-D)]}{96(1-d^2)^3},$$

and

$$|a_5| \le \frac{24(C-D)(1+d)^2[2(A-B)+(1+d)]}{600(1-d^2)^4} + \frac{(A-B)[(A-B)+(1+d)]\{36(C-D)(1+d)+[(A-B)+2(1+d)][5(A-B)+16(C-D)+15(1+d)]\}}{600(1-d^2)^4}.$$

Proof. With the application of principle of subordination in Definition 1.4 and using Lemma 2.1 and Lemma 2.2, the results of Theorem 3.4 can be easily obtained by following the procedure of Theorem 3.3. □

For $A = 1, B = -1$, Theorem 3.4 gives the following result:

Corollary 3.7. *If $f \in C_1^*(w; C, D)$, then*

$$|a_2| \leq \frac{4 + (C - D)}{4(1 - d^2)},$$

$$|a_3| \leq \frac{(3 + d)[3 + (C - D)] + 2(C - D)}{9(1 - d^2)^2},$$

$$|a_4| \leq \frac{3(1 + d)(5 + d)(C - D) + 8(2 + d)(3 + d)}{48(1 - d^2)^3},$$

and

$$|a_5| \leq \frac{6(C - D)(5 + d)(1 + d)^2 + (3 + d)[18(C - D)(1 + d) + (2 + d)\{16(C - D) + 25 + 15d\}]}{150(1 - d^2)^4}.$$

For $A = 1, B = -1, C = 1, D = -1$, Theorem 3.4 gives the following result:

Corollary 3.8. *If $f \in C_1^*(w)$, then*

$$|a_2| \leq \frac{3}{2(1 - d^2)},$$

$$|a_3| \leq \frac{19 + 5d}{9(1 - d^2)^2},$$

$$|a_4| \leq \frac{7d^2 + 38d + 39}{24(1 - d^2)^3},$$

and

$$|a_5| \leq \frac{4(5 + d)(1 + d)^2 + (3 + d)[5d^2 + 41d + 50]}{50(1 - d^2)^4}.$$

4. Conclusion

In the present work, we have estimated the bounds for the first four coefficients of the new defined subclasses of close-to-convex functions and quasi-convex functions with fixed point. Till now, no researcher has worked on such classes and so this work will pave the way for the other researchers to study some more interesting classes of analytic functions with fixed point.

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A NOTE ON THE DISTRIBUTION OF ZEROS OF POLYNOMIALS AND CERTAIN CLASS OF TRANSCENDENTAL ENTIRE FUNCTIONS

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Abstract

In the paper we wish to find a region containing all the zeros of a polynomial. Our result in some special case sharpen some very well known results obtained for this purpose. Also, we obtain a zero free region about an arbitary point for a certain class of transcendental entire functions by restricting the coefficients of its Taylor's series expansions to some conditions.
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1. Introduction

To find a region containing all the zeros of a polynomial, Cauchy {cf. [15]} introduced the following classical result:

Theorem A ([15]). *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n, then all the zeros of $P(z)$ lie in* $|z| \leq 1 + max_{0\leq j\leq(n-1)}|\frac{a_j}{a_n}|$.

Theorem A was improved in several ways by many researchers {cf. [6], [12] & [16]}. As an improvement of Theorem A, Joyal et al. [10] gave the following theorem:

Theorem B ([10]). *If $P(z) = \sum_{j=0}^{n} a_j z^j$ $(a_n = 1)$ is a polynomial of degree n and $\beta = \max\limits_{0\leq j<n-1} |a_j|$, then all the zeros of $P(z)$ lie in*

$$|z| \leq \frac{1}{2}\left\{1+|a_{n-1}| + \sqrt{(1-|a_{n-1}|)^2 + 4\beta}\right\}.$$

Theorem B gives no improvement of Theorem A if $\beta = |a_{n-1}|$.

Again, in a different direction G. Enström and S. Kakeya [8] established following result known as Enström-Kakeya theorem.

Theorem C ([8]). *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying* $0 \leq a_0 \leq a_1 \leq ... \leq a_n$, *then all the zeros of $P(z)$ lie in* $|z| \leq 1$.

Many improvements and generalizations of Theorem C for polynomials and analytic functions are seen in the existing literature {cf. [1]- [5],[7],[8],[10], [11] & [14]}.

We recall that an entire function f of one complex variable z is a function analytic in the finite complex plane $\mathbb{C}$ and therefore it can be represented by an everywhere convergent power series like

$$f(z) = c_0 + c_1 z + ... + c_n z^n + ...$$

where $c_i, i = 0, 1, ..., n, ...$ are real or complex constants. Thus entire functions can be thought of as the natural generalization of polynomials.

The prime concern of this paper is to improve Theorem A as well as Theorem B in some special case and also derive a zero free region of a certain class of transcendental entire functions with restricted coefficients. We do not explain the standard theories, notations and definitions of entire functions as those are available in [17].

2. Lemma

In this section we present a lemma which will be needed in the sequel.

Lemma 2.1. *Let $\{f_n(z)\}, n = 1, 2, ...$ be a sequence of functions that are analytic in a region D and that converges uniformly to a function $f(z)$ in every closed sub region of D. Let z_0 be an interior point of D. If z_0 is a limit point of the zeros of $f_n(z)$, then z_0 is a zero of $f(z)$. Conversely, if z_0 is an m-fold zero of $f(z)$, every sufficiently small neighborhood of z_0 contains exactly m zeros (counted with their multiplicities) of each f_n with $n > N$ for a sufficiently large integer N.*

Remark 2.1. Lemma 2.1 is known as Hurwitz theorem in $\mathbb{C}$.

3. Theorems

In this section we present the main results of the paper.

Theorem 3.1. *Let* $P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ *be a polynomial of degree* $n > 1$ *and* $M = \max\{|a_0|, |a_1 - a_0|, ..., |a_{n-1} - a_{n-2}|\}$*.Then all the zeros of* $P(z)$ *are contained in the closed disc* $|z| \leq R$ *where*

$$R = \frac{1}{2|a_n|}\left\{|a_n|+|a_n - a_{n-1}| + \sqrt{(|a_n|-|a_n - a_{n-1}|)^2 + 4|a_n|.M}\right\}.$$

Proof. Let

$$Q(z) = (1 - z)P(z)$$

$$\text{i.e, } Q(z) = -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... + (a_1 - a_0)z + a_0.$$

Then,

$$\begin{aligned}|Q(z)| &\geq |a_n||z|^{n+1} - |(a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + ... + (a_1 - a_0)z + a_0| \\ &\geq (|a_n||z| - |a_n - a_{n-1}|)|z|^n - |(a_{n-1} - a_{n-2})z^{n-1} + ... + (a_1 - a_0)z + a_0|. \end{aligned} \tag{3.1}$$

Now for $|z| = r(> 1)$, it follows that

$$\begin{aligned}&|(a_{n-1} - a_{n-2})z^{n-1} + (a_{n-2} - a_{n-3})z^{n-2} + ... + (a_1 - a_0)z + a_0| \\ &\leq |a_{n-1} - a_{n-2}|r^{n-1} + |a_{n-2} - a_{n-1}|r^{n-2} + ... + |a_1 - a_0|r + |a_0| \\ &\leq M.r^n\left\{\frac{1}{r} + \frac{1}{r^2} + ... + \frac{1}{r^n}\right\} \text{ where } M = \max\{|a_0|, |a_1 - a_0|, ..., |a_{n-1} - a_{n-2}|\} \\ &\leq M.r^n \textstyle\sum_{j=1}^{\infty} \frac{1}{r^j} \\ &= M.r^n \frac{1}{r - 1}.\end{aligned}$$

Hence from (3.1), we get for $|z| = r(> 1)$ that

$$\begin{aligned}|Q(z)| &\geq (|a_n|r - |a_n - a_{n-1}|)r^n - M.r^n\frac{1}{r-1} \\ &= \frac{r^n}{r-1}\left\{|a_n|r^2 - (|a_n|+|a_n - a_{n-1}|)r + |a_n - a_{n-1}| - M\right\} > 0\end{aligned}$$

$$\text{if } |a_n|r^2 - (|a_n|+|a_n - a_{n-1}|)r + |a_n - a_{n-1}| - M > 0$$

$$\text{i.e, if } r > \frac{1}{2|a_n|}\left\{|a_n|+|a_n - a_{n-1}| + \sqrt{(|a_n|-|a_n - a_{n-1}|)^2 + 4|a_n|.M}\right\}.$$

Thus, no zero of $Q(z)$ outside the unit circle lies in

$$|z| > \frac{1}{2|a_n|}\left\{|a_n|+|a_n - a_{n-1}| + \sqrt{(|a_n|-|a_n - a_{n-1}|)^2 + 4|a_n|.M}\right\}.$$

Since zeros of $P(z)$ are the zeros of $Q(z)$, all the zeros of $P(z)$ lie in

$$|z| \leq \frac{1}{2|a_n|}\left\{|a_n|+|a_n - a_{n-1}| + \sqrt{(|a_n|-|a_n - a_{n-1}|)^2 + 4|a_n|.M}\right\}.$$

This proves the theorem. □

Remark 3.1. If $M > |a_n - a_{n-1}|$, Theorem 3.1 is an improvement of both Theorem A & Theorem B. The following example justifies the validity of the sharpness of Theorem 3.1.

Example 3.1. Let $P(z) = 4z^6 + 3z^5 + 2z^4 - z^2 - z - 4$.

Here, $a_0 = -4,\ a_1 = -1,\ a_2 = -1,\ a_3 = 0,\ a_4 = 2,\ a_5 = 3\ \&\ a_6 = 4$.

Hence all the zeros of $P(z)$ by Theorem 3.1 lie in $|z| \leq 1.69$ whereas by Theorem B lie in $|z| \leq 1.88$ and by Theorem A lie in $|z| \leq 2$.

Theorem 3.2. *Let* $f(z) = a_0 + a_{n_1}z^{n_1} + ... + a_{n_l}z^{n_l} + a_{n_m}z^{n_m} + ...$ *be an entire function with* $a_0 \neq 0$ *and* $n_1, n_2, ..., n_l, n_m, ...$ *are positive integers such that* $1 \leq n_1 < ... < n_l < n_m < ....$ *Also let for some positive integer* l

$$|a_{n_l}| \geq |a_{n_m}| \geq ... .$$

Then no zero of $f(z)$ *lies in*

$$|z| < \frac{|a_{n_0}|}{|a_{n_0}| + M}$$

where $M = \max\{|a_{n_1}|, |a_{n_2}|, ..., |a_{n_l}|\}$.

Proof. Let

$$f_k(z) = a_0 + a_{n_1} z^{n_1} + ... + a_{n_l} z^{n_l} + a_{n_m} z^{n_m} + ... + a_{n_k} z^{n_k} .$$

Also, let

$$F(z) = z^{n_k} f_k(\frac{1}{z})$$

$$\text{i.e, } F(z) = a_0 z^{n_k} + a_{n_1} z^{n_k - n_1} + ... + a_{n_l} z^{n_k - n_l} + a_{n_m} z^{n_k - n_m} + ... + a_{n_k} .$$

Now for $|z| = r(> 1)$, we get that

$$\begin{aligned}
&|a_{n_1} z^{n_k - n_1} + a_{n_2} z^{n_k - n_2} + ... + a_{n_l} z^{n_k - n_l} + a_{n_m} z^{n_k - n_m} + ... + a_{n_k}| \\
&\leq |a_{n_1}| r^{n_k - n_1} + |a_{n_2}| r^{n_k - n_2} + ... + |a_{n_l}| r^{n_k - n_l} + |a_{n_m}| r^{n_k - n_m} + ... + |a_{n_k}| \\
&\leq M r^{n_k} \left\{ \frac{1}{r^{n_1}} + \frac{1}{r^{n_2}} + ... + \frac{1}{r^{n_l}} + \frac{1}{r^{n_m}} + ... + \frac{1}{r^{n_k}} \right\} \text{ where } M = \max\{|a_{n_1}|, |a_{n_2}|, ..., |a_{n_l}|\} \\
&\leq M r^{n_k} \left\{ \frac{1}{r} + \frac{1}{r^2} + ... + \frac{1}{r^l} + \frac{1}{r^m} + ... + \frac{1}{r^k} \right\} \text{ since } n_i \geq i \;\&\; \frac{1}{r^{n_i}} \leq \frac{1}{r^i} \text{ for } i = 1,\ 2,\ 3, ...,\ k \\
&\leq M r^{n_k} \textstyle\sum_{j=1}^{\infty} \frac{1}{r^j} \\
&= M . r^{n_k} \frac{1}{r - 1} .
\end{aligned}$$

Hence for $|z| = r(> 1)$, it follows that

$$|F(z)| \geq |a_0| r^{n_k} - M r^{n_k} \frac{1}{r - 1} > 0 \text{ if } r > \frac{|a_{n_0}| + M}{|a_{n_0}|} .$$

Therefore,

$$|F(z)| > 0 \text{ if } |z| > \frac{|a_{n_0}| + M}{|a_{n_0}|} .$$

Consequently,

$$|f_k(z)| > 0 \text{ if } |z| < \frac{|a_{n_0}|}{|a_{n_0}| + M} .$$

Thus, no zero of the partial sum $f_k(z)$ is contained in $|z| < \frac{|a_{n_0}|}{|a_{n_0}|+M}$. Hence by Lemma 2.1, $f(z)$ does not vanish in

$$|z| < \frac{|a_{n_0}|}{|a_{n_0}| + M} .$$

Thus the theorem is established. □

Remark 3.2. The following example with related figure ensures the validity of Theorem 3.2.

Example 3.2. Let $f(z) = z \sin z^2 + 4z^3 - 3z^2 + z + 4$.
Then the Taylor's series expansion of $f(z)$ is

$$f(z) = 4 + z - 3z^2 + 5z^3 - \frac{z^7}{3!} + \frac{z^{11}}{5!} - \frac{z^{15}}{7!} + ...$$

Here, $a_0 = 4$ & $M = 5$.
Hence by Theorem 3.2, $f(z)$ does not vanish in

$$|z| < 0.44 .$$

Justification

Taking $f_1(z) = 4$ & $f_2(z) = z \sin z^2 + 4z^3 - 3z^2 + z$, it follows that

$$|f_2(z)| < |f_1(z)| \text{ for } |z| = 0.44.$$

Hence by very well known Rouche's theorem, $f(z)$ has no zero in $|z| < 0.44$.

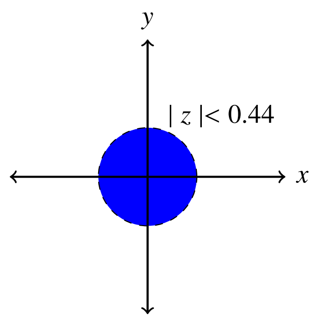

Figure 3.1: Zero free region of $f(z) = z\sin z^2 + 4z^3 - 3z^2 + z + 4$ about origin

Theorem 3.3. *Let $f(z) = a_0 + a_1 z + a_2 z^2 + ...$ be an entire function with $a_0 \neq 0$. Also, let for some positive integers l, m*

$$|a_{2l}| \geq |a_{2l+2}| \geq |a_{2l+4}| \geq ...$$

and

$$|a_{2m+1}| \geq |a_{2m+3}| \geq |a_{2m+5}| \geq .... .$$

Then $f(z)$ does not vanish in $|z| < \frac{|a_0|}{|a_0|+2M}$ where $M = \max\limits_{\substack{0\leq i\leq l \\ 0\leq j\leq m}} \{|a_{2i}|,\ |a_{2j+1}|\}$.

Proof. Let

$$f_n(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$$

and

$$F(z) = z^n f_n(\frac{1}{z}).$$

Again, let

$$\begin{aligned}
&Q(z) = (z^2-1)F(z) \\
\text{i.e, } &Q(z) = (z^2-1)(a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + ... + a_{2l} z^{n-2l} + a_{2l+1} z^{n-2l-1} + ... + a_{2m} z^{n-2m} \\
&\quad + a_{2m+1} z^{n-2m-1} + ... + a_n) \\
\text{i.e, } &Q(z) = a_0 z^{n+2} + a_1 z^{n+1} + (a_2 - a_0) z^n + (a_3 - a_1) z^{n-1} + ... + (a_{2l+1} - a_{2l-1}) z^{n-2l+1} \\
&\quad + (a_{2l+2} - a_{2l}) z^{n-2l} + ... + (a_{2m+1} - a_{2m-1}) z^{n-2m+1} + (a_{2m+2} - a_{2m}) z^{n-2m} + ... - a_{n-1} z - a_n \\
\text{i.e, } &Q(z) = a_0 z^{n+2} + P(z)\ . \qquad (3.2)
\end{aligned}$$

Now for $|z| = r(> 1)$, we get that

$$\begin{aligned}
|P(z)| =& |a_1 z^{n+1} + (a_2 - a_0) z^n + (a_3 - a_1) z^{n-1} + ... + (a_{2l+1} - a_{2l-1}) z^{n-2l+1} + (a_{2l+2} - a_{2l}) z^{n-2l} + \\
& ... + (a_{2m+1} - a_{2m-1}) z^{n-2m+1} + (a_{2m+2} - a_{2m}) z^{n-2m} + ... - a_{n-1} z - a_n| \\
\leq& |a_1| r^{n+1} + (|a_2| + |a_0|) r^n + (|a_3| + |a_1|) r^{n-1} + ... + (|a_{2l+1}| + |a_{2l-1}|) r^{n-2l+1} + (|a_{2l+2}| + |a_{2l}|) r^{n-2l} + \\
& ... + (|a_{2m+1}| + |a_{2m-1}|) r^{n-2m+1} + (|a_{2m+2}| + |a_{2m}|) r^{n-2m} + ... + |a_{n-1}| r + |a_n| \\
\leq& 2Mr^{n+2} \left\{ \frac{1}{r} + \frac{1}{r^2} + ... + \frac{1}{r^{n+2}} \right\} \text{ where } M = \max_{\substack{0\leq i\leq l \\ 0\leq j\leq m}} \{|a_{2i}|,\ |a_{2j+1}|\} \\
\leq& 2Mr^{n+2} \sum_{k=1}^{\infty} \frac{1}{r^k} \\
=& 2Mr^{n+2} \frac{1}{r-1}\ .
\end{aligned}$$

Hence for $|z| = r(> 1)$, it follows from (3.2) that

$$|Q(z)| \geq |a_0| r^{n+2} - 2Mr^{n+2}.\frac{1}{r-1} > 0 \text{ if } r > \frac{|a_0| + 2M}{|a_0|}$$

$$\text{i.e, } |Q(z)| > 0 \text{ if } |z| > \frac{|a_0| + 2M}{|a_0|}\ .$$

Therefore,

$$|F(z)| > 0 \text{ if } |z| > \frac{|a_0| + 2M}{|a_0|}$$
$$\text{i.e, } |f_n(\frac{1}{z})| > 0 \text{ if } |z| > \frac{|a_0| + 2M}{|a_0|}$$
$$\text{i.e, } |f_n(z)| > 0 \text{ if } |z| < \frac{|a_0|}{|a_0| + 2M} .$$

Thus, it follows by Lemma 2.1 that

$$|f(z)| > 0 \text{ if } |z| < \frac{|a_0|}{|a_0| + 2M} .$$

This completes the proof of the theorem. □

Remark 3.3. The following example with related figure ensures the validity of Theorem 3.3.

Example 3.3. Let $f(z) = z^2 \sin 2z + \cos z$.
Now the Taylor's series expansion of $f(z)$ is

$$f(z) = 1 - \frac{z^2}{2} + 2z^3 + \frac{z^4}{24} - \frac{4z^5}{3} - ... .$$

Here, it follows that

$$|a_0| \geq |a_2| \geq |a_4| \geq ...$$

and

$$|a_3| \geq |a_5| \geq |a_7| \geq ... .$$

Hence by Theorem 3.3, $f(z)$ does not vanish in

$$|z| < 0.2 .$$

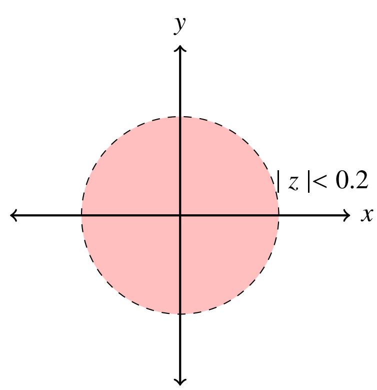

Figure 3.2: Zero free region of $f(z) = z^2 \sin 2z + \cos z$ about origin

4. Future prospect

In the line of the works as carried out in the paper one may think of the zero free region of transcendental entire functions having general infinite series except Taylor series.

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COMPARISON OF SUMMABILITY AND CESÅRO $|(C,\ \alpha)|_p$ SUMMABILITY

By

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Abstract

Summability is a branch of mathematical analysis in which an innite series which is usually divergent can converge to a finite sum s (say) by ordinary summation techniques and become summable with the help of dierent summation means or methods. C method was given by Ernesto Cesåro such that ordinary Cesåro summation was written as (C, l) summation whereas generalised Cesåro summation was given as (C, α). In 1913, Hardy [3] proved a theorem on (C, a), $a > 0$ summability of the series. In this paper, comparison of relative strength of absolute summabilities for functions has been investigated on different sets of parameters.

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Keywords and Phrases: (D, k) means, (C, α) means, (C, α, b) means, $(D, k)(C, \alpha)$ product means, Fourier Series,Conjugate Series, Lebesgue Integral.

1. Introduction

Kuttner [4] introduced the summability method (D, α) for functions and investigated some of its properties. Pathak [13] discussed relative strength of summability $|(D, k)(C, l)|_p$ and absolute Cesro summability. Mishra and Srivastava [7] introduced the Summability method (C, α, β) for functions by generalizing (C, α) summability method. In this paper, we discuss relative strength of summability $|(D, k)(C, \alpha, \beta)|_p$ and absolute Cesro summability for functions and investigate a relation between different sets of parameters $(\alpha \geq 0, p \geq 1, \beta > -1)$.

2. Some Definitions

Let $f(x)$ be any function which is Lebesgue-measurable, and that$f$: $[0, +\infty) \to R$, and integrable in $(0, x)$ for any finite$x$ and which is bounded in some right hand neighborhood of origin. Integrals of the form $\int_0^\infty$ are throughout to be taken as $\lim_{x\to\infty} \int_0^x, \int_0^x$ being a Lebesgue integral.

Let $k > 0$. If, for $t > 0$, the integral

$$g(t) = g^{(k)}(t) = kt\int_0^\infty \frac{x^{k-1}}{(x+t)^{k+1}} f(x)dx \tag{2.1}$$

exists and if $g(t) \to s$ as $t \to \infty$, we say that function $f(x)$ is summable (D, k) to the sum s and we write $f(x) \to s(D, k)$ as $x \to \infty$.

We note that, for any fixed $t > 0$, $k > 0$, it is necessary and sufficient for convergence of (2.1) that

$$\int_1^\infty \frac{f(x)}{x^2}\text{dx}, \tag{2.2}$$

should converge.

The (C, α, β) transform of $f(x)$, which we denote by $\partial_{\alpha,\beta}(x)$ is given by

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} \frac{1}{x^{\alpha+\beta}} \int_0^x (x-y)^{\alpha-1} y^\beta f(y)dy,\ (\alpha > 0, \beta > -1) \tag{2.3}$$

If this exists for $x > 0$ and $\partial_{\alpha,\beta}(x)$ tends to a limit $s$as $x \to \infty$, we say that$f(x)$ is summable (C, α, β) to s, and we write $f(x) \to s(C, \alpha, \beta)$. We also write

$$U_{k,\alpha,\beta}(t) = kt\int_0^\infty \frac{x^{k-1}}{(x+t)^{k+1}} \partial_{\alpha,\beta}(x)dx, \tag{2.4}$$

If this exists, and tends to a limit s as $t \to \infty$, we say that the function $f(x)$ is summable $(D, k)(C, \alpha, \beta)$ to s.

When $\beta = 0$, $(D, k)(C, \alpha, \beta)$ and $(D, k)(C, \alpha)$ denote the same method .

If $\alpha \geq 0, p \geq 1, \beta > -1$, we say that $f(y)$ is is summable $|C, \alpha, \beta|_p$ (absolutely summable (C,α,β)) with index p given by a result due to Deepmala et al. [1], if

$$\int_T^\infty y^{p-1} \left| \frac{d}{\text{dy}} \partial_{\alpha,\beta}(y) \right|^p dy < \infty \text{ for some } T \geq 0. \tag{2.5}$$

This is analogue (The (C,α,β) transform of $f(x)$) for functions of definition for sequences given by Flett [2]. In any result involving $|C,\alpha,\beta|_p$ for values of $\alpha < 1$, we restrict ourselves to the case in which $f(y)$ is an indefinite Lebesgue integral of a function $a(y)$, say; this ensures that the derivative $\left(\frac{d}{\mathrm{dy}}\partial_{\alpha,\beta}(y)\right)$ exists almost everywhere (by a result due to Mishra and Mishra [6]).

Such a restriction is not, however, needed when $\alpha \geq 1$. By analogy with Flett [2], it might at first sight appear and one should define $|C,\alpha,\beta|_p$-summability by

$$\int_0^\infty y^{p-1}\left|\frac{d}{dy}\partial_{\alpha,\beta}(y)\right|^p dy < \infty, (\alpha \geq 0, \beta > -1, p > 1). \tag{2.6}$$

Further suppose that $k > 0$, $\beta > -1$, $\alpha > 0$ and $p \geq 1$. Then we say that the function $f(y)$ is summable $|(D,k)(C,\alpha,\beta)|_p$ or absolutely summable $(D,k)(C,\alpha,\beta)$ with index p, if the integral defined by

$$U_{k,\alpha,\beta}(y) = ky\int_0^\infty \frac{x^{k-1}}{(x+y)^{k+1}}\partial_{\alpha,\beta}(x)dx$$

converges for all $y > 0$, and

$$\int_1^\infty y^{-1}\left|y\frac{d}{dy}U_{k,\alpha,\beta}\right|^p dy < \infty. \tag{2.7}$$

3. Main Results

In this section, we have the following theorems on the relative strength between $|C,\gamma,\beta|_p$and $|(D,k)(C,\alpha,\beta)|_p$.

Theorem 3.1. *Let $\alpha > \gamma \geq 0, p \geq 1, \beta > -1$. If $f(x)$ is summable $|C,\gamma,\beta|_p$, then it is summable$|C,\alpha,\beta|_p$.*

Theorem 3.2. *$\alpha \geq 0, p \geq 1, \gamma \geq 0$. If $f(x)$ is summable $|C,\gamma,\beta|_p$, and the integral defined by $U_{k,\alpha-1,\beta}(y)$ exists for all $y > 0$,then $f(x)$ is summable $|(D,k)(C,\alpha,\beta)|_p$if $k \leq 1$. Also the convergence of $\int_1^\infty \frac{\partial_{\alpha,\beta}(x)}{x^2}dx$is implied by $|C,\gamma,\beta|_p$summability of $f(x)$.*

We first prove this theorem under definition (2.7). However ,if the result holds with (2.7), then it must also hold under the definition of (2.5). This follows from the following two Lemmas

Lemma 3.1. *Let $p \geq 1, \gamma > 1$. Suppose that $f(x) \in L(0,x)$ for finite $x > 0$.Suppose that $|C,\gamma,\beta|_p$,according to the definition* (2.5)*. Define*

$$\bar{f}(x) = \begin{cases} f(x) & \text{for } x \geq T \\ 0 & \text{for } x < T \end{cases}. \tag{3.1}$$

Let $\bar{\partial}_{\gamma,\beta}(y)$ denote the expression corresponding to $\partial_{\gamma,\beta}(y)$ with $f(x)$ replaced by $\bar{f}(x)$. Then

$$\int_0^\infty y^{p-1}\left|\frac{d}{dy}\partial_{\gamma,\beta}(y)\right|^p dy < \infty. \tag{3.2}$$

Thus$\bar{f}(x)$ is summable $|C,\gamma,\beta|_p$under the definition (2.7)*.(By a result due to Mishra and Mishra* [5]*).*

Lemma 3.2. *Let the hypothesis be as in Lemma* 3.1*, and define $f(x)$ as above. Let $k > 0, \beta > -1$and$\alpha > 0$.Then $|(D,k)(C,\alpha,\beta)|_p$summability of $\{f(x)\}$ and $\left\{\overline{f}(x)\right\}$are equivalent.*

Proof of Lemma 3.1. It is given that , for some $T > 0$,

$$\int_T^\infty x^{p-1}\left|\frac{d}{dx}\partial_{\alpha,\beta}(x)\right|^p dx < \infty. \tag{3.3}$$

Since, if (3.3) holds for given T, it holds for any greater T, it must hold for all sufficiently large T. Now since $\gamma > 1$, therefore by standard properties of fractional integrals, we have

$$\int_0^T (T-u)^{\gamma-2}u^\beta\,|f(u)|\,du < \infty, \tag{3.4}$$

for almost all T (and thus , in particular, for some arbitrary large T), we may thus suppose that T should be chosen so that (3.3) and (3.4) hold. Since $\bar{\partial}_{\gamma,\beta}(x) = 0$ for $x < T$, (3.2) will follow if

$$\int_T^\infty x^{p-1}\left|\frac{d}{dx}\partial_{\gamma,\beta}(x)\right|^p dx < \infty.$$

Since (3.3) holds, this will follow from Minkowski's inequality if we prove that

$$\int_T^\infty x^{p-1}\left|\frac{d}{dx}\left\{\bar{\partial}_{\gamma,\beta}(x) - \partial_{\gamma,\beta}(x)\right\}\right|^p dx < \infty. \tag{3.5}$$

Now , it follows at once from the definition that, for$x > T$,

$$\bar{\partial}_{\gamma,\beta}(x) - \partial_{\gamma,\beta}(x) = \frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)}\frac{1}{x^{\gamma+\beta}}\int_0^T (x-y)^{\gamma-1}y^\beta \bar{f}(y)\mathrm{dy} - \frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)}\frac{1}{x^{\gamma+\beta}}\int_0^T (x-y)^{\gamma-1}y^\beta \bar{f}(y)dy$$
$$= \frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)}\frac{1}{x^{\gamma+\beta}}\int_0^T (x-y)^{\gamma-1}\left\{\bar{f}(y) - f(y)\right\}\mathrm{dy}$$
$$= \frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)}\frac{1}{x^{\gamma+\beta}}\int_0^T (x-y)^{\gamma-1}y^\beta f(y)dy.$$

It follows easily that

$$\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)}\frac{1}{x^{\gamma+\beta+1}}\int_0^T [\beta(x-y)+(x-\gamma y)]\,(x-y)^{\gamma-2}y^\beta f(y)dy.$$

For relevant values of variables

$$\left|x-\mathrm{y}\right| \le x+\gamma y \le x+\gamma x,\ \text{so that}$$

$$\left|\frac{d}{\mathrm{dx}}\left\{\bar{\partial}_{\gamma,\beta}(x) - \partial_{\gamma,\beta}(x)\right\}\right| \le \left|\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)}\frac{1}{x^{\gamma+\beta+1}}\int_0^T [\beta(x-y)+(x-\gamma y)]\,(x-y)^{\gamma-2}y^\beta f(y)dy\right|$$
$$\le \frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)}\frac{(\beta+\gamma+1)x}{x^{\gamma+\beta+1}}\int_0^T (x-y)^{\gamma-2}y^\beta\,|f(y)|\,\mathrm{dy}.$$

If $\gamma \le 2$, then for $x > T$, we have $(x-y)^{\gamma-2} \le (T-y)^{\gamma-2}$, so that

$$\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)}\frac{(\beta+\gamma+1)x}{x^{\gamma+\beta}}\int_0^T (x-y)^{\gamma-2}y^\beta\,|f(y)|\,dy$$

$= \frac{\text{Const.}}{x^{\beta+\gamma}}$ by (3.4).

If $\gamma \ge 2$, then $(x-y)^{\gamma-2} \le x^{\gamma-2}$, so that

$$\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)}\frac{(\beta+\gamma+1)x}{x^{\beta+2}}\int_0^T |f(y)|\,dy$$

$= \frac{\text{Const.}}{x^{\beta+2}}$.

Since $\gamma > 1$, (3.5) will follow in any case.

Proof of Lemma 3.2. We use notations as in Lemma 3.1, and write further $\bar{U}_{k,\alpha,\beta}(y)$ for the expression corresponding to $U_{k,\alpha,\beta}(y)$ but with $f(x)$ replaced by $\bar{f}(x)$ (by a result due to Mishra et al. [9]).

We know that for any fixed $y > 0, k > 0, \beta > -1, \alpha > 0$ convergence of

$U_{k,\alpha,\beta}(y) = ky\int_0^x \frac{x^{k-1}}{(x+y)^{k+1}}\partial_{\alpha,\beta}(x)dx$, is equivalent to the convergence of $\int_1^\infty \frac{\partial_{\alpha,\beta}(x)}{x^2}dx$.Then the conclusion will follow from Minkowski's inequality, if we show that

$$\int_1^\infty y^{p-1}\left|\frac{d}{dy}\left\{U_{k,\alpha,\beta}(y) - \bar{U}_{k,\alpha,\beta}(y)\right\}\right|^p dy < \infty, \tag{3.6}$$

where we take (3.6) as including the assertion that the integral defined by $U_{k,\alpha,\beta}(y) - \bar{U}_{k,\alpha,\beta}(y)$ converges for all $y > 0$. For large y,we have

$$\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}\frac{1}{y^{\alpha+\beta}}\int_0^T (y-x)^{\alpha-1}x^\beta f(x)dx = \frac{1}{y^{\alpha+\beta}}y^{\alpha-1}\int_0^T (y-x)^{\alpha-1}x^\beta\,|f(x)|\,dx$$
$$= O(\frac{1}{y^{\alpha+\beta}})\int_0^T x^\beta dx$$
$$= O(\frac{T}{y^{\alpha+\beta}})^{\beta+1} = O(\frac{1}{y})^{\beta+1}, (T < y). \tag{3.7}$$

Hence the convergence of $ky\int_0^x \frac{x^{k-1}}{(x+y)^{k+1}}\partial_{\alpha,\beta}(x)\left\{\partial_{\alpha,\beta}(x) - \bar{\partial}_{\alpha,\beta}(x)\right\}dx$, follows at once by a result due to Mishra and Mishra [5]. Now (3.6) is equivalent to

$$\int_1^\infty y^{p-1}dy\left|c\int_0^\infty \frac{x^{k-1}}{(x+y)^{k+2}}(x-ky)\left\{\partial_{\alpha,\beta}(x) - \bar{\partial}_{\alpha,\beta}(x)\right\}dx\right|^p < \infty. \tag{3.8}$$

Let T_0 be any sufficiently large constant. Then (3.8) will follow from Minkowski's inequality, if we show that

$$\int_1^\infty y^{p-1}dy\left|c\int_0^{T_0} \frac{x^{k-1}}{(x+y)^{k+2}}(x-ky)\left\{\partial_{\alpha,\beta}(x) - \bar{\partial}_{\alpha,\beta}(x)\right\}dx\right|^p < \infty. \tag{3.9}$$

$$\int_1^\infty y^{p-1}dy\left|c\int_{T_0}^\infty \frac{x^{k-1}}{(x+y)^{k+2}}(x-ky)\left\{\partial_{\alpha,\beta}(x)-\bar{\partial}_{\alpha,\beta}(x)\right\}dx\right|^p<\infty. \tag{3.10}$$

For $x < T_0$, we have $|x-ky| \le x+kx \le x(k+1) \le T_0(k+1) = C$ (Const.) .

By (3.9), we have

$$\int_1^\infty y^{p-1}dy\left|c\int_0^{T_0} \frac{x^{k-1}}{(x+y)^{k+2}}(x-ky)\left\{\partial_{\alpha,\beta}(x)-\bar{\partial}_{\alpha,\beta}(x)\right\}dx\right|^p$$
$$= O(1)\int_1^\infty y^{p-1}dy\left|y^{-k-2}T_0^k\right|^p$$
$$= O(1)\int_1^\infty y^{-kp-p-1}dy$$
$$= O(1)\left[y^{-kp-p}\right]_1^\infty = O(1).$$

Hence (3.9) follows .

By (3.7), the expression on the left of (3.10) does not exceed a constant. Thus

$$\int_1^\infty y^{p-1}dy\left|c\int_{T_0}^\infty \frac{x^{k-1}}{(x+y)^{k+2}}(x-ky)\left\{\partial_{\alpha,\beta}(x)-\bar{\partial}_{\alpha,\beta}(x)\right\}dx\right|^p = \int_1^\infty y^{p-1}dy\left|c\int_{T_0}^\infty (x+y)^{-2}o(\frac{1}{x})^{\beta+1}dx\right|^p$$
$$= \int_1^\infty y^{p-1}dy\left|c\int_{T_0}^\infty (x+y)^{-2}o(\frac{1}{x})^{\beta+1}dx\right|^p$$
$$= o(1)\int_1^\infty y^{p-1}dy\left|\int_{T_0}^\infty (x+y)^{-2}x^{-\beta-1}dx\right|^p \tag{3.11}$$

By an obvious change of variables the expression (3.11) is equal to

$$o(1)\int_1^\infty y^{p-1}dy\left|\int_y^\infty t^{-2}(t-y)^{-\beta-1}dt\right|^p$$
$$= o(1)\int_1^\infty y^{\beta p-p-1}dy$$
$$= o(1)C = C.$$

The result follows.

Proof of Theorem 3.2. We divide the proof into the following cases .

Case I. $\alpha > \gamma$,

Case II. $\alpha = \gamma$,

Case III. $\alpha < \gamma$.

Here we observe that Case I and II follow from Case III, with the aid of Theorem 3.1 .

For, if $\alpha \ge \gamma$, Choose any $\gamma' > \alpha$, summability $|C,\gamma,\beta|_p$implies summability $\left|C,\gamma',\beta\right|_p$by Theorem 3.1, and it follows from Case III, that this implies $|(D,k)(C,\alpha,\beta)|_p$. Hence it is sufficient to consider the Case III only.

Proof of Case III. Since $f(x) \to s(C,\alpha,\beta)$ implies that $f(x) \to s(C,\alpha',\beta)$ for $\alpha' > \alpha > o$, there is no loss of generality in considering the Case $\gamma = \alpha + k$, $k$is a positive integer. We have, by Mishra & Mishra [5]

$$\frac{d}{\mathrm{dy}}U_{k,\alpha,\beta}(y) = C\int_{T_0}^\infty \frac{x^{k-1}}{(x+y)^{k+2}}(x-ky)\partial_{\alpha,\beta}(x)dx. \tag{3.12}$$

Now, by definition

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+p+\gamma)(\gamma+\beta+1)}\frac{1}{y^{\alpha+\beta+p}}\int_0^x (x-t)^{\alpha-\gamma+p-1}t^{\gamma+\beta}\partial_{\alpha,\beta}(t)dt.$$

Putting $p = 1$ and $\alpha = \gamma$, we see that

$$\frac{(\alpha+\beta+1)}{x^{\alpha+\beta+1}}\int_0^x t^{\alpha+\beta}\partial_{\alpha,\beta}(t)dt. \tag{3.13}$$

We also write $\int_x^\infty \frac{\partial_{\alpha,\beta}(t)}{t^2}dt$.

It is clear that, whenever $\int_1^\infty \frac{\partial_{\alpha,\beta}(x)}{x^2}dx$ converges,

$R_{\alpha,\beta}(x)$ is defined for $x > 0$, and that $R_{\alpha,\beta}(x) \to 0$ as $x \to \infty$.

It follows immediately from (3.13) that

$$\partial_{\alpha+1,\beta}(x) = -\frac{(\alpha+\beta+1)}{x^{\alpha+\beta+1}}\int_0^x t^{\alpha+\beta}t^2 dR_{\alpha,\beta}(t)dt$$
$$= o(x^1)$$

and hence that, for $p \geq 1$.

$$\partial_{\alpha+1,\beta}(x) = o(x^1). \tag{3.14}$$

Now by (3.12), we have

$$\frac{d}{dy}U_{k,\alpha,\beta}(y) = C\int_0^\infty \frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+2}}(x-ky)x^{\alpha+\beta}\partial_{\alpha,\beta}(x)dx. \tag{3.15}$$

Integrating (3.15) by parts k times,we deduce with the help of (3.14) that

$$\frac{d}{dy}U_{k,\alpha,\beta}(y) = (-1)^k C\int_0^\infty x^{\alpha+\beta+k}\partial_{\alpha+k,\beta}(x)\left\{\frac{d^k}{dx^k}\left[\frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+2}}(x-ky)\right]\right\}dx.. \tag{3.16}$$

It is easily verified that the expression in curly brackets (3.16) is

$$o\left(\frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+1}}\right). \tag{3.17}$$

Let $\int_0^x t^{\alpha+\beta+k}\frac{d^k}{dx^k}\left[\frac{t^{k-\alpha-\beta-1}}{(t+y)^{k+2}}(t-ky)\right]\mathrm{dt}$.

In fact, for fixed $k > 0$, due to result of Mishra et al. [11], we have uniformly for $x > 0, y > 0$,

$$R(x,y) = 0\left(\frac{x^k}{(x+y)^{k+1}}\right). \tag{3.18}$$

This may be proved by induction on$k$, if $k = 0$, we have

$$\int_0^x t^{\alpha+\beta}\left[\frac{t^{k-\alpha-\beta-1}}{(t+y)^{k+2}}(t-ky)\right]dt$$

$$= \int_0^x \frac{t^{k-1}}{(t+y)^{k+2}}(t-ky)dt$$

$$= \int_0^x \frac{d}{\mathrm{dt}}\left(-\frac{t^k}{(t+y)^{k+1}}\right)\mathrm{dt}$$

$$= \frac{x^k}{(x+y)^{k+1}},$$

hence the result is evident. Suppose that $k \geq 1$, and assume the result true for $k-1$. Integrating by parts, we have

$$R(x,y) = x^{\alpha+\beta+k}\frac{d^{k-1}}{dx^{k-1}}\left[\frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+2}}(x-ky)\right] - (\alpha+\beta+k)\int_0^x t^{\alpha+\beta+k+1}\frac{\partial^{k-1}}{\partial t^{k-1}}\left\{\frac{t^{k-\alpha-\beta-1}}{(t+y)^{k+2}}(t-ky)\right\}dt.$$

the first term is of required order by (3.17) (with k replaced by k-1), and the second by induction hypothesis. Mishra et al. [12] also obtained a result of this type in degree of approximation using variation spaces.

Now integrating (3.16) by parts, we have

$$\frac{d}{\mathrm{dy}}U_{k,\alpha,\beta}(y)$$

$$= \int_0^\infty R(x,y)\left(\frac{d}{dx}\partial_{\alpha+k,\beta}(x)\right)\mathrm{dx}$$

$$= \int_0^\infty R(x,y)\left(\frac{d}{dx}\partial_{\gamma,\beta}(x)\right)dx.$$

Since the integrated term tends to 0 as $\partial_{\gamma,\beta}(x)$ is bounded and $R(x,y) \to 0 as x \to \infty$.

Now we have

$$\left|\frac{d}{dy}U_{k,\alpha,\beta}(y)\right|^p \leq c\left|\int_0^\infty \left\{R(x,y)x^{p-1}\right\}^{\frac{1}{p}}\left(\frac{d}{dx}\partial_{\gamma,\beta}(x)\right)\left\{\frac{R(x,y)}{x}\right\}^{\frac{1}{q}}dx\right|^p.$$

Applying Holder's inequality with indices p and $\frac{p}{p-1}$, we have

$$\left|\frac{d}{dy}U_{k,\alpha,\beta}(y)\right|^p \leq c\int_0^\infty \left\{R(x,y)x^{p-1}\right\}\left|\frac{d}{dx}\partial_{\gamma,\beta}(x)\right|^p\left\{\int_0^\infty \frac{|R(x,y)|}{x}\mathrm{dx}\right\}^{p-1}.$$

Using (3.18) and putting $x = ty$, we see that the expression in curly brackets

$$\leq C\int_0^x \frac{x^{k-1}}{(x+y)^{k+1}}dx = \frac{C}{y}\int_0^x \frac{t^{k-1}}{(1+t)^{k+1}}dt = \frac{C}{y},$$

(Since the integral converges). Hence

$$\int_0^\infty y^{p-1}\left|\frac{d}{dy}U_{k,\alpha,\beta}(y)\right|^p \leq \int_0^\infty dy\int_0^\infty x^{p-1}\left|\frac{d}{dx}\partial_{\gamma,\beta}(x)\right|^p |R(x,y)|\,dx$$

$$= C\int_0^\infty x^{p-1}\left|\frac{d}{dx}\partial_{\gamma,\beta}(x)\right|^p dx\,|R(x,y)|\,dy.$$

Again using (3.18) , the inner integral

$$\leq Cx^k\int_0^\infty \frac{1}{(x+y)^{k+1}}dy, \tag{3.19}$$

on putting $y = xt$, the expression on the right of (3.19) is equal to

$$C\int_0^\infty \frac{1}{(1+t)^{k+1}}dt = C.$$

Now

$$\int_1^\infty \frac{\partial_{\alpha,\beta}(x)}{x^2}dx = \int_1^x \frac{x^{\alpha+\beta}\partial_{\alpha+\beta}(x)}{x^{\alpha+\beta+2}}dx$$

$$= \frac{\partial_{\alpha+1,\beta}(x)}{(\alpha+\beta+1)x} - \frac{\partial_{\alpha+1,\beta}(1)}{(\alpha+\beta+1)} + \frac{(\alpha+\beta+2)}{(\alpha+\beta+1)}\int_1^x \frac{\partial_{\alpha+1,\beta}(x)}{x^2}dx.$$

But we have

$$\int_1^\infty x^{p-1}\left|\frac{d}{dx}\partial_{\alpha+1,\beta}(x)\right|^p dx < \infty.$$

Also, we have

$$\int_1^x \left(\frac{d}{dx}\partial_{\alpha+1,\beta}(x)\right)dx. \tag{3.20}$$

By Hölder's inequality with indices p and q, we have

$$\left|\int_1^x \left(\frac{d}{dx}\partial_{\alpha+1,\beta}(x)\right)dx\right| \leq \left(\int_1^x x^{p-1}\left|\frac{d}{dx}\partial_{\alpha+1,\beta}(x)\right|^p dx\right)^{\frac{1}{p}}\left(\int_1^x \frac{1}{x}dx\right)^{\frac{1}{q}} = O(\log x)^{\frac{1}{q}}. \tag{3.21}$$

From (3.20) and (3.21), we see that
$\int_1^\infty \frac{\partial_{\alpha,\beta}(x)}{x^2}$dx is convergent.

4. Conclusion

The paper concludes that, if function $f(x)$ is summable $|C,\gamma,\beta|_p$, then it is summable$|C,\alpha,\beta|_p$ under the conditions $\alpha > \gamma \geq 0, p \geq 1, \beta > -1$.This gives comparison of two absolute summabilities .

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FIXED POINT FOR GENERALIZED RATIONAL TYPE CONTRACTION IN PARTIALLY ORDERED METRIC SPACES

By

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Abstract

In this paper, we prove some fixed point result for mapping satisfying a generalized contractive condition of rational type in partially ordered metric space. Our results generalize and extend some already proved result in the literature.

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Keywords and Phrases: Fixed point, Generalized contraction mapping, Ordered metric space, Non-decreasing mapping.

1. Introduction

Fixed point theory is one of the most important tool in the study of nonlinear analysis in which Banach contraction principle (BCP) the most versatile result. The BCP plays an important role for solving functional equation in different fields of mathematics. Many authors extended and generalized the BCP. In recent times, fixed point results in partially ordered metric space have been used to solve nonlinear equations in different branches of mathematics . The first result in partially ordered set was given by Wolk [17] in 1975 and later by Monjardet [10] in 1981. Ran and Reurings [15] studied the existence of fixed point result in partially ordered metric space and gave some application to solve linear as well as nonlinear matrix equations. Nieto et al. [[11], [12], [13]] extended their results for non-decreasing mapping and found the solution of first order ordinary differential equation with periodic boundary conditions. Agarwal et al. [1] also generalized contraction conditions in partially ordered metric space and gave some new fixed point results. In partially ordered metric space, the concept of mixed monotone mappings was introduced by Bhaskar and Lakshmikantham [3]. They obtained some coupled fixed point results and also applied their result to obtain unique solution for first order ordinary differential equation with periodic boundary conditions. Many authors have obtained fixed point, coupled fixed point, common fixed point and coupled common fixed point results in partially ordered metric space (see [2], [4], [5], [7], [8], [9], [14], [16]).

In [6] Dass and Gupta proved the following fixed point theorem.

Theorem 1.1. *Let (U,ϱ) be a comlplete metric space and $Q : U \longrightarrow U$ be a self mapping such that their exist $\beta_1,\beta_2 \geq 0$ with $\beta_1+\beta_2 < 1$ satisfying*

$$\varrho(Q\mu, Q\nu) \leq \beta_1 \frac{\varrho(\nu, Q\nu)[1+\varrho(\mu, Q\mu)]}{1+\varrho(\mu,\nu)} + \beta_2\varrho(\mu,\nu)$$

for all $\mu,\nu \in U$. Then Q has a fixed point.

Cabrera et al. [4] extended the result of Dass and Gupta [6] and established a fixed point result in partially ordered metric spaces.

The purpose of this paper is to establish some fixed point result for generalized contraction mapping satisfying a generalized rational type expression in the framework of partially ordered metric space. Also, we establish a result for existence and uniqueness of fixed point for such class of mappings.

The following definitions are needed to prove our main result.

Definition 1.1. *Suppose $(U,\leq)$ be a partially ordered set and let $Q : U \to U$ be a self mapping. Q is said to be monotone non-decreasing if for all $\mu,\nu \in U$,*

$$\mu \leq \nu \text{ implies } Q\mu \leq Q\nu.$$

Definition 1.2. *Let $(U,\leq)$ be a partially ordered set and let $Q : U \longrightarrow U$ be a self mapping. Then*

(1) *Element $\mu,\nu \in U$ are comparable, if $\mu \leq \nu$ or $\nu \leq \mu$ holds.*

(2) *A non-empty set U is called well ordered set, If two elements of it are comparable.*

2. Main Result

Theorem 2.1. *Let $(U, \leq)$ be a partially ordered set and suppose that there exist a metric ϱ in U such that (U, ϱ) is a complete metric space. Let Q is a continuous self mapping on U, Q is monotone non-decreasing mapping such that*

$$\varrho(Q\mu, Q\nu) \leq \beta_1 \frac{\varrho(\nu, Q\nu)[1+\varrho(\mu, Q\mu)]}{1+\varrho(\mu,\nu)} + \beta_2 \frac{\varrho(\mu, Q\mu).\varrho(\nu, Q\nu)}{\varrho(\mu,\nu)} + \beta_3[\varrho(\mu, Q\mu)+\varrho(\nu, Q\nu)] + \beta_4[\varrho(\mu, Q\nu)+\varrho(\nu, Q\mu)] + \beta_5\varrho(\mu,\nu) \quad (2.1)$$

for all $\mu, \nu \in U$, $\mu \neq \nu$, $\mu \geq \nu$ and for some β_1, β_2, β_3, β_4, $\beta_5 \in [0,1)$ with $0 \leq \beta_1+\beta_2+2\beta_3+2\beta_4+\beta_5 < 1$, if there exist $\mu_0 \in U$ with $\mu_0 \leq Q\mu_0$, then Q has a fixed point.

Proof. If $Q\mu_0 = \mu_0$, then the theorem is proved. So, suppose that $\mu_0 < Q\mu_0$. Since, Q is monotone non-decreasing mapping. Therefore, by using mathematical induction, we get

$$\mu_0 < Q\mu_0 \leq Q^2\mu_0 \leq ... \leq Q^n\mu_0 \leq Q^{n+1}\mu_0 \leq ... .$$

This gives a sequence $\{\mu_n\}$ in U such that $\mu_{n+1} = Q\mu_n$, for every $n \geq 0$.
Since, Q is monotone non-decreasing mapping, we have

$$\mu_0 \leq \mu_1 \leq \mu_2 \leq ... \leq \mu_n \leq \mu_{n+1} \leq ... .$$

If there exist $n \geq 1$ such that $\mu_{n+1} = Q\mu_n = \mu_n$, μ_n is a fixed point then the proof is finished.
So, Suppose that $\mu_{n+1} \neq \mu_n$, for all $n \geq 0$.
Since $\mu_n \leq \mu_{n+1}$ for any $n \in N$.
For $n \geq 1$, using contractive condition (2.1), we get

$$\begin{aligned}
\varrho(\mu_n, \mu_{n+1}) &= \varrho(Q\mu_{n-1}, Q\mu_n) \\
&\leq \frac{\beta_1\varrho(\mu_n, Q\mu_n)[1+\varrho(\mu_{n-1}, Q\mu_{n-1})]}{1+\varrho(\mu_{n-1},\mu_n)} + \frac{\beta_2\varrho(\mu_{n-1}, Q\mu_{n-1}).\varrho(\mu_n, Q\mu_n)}{\varrho(\mu_{n-1},\mu_n)} \\
&+ \beta_3[\varrho(\mu_{n-1}, Q\mu_{n-1})+\varrho(\mu_n, Q\mu_n)] + \beta_4[\varrho(\mu_{n-1}, Q\mu_n)+\varrho(\mu_n, Q\mu_{n-1})] + \beta_5\varrho(\mu_{n-1},\mu_n) \\
&\leq \frac{\beta_1\varrho(\mu_n,\mu_{n+1})[1+\varrho(\mu_{n-1},\mu_n)]}{1+\varrho(\mu_{n-1},\mu_n)} + \frac{\beta_2\varrho(\mu_{n-1},\mu_n).\varrho(\mu_n,\mu_{n+1})}{\varrho(\mu_{n-1},\mu_n)} \\
&+ \beta_3[\varrho(\mu_{n-1},\mu_n)+\varrho(\mu_n,\mu_{n+1})] + \beta_4[\varrho(\mu_{n-1},\mu_{n+1})+\varrho(\mu_n,\mu_n)] + \beta_5\varrho(\mu_{n-1},\mu_n).
\end{aligned}$$

Finally, we obtain

$$\varrho(\mu_n,\mu_{n+1}) \leq \left(\frac{\beta_3+\beta_4+\beta_5}{1-(\beta_1+\beta_2+\beta_3+\beta_4)}\right)\varrho(\mu_{n-1},\mu_n).$$

Now, Using mathematical induction, we have

$$\varrho(\mu_n,\mu_{n+1}) \leq \left(\frac{\beta_3+\beta_4+\beta_5}{1-(\beta_1+\beta_2+\beta_3+\beta_4)}\right)^n \varrho(\mu_0,\mu_1).$$

Put $K = \frac{\beta_3+\beta_4+\beta_5}{1-(\beta_1+\beta_2+\beta_3+\beta_4)} < 1$. Now, suppose that $\{\mu_n\}$ is a cauchy sequence. For $m \geq n$, we have

$$\begin{aligned}
\varrho(\mu_n,\mu_m) &\leq \varrho(\mu_n,\mu_{n+1}) + \varrho(\mu_{n+1},\mu_{n+2}) + ... + \varrho(\mu_{m-1},\mu_m) \\
&\leq (K^n + K^{n+1} + ... + K^{m-1})\varrho(\mu_0,\mu_1) \\
&\leq \left(\frac{K^n}{1-K}\right)\varrho(\mu_0,\mu_1).
\end{aligned}$$

Taking Lim $n, m \to \infty$, we get

$$\lim_{n,m\to\infty} \varrho(\mu_n,\mu_m) = 0 \qquad (\because K < 1).$$

Thus, $\{\mu_n\}$ is a cauchy sequence.
Now, since (U,ϱ) is a complete metric space. Therefore,

$$\lim_{n\to\infty} \mu_n = \mu, \text{ for some } \mu \in U.$$

Also, Q is continuous. Therefore,

$$Q\mu = Q(\lim_{n\to\infty} \mu_n) = \lim_{n\to\infty} Q\mu_n = \lim_{n\to\infty} \mu_{n+1} = \mu.$$

Hence, μ is a fixed point. □

Now, we will show that Theorem 2.1 is still valid for Q not necessarily continuous, assuming the following hypothesis in U:

$$\text{If } \mu_n \text{ is a non-decreasing sequence in } U \text{ such that } \mu_n \to \mu, \text{ then } \mu = \sup\{\mu_n\}. \tag{2.2}$$

Theorem 2.2. *Let $(U, \leq)$ be a partially ordered set and suppose that there exist a metric ϱ on U such that (U, ϱ) is a complete metric space. Suppose that Q be a self mapping on U. Q is monotone non-decreasing mapping and*

$$\varrho(Q\mu, Qv) \leq \beta_1 \frac{\varrho(v, Qv)[1 + \varrho(\mu, Q\mu)]}{1 + \varrho(\mu, v)} + \beta_2 \frac{\varrho(\mu, Q\mu).\varrho(v, Qv)}{\varrho(\mu, v)} + \beta_3[\varrho(\mu, Q\mu) + \varrho(v, Qv)] + \beta_4[\varrho(\mu, Qv) + \varrho(v, Q\mu)] + \beta_5\varrho(\mu, v) \tag{2.3}$$

for all $\mu, v \in U$, $\mu \neq v$, $\mu \geq v$ and for some $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in [0, 1)$ with $0 \leq \beta_1 + \beta_2 + 2\beta_3 + 2\beta_4 + \beta_5 < 1$. And assume that $\{\mu_n\}$ is a non-decreasing sequence in U such that $\mu_n \to \mu$, then $\mu = sup\{\mu_n\}$. If there exist $\mu_0 \in U$ with $\mu_0 \leq Q\mu_0$. Then Q has a fixed point.

Proof. Following the proof of the Theorem 2.1, we have $\{\mu_n\}$ is a cauchy sequence. Since, $\{\mu_n\}$ is a non-decreasing sequence in U such that $\mu_n \to \mu$, then $\mu = \sup\{\mu_n\}$. Particularly, $\mu_n \leq \mu$, for all $n \in N$.

Since, Q is a monotone non-decreasing mapping $Q\mu_n \leq Q\mu$, for all $n \in N$. Moreover, as $\mu_n < \mu_{n+1} \leq Q\mu$ and $\mu = \sup\{\mu_n\}$, we get $\mu \leq Q\mu$.

Construct a sequence $\{v_n\}$ as $v_0 = \mu$, $v_{n+1} = Qv_n$, for all $n \geq 0$. Since, $v_0 \leq Qv_0$ aruguing as above we obtain that $\{v_n\}$ is non-decreasing sequence and $\lim_{n\to\infty} v_n = v$ for some $v \in U$. So, we have $v = sup\{v_n\}$.

Since $\mu_n \leq \mu = v_0 \leq Q\mu = Qv_0 \leq v_n \leq v$ for all n. Using (2.3), we have

$$\begin{aligned}
\varrho(\mu_{n+1}, v_{n+1}) &= \varrho(Q\mu_n, Qv_n) \\
&\leq \frac{\beta_1\varrho(v_n, Qv_n)[1 + \varrho(\mu_n, Q\mu_n)]}{1 + \varrho(\mu_n, v_n)} + \frac{\beta_2\varrho(\mu_n, Q\mu_n).\varrho(v_n, Qv_n)}{\varrho(\mu_n, v_n)} \\
&+ \beta_3[\varrho(\mu_n, Q\mu_n) + \varrho(v_n, Qv_n)] + \beta_4[\varrho(\mu_n, Qv_n) + \varrho(v_n, Q\mu_n)] + \beta_5\varrho(\mu_n, v_n) \\
&\leq \beta_1 \frac{\varrho(v_n, v_{n+1})[1 + \varrho(\mu_n, \mu_{n+1})]}{1 + \varrho(\mu_n, v_n)} + \frac{\beta_2\varrho(\mu_n, \mu_{n+1}).\varrho(v_n, v_{n+1})}{\varrho(\mu_n, v_n)} \\
&+ \beta_3[\varrho(\mu_n, \mu_{n+1}) + \varrho(v_n, v_{n+1})] + \beta_4[\varrho(\mu_n, v_{n+1}) + \varrho(v_n, \mu_{n+1})] + \beta_5\varrho(\mu_n, v_n).
\end{aligned}$$

Taking $n \to \infty$ in the last inequality, we have $\varrho(\mu, v) \leq (\ 2\beta_4 + \beta_5)\varrho(\mu, v)$.

As $2\beta_4 + \beta_5 < 1$. We have $\varrho(\mu, v) = 0$. Particularly $\mu = v = \sup\{v_n\}$ and consquently $\mu \leq Q\mu \leq \mu$. Hence, we conclude that μ is a fixed point of Q. □

Now, we will give an example where it can be proved that assumption in Theorem 2.1 do not guarantee the uniqueness of the fixed point.

Example 2.1. Let $U = \{(1, 2), (2, 1)\} \subset R^2$ and consider the usual order given by

$$(\mu, v) \leq (\omega, \chi) \Longleftrightarrow \mu \leq \omega,\ v \leq \chi.$$

Hence, $(U, \leq)$ is a partially ordered set in which distinct non-comparable elements. Besides (U, ϱ_2) is a complete metric space, where ϱ_2 is the Euclidean distance. The identity map $Q(\mu, v) = (\mu, v)$ is obviously non-decreasing and continuous and assumption (2.1) of Theorem 2.1 is satified because elements in U are only comparable to themselves. Moreover, $(1,2) \leq Q(1, 2)$ and Q has two fixed point in U.

In what follow, we can give a sufficient condition for the uniqueness of the fixed point in Theorem 2.1 and Theorem 2.2. This condition appear in [15].

For $\mu, v \in U$, their exist a lower bound or an upper bound.

In [11], it is proved that the above condition is equivalent to

$$\text{For } \mu, v \in U, \text{ their exist } \omega \in U \text{ which is comparable to } \mu \text{ and } v. \tag{2.4}$$

Theorem 2.3. *Adding condition* (2.4) *to the assumption of Theorem* 2.1 *and Theorem* 2.2*. We obtain uniqueness of the fixed point of Q.*

Proof. Suppose that their exist $\mu, v \in U$ which are two fixed point of Q. Now, we have two different cases.

Case 1. If $\mu \neq v$, μ and v are comparable. Then using (2.1), we have

$$\varrho(\mu, v) = \varrho(Q\mu, Qv)$$

$$\begin{aligned}
&\leq \frac{\beta_1\varrho(\nu, Q\nu)[1+\varrho(\mu, Q\mu)]}{1+\varrho(\mu,\nu)} + \frac{\beta_2\varrho(\mu, Q\mu).\varrho(\nu, Q\nu)}{\varrho(\mu,\nu)} \\
&+ \beta_3[\varrho(\mu, Q\mu)+\varrho(\nu, Q\nu)] + \beta_4[\varrho(\mu, Q\nu)+\varrho(\nu, Q\mu)] + \beta_5\varrho(\mu,\nu) \\
&\leq \frac{\beta_1\varrho(\nu,\nu)[1+\varrho(\mu,\mu)]}{1+\varrho(\mu,\nu)} + \frac{\beta_2\varrho(\mu,\mu).\varrho(\nu,\nu)}{\varrho(\mu,\nu)} \\
&+ \beta_3[\varrho(\mu,\mu)+\varrho(\nu,\nu)] + \beta_4[\varrho(\mu,\nu)+\varrho(\nu,\mu)] + \beta_5\varrho(\mu,\nu) \\
\varrho(\mu,\nu) &\leq (2\beta_4+\beta_5)\varrho(\mu,\nu).
\end{aligned}$$

As $2\beta_4+\beta_5 < 1$.

Therefore, $\lim_{n\to\infty}\varrho(\mu,\nu) = 0$. Hence $\mu = \nu$.

Case 2. if μ is not comparable to ν, then their exist $\omega \in U$ which is comparable to μ and ν. Monotonicity implies that $Q^n\omega$ is comparable to $Q^n\mu = \mu$ and $Q^n\nu = \nu$ for n = 0,1,2,3,..... If their exist $n_0 \geq 1$ such that $Q^{n_0}\omega = \mu$, then as μ is a fixed point, the sequence $\{ Q^n\omega : n \geq n_0\}$ is constant and consequently, $\lim_{n\to\infty} Q^n\omega = \mu$.

On the other hand, if $Q^n\omega \neq \mu$, for $n \geq 1$, using the contractive condition, we obtain for $n \geq 2$

$$\begin{aligned}
\varrho(Q^n\omega,\mu) &= \varrho(Q^n\omega, Q^n\mu) \\
&= \varrho(Q(Q^{n-1}\omega), Q(Q^{n-1}\mu)) \\
&\leq \frac{\beta_1\varrho(Q^{n-1}\mu, Q^n\mu)[1+\varrho(Q^{n-1}\omega, Q^n\omega)]}{1+\varrho(Q^{n-1}\omega, Q^{n-1}\mu)} + \frac{\beta_2\varrho(Q^{n-1}\omega, Q^n\omega).\varrho(Q^{n-1}\mu, Q^n\mu)}{\varrho(Q^{n-1}\omega, Q^{n-1}\mu)} \\
&+ \beta_3[\varrho(Q^{n-1}\omega, Q^n\omega)+\varrho(Q^{n-1}\mu, Q^n\mu)] + \beta_4[\varrho(Q^{n-1}\omega, Q^n\mu)+\varrho(Q^{n-1}\mu, Q^n\omega)] \\
&+ \beta_5\varrho(Q^{n-1}\omega, Q^{n-1}\mu) \\
&\leq \beta_1\frac{\varrho(\mu,\mu)[1+\varrho(Q^{n-1}\omega, Q^n\omega)]}{1+\varrho(Q^{n-1}\omega,\mu)} + \frac{\beta_2\varrho(Q^{n-1}\omega, Q^n\omega).\varrho(\mu,\mu)}{\varrho(Q^{n-1}\omega,\mu)} \\
&+ \beta_3[\varrho(Q^{n-1}\omega, Q^n\omega)+\varrho(\mu,\mu)] + \beta_4[\varrho(Q^{n-1}\omega,\mu)+\varrho(\mu, Q^n\omega)] + \beta_5\varrho(Q^{n-1}\omega,\mu).
\end{aligned}$$

Finally, we obtain

$$\varrho(Q^n\omega,\mu) \leq \left(\frac{\beta_3+\beta_4+\beta_5}{1-\beta_3-\beta_4}\right)\varrho(Q^{n-1}\omega,\mu).$$

Using mathematicial induction, we have

$$\varrho(Q^n\omega,\mu) \leq \left(\frac{\beta_3+\beta_4+\beta_5}{1-\beta_3-\beta_4}\right)^n \varrho(\omega,\mu)$$

and as $\frac{\beta_3+\beta_4+\beta_5}{1-\beta_3-\beta_4} < 1$. Therefore, We have $\lim_{n\to\infty} Q^n\omega = \mu$. Using a similar argument, we can show that $\lim_{n\to\infty} Q^n\omega = \nu$. Now, the uniqueness of the limit implies $\mu = \nu$. Hence, Q has a unique fixed point. □

Remark 2.1. If $\beta_1 = \beta_3 = \beta_4 = 0$ in Theorems 2.1, 2.2 and 2.3 then we have Theorems 2.2, 2.3 and 2.4 of Harjani et al. [8].

Remark 2.2. If $\beta_2 = \beta_3 = \beta_4 = 0$ in Theorems 2.1, 2.2 and 2.3 then we have Theorems 2, 3 and 4 of Cabrera et al. [4].

Remark 2.3. If $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 0$ in Theorems 2.1, 2.2 and 2.3 then we have Theorems 2.1, 2.2 and 2.3 of Nieto et al. [11].

Finally, we will give an example where Theorem 2.1 can be applied.

Example 2.2. Let $U = \{(1,2),(2,1),(2,2)\}$ and consider in U the partial order given by $R = \{(\mu,\mu) : \mu \in U\}$. Notice that elements in U are only comparable to themselves. Besides, (U,ϱ_2) is a complete metric space considering ϱ_2, is the Euclidean distance.

Let $Q : U \longrightarrow U$ be defined by

Q(1,2) = (2,1) , Q(2,1) = (1,2) , Q(2,2) = (2,2)

Q is obiviously non-decreasing and continuous and assumption (2.1) of Theorem 2.1 is satisfied. Because, elements in U are only comparable to themselves. Moreover, (2,2) $\leq Q$(2,2) = (2,2) and by Theorem 2.1 Q has a fixed point (Obiviously this fixed point is (2,2)).

If we take μ = (1,2) and ν = (2,1) $\in U$, then

$\varrho(Q\mu, Q\nu) = \sqrt{2}, \quad \varrho(\mu, Q\mu) = \sqrt{2}, \quad \varrho(\nu, Q\nu) = \sqrt{2}, \quad \varrho(\mu, \nu) = \sqrt{2}$
and the contractive condition appearing in Theorem 1.1 is not satisfied, beacause

$$\begin{aligned}
\varrho(Q\mu, Q\nu) &= \sqrt{2} \leq \frac{\beta_1\, \varrho(\nu, Q\nu).[1 + \varrho(\mu, Q\mu)]}{1 + \varrho(\mu, \nu)} + \beta_2 \varrho(\mu, \nu). \\
\sqrt{2} &\leq \frac{\beta_1\, \sqrt{2}\,(1 + \sqrt{2})}{(1 + \sqrt{2})} + \beta_2\, \sqrt{2}. \\
\sqrt{2} &\leq \beta_1\, \sqrt{2} + \beta_2\, \sqrt{2}.
\end{aligned}$$

and thus $\beta_1 + \beta_2 \geq 1$.

3. Conclusion

In this article, we extend and generalized a contraction condition of Dass and Gupta [6] and proved some fixed point Theorems for new generalized contraction mapping in partially ordered metric space. Also, we establish a result for existence and uniqueness of fixed point for such class of mappings. We also give an example in support of our results.

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A STUDY ON THE GROWTH OF GENERALIST ITERATED ENTIRE FUNCTIONS IN TERMS OF ITS MAXIMUM TERM

By

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Abstract

In this paper we consider iteration of three entire functions and study comparative growth of the maximum term of generalist iterated entire functions with that of the maximum term of the related functions.

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1. Introduction

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function. Then $M(r, f) = \max_{|z|=r} |f(z)|$ and $\mu(r, f) = \max_n |a_n| r^n$ are respectively called the maximum modulus and maximum term of $f(z)$ on $|z| = r$.

Definition 1.1. *The order ρ_f and lower order λ_f of an entire function $f(z)$ is defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log\log M(r, f)}{\log r}.$$

A simple but useful relation between $M(r, f)$ and $\mu(r, f)$ is given in the following theorem.

Theorem 1.1 ([10]). *For* $0 \le r < R$,

$$\mu(r, f) \le M(r, f) \le \frac{R}{R - r}\mu(R, f).$$

Taking $R = 2r$, for all sufficiently large values of r, we have

$$\mu(r, f) \le M(r, f) \le 2\mu(2r, f). \tag{1.1}$$

Taking two times logarithms in (1.1), we can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log log\mu(r, f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log log\mu(r, f)}{\log r}.$$

Definition 1.2 ([2]). *Let f(z), g(z) and h(z) be three entire functions defined in the open complex plane. Then the generalized iterations of f with respect to g and h are defined as follows:*

$$\begin{aligned}
f_1(z) &= f(z)\\
f_2(z) &= f(g(z)) = f(g_1(z))\\
f_3(z) &= f(g(h(z))) = f(g(h_1(z))) = f(g_2(z))\\
f_4(z) &= f(g(h(f(z)))) = f(g(h_2(z))) = f(g_3(z))\\
&\vdots\\
f_n(z) &= f(g(h(f..(f(z)\ or\ g(z)\ or\ h(z)\ according\ as\ n = 3m-2\ or\ 3m-1\\
&\quad or\ 3m)...)))\\
&= f(g_{n-1}(z)) = f(g(h_{n-2}(z))).
\end{aligned}$$

Similarly,

$$\begin{aligned}
g_1(z) &= g(z)\\
g_2(z) &= g(h(z)) = g(h_1(z))\\
g_3(z) &= g(h(f(z))) = g(h(f_1(z))) = g(h_2(z))\\
g_4(z) &= g(h(f(g(z)))) = g(h(f_2(z))) = g(h_3(z))\\
&\vdots\\
g_n(z) &= g(h(f(g...(g(z)\ \mathit{or}\ h(z)\ \mathit{or}\ f(z)\ \mathit{according\ as}\ n = 3m-2\ \mathit{or}\ 3m-1\\
&\quad \mathit{or}\ 3m)...)))\\
&= g(h_{n-1}(z)) = g(h(f_{n-2}(z)))
\end{aligned}$$

and

$$\begin{aligned}
h_1(z) &= h(z)\\
h_2(z) &= h(f(z)) = h(f_1(z))\\
h_3(z) &= h(f(g(z))) = h(f(g_1(z))) = h(f_2(z))\\
h_4(z) &= h(f(g(h(z)))) = h(f(g_2(z))) = h(f_3(z))\\
&\vdots\\
h_n(z) &= h(f(g(h...(h(z)\ \mathit{or}\ f(z)\ \mathit{or}\ g(z)\ \mathit{according\ as}\ n = 3m-2\ \mathit{or}\ 3m-1\\
&\quad \mathit{or}\ 3m)...)))\\
&= h(f_{n-1}(z)) = h(f(g_{n-2}(z))).
\end{aligned}$$

Clearly all $f_n(z)$, $g_n(z)$ and $h_n(z)$ are entire functions.

Notations 1.1 ([9]). Let $log^{[0]}x = x$, $exp^{[0]}x = x$ and for positive integer m, $log^{[m]}x = log(log^{[m-1]}x)$, $exp^{[m]}x = exp(exp^{[m-1]}x)$.

In 1989, A. P. Singh [10] studied the growth of composite entire function in terms of maximum term. Later on Lahiri and Sarma [8] studied growth property of composite entire function in terms of maximum terms. Recently Banerjee and Dutta [1], Dutta [4], [5], Dutta and Mandal [6] investigated growth of iterated entire function in terms of maximum terms and improve some earlier results.

In this paper, we study growth properties of the maximum term of generalist iterated entire functions as compared to the growth of the maximum term of the related function to generalist some earlier results. Throughout the papers we denote by $f(z)$, $g(z)$, $h(z)$ etc. non-constant entire functions of order (lower order) $\rho_f(\lambda_f)$, $\rho_g(\lambda_g)$, $\rho_h(\lambda_h)$ etc. We do not explain the standard notations and definitions of the theory of entire functions as those are available in [7], [11], [12].

2. Main results

The following lemmas will be needed in the sequel.

Lemma 2.1 ([3]). *If f and g are any two entire functions, for all sufficiently large values of r,*

$$M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - |g(0)|, f\right) \leq M(r, f \circ g) \leq M(M(r, g), f).$$

Lemma 2.2 ([5]). *If λ_g is finite, then*

$$\liminf_{r\to\infty} \frac{\log M(r, g)}{\log \mu(r, g)} \leq 2^{\lambda_g}.$$

Lemma 2.3. *If ρ_f, ρ_g and ρ_h are finite, then for any $\varepsilon > 0$,*

$$\log^{[n]} \mu(r, f_n) \leq \begin{cases} (\rho_g + \varepsilon)\log M(r, h) + O(1) & \mathit{when}\ n = 3k,\\ (\rho_h + \varepsilon)\log M(r, f) + O(1) & \mathit{when}\ n = 3k+1,\\ (\rho_f + \varepsilon)\log M(r, g) + O(1) & \mathit{when}\ n = 3k+2 \end{cases}$$

for all sufficiently large values of r and a positive integer k.

Proof. First suppose that $n = 3k$ then in view of (1.1) and by Lemma 2.1 it follows that for all sufficiently large values of r,

$$\begin{aligned}
\mu(r, f_n) &\leq M(r, f_n) \\
&\leq M(M(r, g_{n-1}), f) \\
\text{i.e., } \log\ \mu(r, f_n) &\leq \log M(M(r, g_{n-1}), f) \\
&\leq [M(r, g_{n-1})]^{\rho_f+\varepsilon}. \\
\text{So, } \log^{[2]}\ \mu(r, f_n) &\leq (\rho_f + \varepsilon)\log M(r, g(h_{n-2})) \\
&\leq (\rho_f + \varepsilon)[M(r, h_{n-2})]^{\rho_g+\varepsilon}. \\
\text{i.e., } \log^{[3]}\ \mu(r, f_n) &\leq (\rho_g + \varepsilon)\log M(r, h_{n-2}) + O(1). \\
\ldots\ldots &\qquad \ldots\ldots \\
\ldots\ldots &\qquad \ldots\ldots \\
\text{Therefore } \log^{[n]} \mu(r, f_n) &\leq (\rho_g + \varepsilon)\log M(r, h) + O(1).
\end{aligned}$$

Similarly for all sufficiently large values of r,

$$\log^{[n]} \mu(r, f_n) \leq (\rho_h + \varepsilon)\log M(r, f) + O(1) \ \text{ for } n = 3k + 1,$$

and

$$\log^{[n]} \mu(r, f_n) \leq (\rho_f + \varepsilon)\log M(r, g) + O(1) \ \text{ for } n = 3k + 2.$$

This proves the lemma 2.3. □

Lemma 2.4. *If λ_f, λ_g and λ_h are non-zero finite, then*

$$\log^{[n]} \mu(r, f_n) \geq \begin{cases} (\lambda_g - \varepsilon)\log M(r, h) + O(1) & \text{when } n = 3k, \\ (\lambda_h - \varepsilon)\log M(r, f) + O(1) & \text{when } n = 3k + 1, \\ (\lambda_f - \varepsilon)\log M(r, g) + O(1) & \text{when } n = 3k + 2 \end{cases}$$

for all sufficiently large values of r and k is natural number.

Proof. First suppose that $n = 3k$. Let $\epsilon(> 0)$ be such that $\epsilon < min\{\lambda_f, \lambda_g, \lambda_h\}$. Now we have from [10] for all sufficiently large values of r,

$$\mu(r, f \circ g) > e^{[M(r,g)]^{\lambda_f-\varepsilon}}.$$

So,

$$\log \mu(r, f \circ g) > [M(r, g)]^{\lambda_f-\varepsilon}. \tag{2.1}$$

Now

$$\begin{aligned}
\log \mu(r, f_n) &= \log \mu(r, f(g_{n-1})) \\
&> [M(r, g_{n-1})]^{\lambda_f-\varepsilon} \quad \text{(using (2.1))} \\
&\geq [\mu(r, g_{n-1})]^{\lambda_f-\varepsilon} \quad \text{(from (1.1))}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\log^{[2]} \mu(r, f_n) &> (\lambda_f - \varepsilon)\log \mu(r, g(h_{n-2})) \\
&> (\lambda_f - \varepsilon)[M(r, h_{n-2})]^{\lambda_g-\varepsilon} \quad \text{(using (2.1))}.
\end{aligned}$$

Hence

$$\begin{aligned}
\log^{[3]} \mu(r, f_n) &> (\lambda_g - \varepsilon)\log[\mu(r, h_{n-2})] + O(1) \\
&> (\lambda_g - \varepsilon)[M(r, f_{n-3})]^{\lambda_h-\varepsilon} + O(1).
\end{aligned}$$

Taking repeated logarithms, we have

$$log^{[n-1]}\mu(r, f_n) \geq (\lambda_f - \varepsilon)[M(r, h)]^{\lambda_g-\varepsilon} + O(1).$$

Therefore

$$\log^{[n]} \mu(r, f_n) \geq (\lambda_g - \varepsilon)\log M(r, h) + O(1).$$

Similarly,

$$\log^{[n]} \mu(r, f_n) \geq (\lambda_h - \varepsilon)\log M(r, f) + O(1) \quad \text{when } n = 3k + 1,$$

and

$$\log^{[n]} \mu(r, f_n) \geq (\lambda_f - \varepsilon)\log M(r, g) + O(1) \quad \text{when } n = 3k + 2.$$

This proves the lemma. □

Theorem 2.1. *If ρ_f, ρ_g and ρ_h are finite, then*

$$\text{(i)}\quad \liminf_{r\to\infty} \frac{\log^{[n]}\mu(r,f_n)}{\log\mu(r,h)} \le \rho_g 2^{\lambda_h} \quad \textit{when } n=3k,$$

$$\text{(ii)}\quad \liminf_{r\to\infty} \frac{\log^{[n]}\mu(r,f_n)}{\log\mu(r,f)} \le \rho_h 2^{\lambda_f} \quad \textit{when } n=3k+1$$

and

$$\text{(iii)}\quad \liminf_{r\to\infty} \frac{\log^{[n]}\mu(r,f_n)}{\log\mu(r,g)} \le \rho_f 2^{\lambda_g} \quad \textit{when } n=3k+2.$$

Proof. When $n=3k$, we have from Lemma 2.3 for all sufficiently large values of r,

$$\begin{aligned} \log^{[n]}\mu(r,f_n) &\le (\rho_g+\varepsilon)\log M(r,h)+O(1). \\ \therefore \quad \liminf_{r\to\infty} \frac{\log^{[n]}\mu(r,f_n)}{\log\mu(r,h)} &\le (\rho_g+\varepsilon)\liminf_{r\to\infty}\frac{\log M(r,h)}{\log\mu(r,h)}. \end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we get from Lemma 2.2,

$$\liminf_{r\to\infty} \frac{\log^{[n]}\mu(r,f_n)}{\log\mu(r,h)} \le \rho_g 2^{\lambda_h}.$$

Similarly we get the other two results.

This proves the theorem. □

Theorem 2.2. *Let $f(z)$, $g(z)$ and $h(z)$ be entire functions of finite order and n=3k then*

$$\text{(i)}\quad \limsup_{r\to\infty} \frac{\log^{[n]}\mu(r,f_n)}{\log\mu(r,f)} = 0 \ \textit{ for } \ \rho_h<\lambda_f$$

and

$$\text{(ii)}\quad \limsup_{r\to\infty} \frac{\log^{[n]}\mu(r,f_n)}{\log\mu(r,g)} = 0 \ \textit{ for } \ \rho_h<\lambda_g.$$

Proof. When $n=3k$, we have from Lemma 2.3 for all sufficiently large values of r,

$$\begin{aligned} \log^{[n]}\mu(r,f_n) &\le (\rho_g+\varepsilon)\log M(r,h)+O(1) \\ &\le (\rho_g+\varepsilon)r^{\rho_h+\varepsilon}+O(1). \end{aligned} \tag{2.2}$$

Also from definition of lower order we have for $r\ge r_0$,

$$\log\mu(r,f)\ge r^{\lambda_f-\varepsilon}. \tag{2.3}$$

So from (2.2) and (2.3) we get for $r\ge r_0$,

$$\frac{\log^{[n]}\mu(r,f_n)}{\log\mu(r,f)} \le \frac{(\rho_f+\varepsilon)r^{\rho_h+\varepsilon}+O(1)}{r^{\lambda_f-\varepsilon}}.$$

Since $\lambda_f>\rho_h$, we can choose $\varepsilon>0$ such that if $\lambda_f-\varepsilon>\rho_h+\varepsilon$ then

$$\limsup_{r\to\infty} \frac{\log^{[n]}\mu(r,f_n)}{\log\mu(r,f)} = 0,$$

which is result (i).

Similarly when $\rho_h<\lambda_g$, we get the result (ii).

This proves the theorem. □

Note 2.1. If we take $n=3k+1$ and $n=3k+2$ we get similar results.

Theorem 2.3. *Let $f(z)$, $g(z)$ and $h(z)$ be entire functions of finite order and $n=3k$ then*

$$\text{(i)}\quad \limsup_{r\to\infty} \frac{\log^{[n]}\mu(r,f_n)}{\log\mu(r,f)} = \infty \ \textit{ for } \ \lambda_h>\rho_f$$

and

$$\text{(ii)}\quad \limsup_{r\to\infty} \frac{\log^{[n]}\mu(r,f_n)}{\log\mu(r,g)} = \infty \ \textit{ for } \ \lambda_h>\rho_g.$$

Proof. When $n = 3k$, we have from Lemma 2.4 for all sufficiently large values of r,

$$\begin{aligned} \log^{[n]} \mu(r, f_n) &\geq (\lambda_g - \varepsilon) \log M(r, h) + O(1) \\ &\geq (\lambda_g - \varepsilon) r^{\lambda_h - \varepsilon} + O(1). \end{aligned} \tag{2.4}$$

Also from definition of lower order we have for $r \geq r_0$,

$$\log \mu(r, f) \leq r^{\rho_f + \varepsilon}. \tag{2.5}$$

So from (2.4) and (2.5) we get for $r \geq r_0$,

$$\frac{\log^{[n]} \mu(r, f_n)}{\log \mu(r, f)} \geq \frac{(\lambda_g - \varepsilon) r^{\lambda_h - \varepsilon} + O(1)}{r^{\rho_f + \varepsilon}}.$$

Since $\lambda_h > \rho_f$, we can choose $\varepsilon > 0$ such that if $\lambda_h - \varepsilon > \rho_f + \varepsilon$ then

$$\limsup_{r \to \infty} \frac{\log^{[n]} \mu(r, f_n)}{\log \mu(r, f)} = \infty,$$

which is result (i).

Similarly when $\lambda_h > \rho_g$ we get result (ii).

This proves the theorem. □

Note 2.2. If we take $n = 3k + 1$ and $n = 3k + 2$ we get similar results.

Theorem 2.4. *Let $f(z)$, $g(z)$ and $h(z)$ be transcendental entire functions of non-zero finite order then*

$$\limsup_{r \to \infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r, f)} = \limsup_{r \to \infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r, g)} = \limsup_{r \to \infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r, h)} = \infty.$$

Proof. First we consider $n = 3k$ then from (2.4) we have for sufficiently large values of r,

$$\log^{[n]} \mu(r, f_n) \geq (\lambda_g - \varepsilon) r^{\lambda_h - \varepsilon} + O(1) \tag{2.6}$$

where $0 < \varepsilon < \min\{\lambda_f, \lambda_g, \lambda_h\}$ and from (2.5)

$$\log^{[2]} \mu(r, f) \leq (\rho_f + \varepsilon) \log r. \tag{2.7}$$

So from (2.6) and (2.7) we get,

$$\frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r, f)} \geq \frac{(\lambda_g - \varepsilon) r^{\lambda_h - \varepsilon} + O(1)}{(\rho_f + \varepsilon) \log r}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\limsup_{r \to \infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r, f)} = \infty.$$

Also when $n = 3k + 1$, from Lemma 2.4 we have for sufficiently large values of r and $0 < \varepsilon < \min\{\lambda_f, \lambda_g, \lambda_h\}$

$$\begin{aligned} \log^{[n]} \mu(r, f_n) &\geq (\lambda_h - \varepsilon) \log M(r, f) + O(1) \\ &\geq (\lambda_h - \varepsilon) r^{\lambda_f - \varepsilon} + O(1). \end{aligned} \tag{2.8}$$

So from (2.7) and (2.8) we get,

$$\frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r, f)} \geq \frac{(\lambda_h - \varepsilon) r^{\lambda_f - \varepsilon} + O(1)}{(\rho_f + \varepsilon) \log r}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\limsup_{r \to \infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r, f)} = \infty.$$

Again for $n = 3k + 2$, from Lemma 2.4 we have for sufficiently large values of r and $0 < \varepsilon < \min\{\lambda_f, \lambda_g, \lambda_h\}$

$$\begin{aligned} \log^{[n]} \mu(r, f_n) &\geq (\lambda_f - \varepsilon) \log M(r, g) + O(1) \\ &\geq (\lambda_f - \varepsilon) r^{\lambda_g - \varepsilon} + O(1). \end{aligned} \tag{2.9}$$

So from (2.7) and (2.9) we get,

$$\frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]} \mu(r, f)} \geq \frac{(\lambda_f - \varepsilon) r^{\lambda_g - \varepsilon} + O(1)}{(\rho_f + \varepsilon) \log r}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\limsup_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]}\mu(r, f)} = \infty.$$

Similarly we have

$$\limsup_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]}\mu(r, g)} = \infty,$$

and

$$\limsup_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log^{[2]}\mu(r, h)} = \infty.$$

This proves the theorem. □

Note 2.3. If we take one more logarithm of the numerator then the expression in Theorem 2.4 is finite. Thus we shall prove the following theorem.

Theorem 2.5. *Let $f(z)$, $g(z)$ and $h(z)$ be three transcendental entire functions of finite order and nonzero lower order then for $n = 3k$,*

$$\text{(i)}\quad \limsup_{r\to\infty} \frac{\log^{[n+1]} \mu(r, f_n)}{\log^{[2]}\mu(r, f)} \le \frac{\rho_h}{\lambda_f},$$

$$\text{(ii)}\quad \limsup_{r\to\infty} \frac{\log^{[n+1]} \mu(r, f_n)}{\log^{[2]}\mu(r, g)} \le \frac{\rho_h}{\lambda_g},$$

for $n = 3k + 1$

$$\text{(iii)}\quad \limsup_{r\to\infty} \frac{\log^{[n+1]} \mu(r, f_n)}{\log^{[2]}\mu(r, g)} \le \frac{\rho_f}{\lambda_g},$$

$$\text{(iv)}\quad \limsup_{r\to\infty} \frac{\log^{[n+1]} \mu(r, f_n)}{\log^{[2]}\mu(r, h)} \le \frac{\rho_f}{\lambda_h},$$

for $n = 3k + 2$

$$\text{(v)}\quad \limsup_{r\to\infty} \frac{\log^{[n+1]} \mu(r, f_n)}{\log^{[2]}\mu(r, f)} \le \frac{\rho_g}{\lambda_f},$$

$$\text{(vi)}\quad \limsup_{r\to\infty} \frac{\log^{[n+1]} \mu(r, f_n)}{\log^{[2]}\mu(r, h)} \le \frac{\rho_g}{\lambda_h}.$$

Proof. First we consider $n = 3k$ then from Lemma 2.3 we have for sufficiently large values of r,

$$\begin{aligned} \log^{[n+1]} \mu(r, f_n) &\le \log^{[2]} M(r, h) + O(1) \\ &\le (\rho_h + \varepsilon) \log r + O(1). \end{aligned} \tag{2.10}$$

Also we have for $r \ge r_0$ and $0 < \varepsilon < \lambda_f$,

$$\log^{[2]} \mu(r, f) \ge (\lambda_f - \varepsilon) \log r. \tag{2.11}$$

Therefore from (2.10) and (2.11) we get for $r \ge r_0$ and $0 < \varepsilon < \lambda_f$,

$$\frac{\log^{[n]} \mu(r, f_n)}{\log \mu(r, f)} \le \frac{(\rho_h + \varepsilon) \log r + O(1)}{(\lambda_f - \varepsilon) \log r}.$$

Since $\varepsilon > 0$ is arbitrary,

$$\limsup_{r\to\infty} \frac{\log^{[n]} \mu(r, f_n)}{\log \mu(r, f)} \le \frac{\rho_h}{\lambda_f},$$

which is result (i).

Similarly we get other parts of the theorem.

This proves the theorem. □

3. Conclusion

Our main goal through this paper is to generalized and extend some previous results on growth properties of the maximum term on iteration of three entire functions of non zero finite order, which have not studied previously. But still there remains some problems to be investigated for future researchers in this field.

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NEIGHBORHOOD TOPOLOGICAL INDICES OF METAL-ORGANIC NETWORKS

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Abstract

A group of chemical compounds containing organic ligands and metal ions(clusters) called as Metal organic networks ($MONs$). These are found as one, two and three dimensional structures of porous and subordinate class of coordination polymers. The characteristics of $MONs$ are high surface area, large pore volume, different morphology and very good chemical stability. The applications of $MONs$ includes gas storage, heterogeneous catalysis and sensing of various gases. The stability and characteristics of these networks have become important because of the above said characteristics. The numerical invariants used to predict the physicochemical characteristics and bio-activities of chemical compounds known as topological indices. In our proposed work, we compute neighborhood redefined first Zagreb index, neighborhood redefined second Zagreb index and Generalized Reciprocal Sanskruti index for two different $MONs$.
2020 Mathematical Sciences Classification: 05C07, 05C09, 05C92.
Keywords and Phrases: Topological indices; Metal-organic networks.

1. Introduction and Terminologies

All planets are the decomposition of various types of elements and each element has significant role in the formation of the earth. Oxygen, hydrogen and nitrogen are the important constituents in the formation of the earth. The echo friendly source of energy is the hydrogen [10, 13]. Hydrogen in the form of a gas, is used in fuel cells and power engines. Out of various categories of available gases, it is difficult for the human beings to recognise the leak of gas because hydrogen has less smell.

A very fast hydrogen detecting device having organic ligands and metal identified as metal-organic network was introduced by Won-Tea et al. [19]. Along with detecting and sensing characteristics, the $MONs$ have other physicochemical characteristics viz., grafting active groups, exchanging of ions, preparation of composites for useful substances. $MONs$ used devices for separation, purification and storage of gases.

Graph theory offers a useful tool in the discipline of chemistry to predict the various properties of chemical compounds(molecules) called topological index*(TI)* [3, 6, 14, 17]. *TI* is the important tool which helps to describe physicochemical and biological characteristics of the chemical compounds.

In the year 2019, Wasson described the concept of linker competition with a MON for topological perception. TIs plays an important role in the *QSAR/QSPR* [8, 12] to link the various chemical compound structures with a biological activity and chemical characteristics.

Recently Hafiz Muhammad Awasis et al. [1] and Gang Hong et al. [5] derived the various expressions of topological indices and computed for $MONs$. In this proposed work, neighborhood redefined first Zagreb index, neighborhood redefined second Zagreb index and Generalized Reciprocal Sanskruti index are computed for both MON_1 and MON_2 networks.

Chemical graph theory is a discipline of mathematical chemistry helps to solve molecular problems. Graph theory is used for mathematical modelling of chemical compounds to know the insights of chemical compounds. In a graph G, the vertices and the edges are represented by atoms and links of a chemical compound. In this paper, G represent simple graph with edges E and vertices V. The vertex degree d_t of t be the total adjoining vertices. For the notations used see [18].

Definition 1.1. *Shanmukha et al.,* [16] *introduced neighborhood version of the redefined first and second Zagreb indices and are defined as*

$$NReZ_1(G) = \sum_{st \in E(G)} \frac{[S_G(s) + S_G(t)]}{[S_G(s) \times S_G(t)]},$$

$$NReZ_2(G) = \sum_{st\in E(G)} \frac{[S_G(s) \times S_G(t)]}{[S_G(s) + S_G(t)]}.$$

Definition 1.2. *Shanmukha et al.* [15] *introduced neighborhood version of the Generalized Reciprocal Sanskruti index and is defined as*

$$RS(G) = \sum_{st\in E(G)} \left(\frac{S_G(s) + S_G(t) - 2}{S_G(s) \times S_G(t)} \right)^3.$$

2. Metal-organic Network(MON)

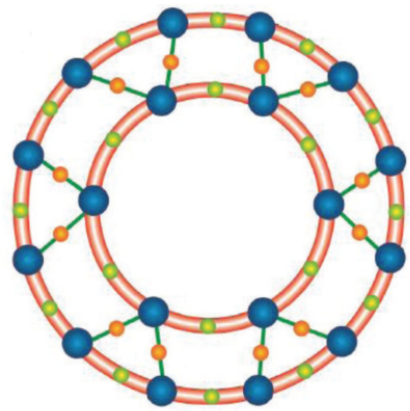


Figure 2.1: Basic Metal organic network.

The formation of MON constitutes metal and organic ligands illustrated in Figure 2.1. The metal and the Zeolite imidazole is represented by the blue vertex and the organic ligand by orange vertex in the figure. The first $MON_1(n)$ which is the first metal organic network is obtained by joining the nodes of the metals of lower layer of the second primary MON. The MON_2 which is the second metal organic network obtained by joining the nodes between the organic ligands of the two primary $MONs$. It is observed that the two organic ligands in the upper layer of MON are joined with lower layer organic ligands.

For dimension $n = 2$, the $MON_1(n)$ and $(MON_2(n))$ are represented in Figure 3.1 and Figure 3.2 respectively. The two $MONs$ have $|V(MON_1(n))| = |V(MON_2(n))| = 48n$ and $|E(MON_1(n))| = |E(MON_2(n))| = 72n - 12$, where $n \geq 2$.

3. Main Results

It has two Subsections. Here we compute different topological indices of $MON_1(n)$ and $MON_2(n)$ in Subsections 3.1 and 3.2 respectively.

3.1. Metal-organic Network $MON_1(n)$

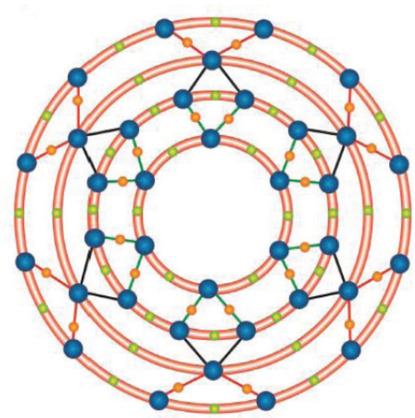


Figure 3.1: $MON_1(n)$ for n=2.

It is noticed from the Figure 3.1, that the Partition of the edges $MON_1(n)$ with respect to neighbour degree sum of end vertices of each edge has eight partitions as below,

$$\begin{aligned}
E_{6,6} &= \{e = st \in E(MON_1(n)) | d_s = 6,\ d_t = 6\}, \\
E_{6,9} &= \{e = st \in E(MON_1(n)) | d_s = 6,\ d_t = 9\}, \\
E_{8,8} &= \{e = st \in E(MON_1(n)) | d_s = 8,\ d_t = 8\}, \\
E_{8,12} &= \{e = st \in E(MON_1(n)) | d_s = 8,\ d_t = 12\}, \\
E_{9,16} &= \{e = st \in E(MON_1(n)) | d_s = 9,\ d_t = 16\}, \\
E_{10,12} &= \{e = st \in E(MON_1(n)) | d_s = 10,\ d_t = 12\}, \\
E_{10,16} &= \{e = st \in E(MON_1(n)) | d_s = 10,\ d_t = 16\}, \\
E_{12,16} &= \{e = st \in E(MON_1(n)) | d_s = 12,\ d_t = 16\}.
\end{aligned}$$

such that

$$\begin{aligned}
|E_{6,6}| &= 24, \\
|E_{6,9}| &= 12, \\
|E_{8,8}| &= 24, \\
|E_{8,12}| &= 12(2n-1), \\
|E_{9,16}| &= 12, \\
|E_{10,12}| &= 12(n-2), \\
|E_{10,16}| &= 12(n-2), \\
|E_{12,16}| &= 24(n-1).
\end{aligned}$$

Theorem 3.1. *The neighborhood version of the redefined first Zagreb index of $MON_1(n)$ with $n \geq 2$ is given by*

$$NReZ_1[MON_1(n)](G) = \frac{1265}{100}n + \frac{6395}{1250}.$$

Proof. The neighborhood version of the redefined first Zagreb index

$$\begin{aligned}
NReZ_1(G) &= \textstyle\sum_{st \in E(G)} \frac{[S_G(s) + S_G(t)]}{[S_G(s) \times S_G(t)]} \\
&= E_{6,6}\left(\frac{6+6}{6\times 6}\right) + E_{6,9}\left(\frac{6+9}{6\times 9}\right) + E_{8,8}\left(\frac{8+8}{8\times 8}\right) + E_{8,12}\left(\frac{8+12}{8\times 12}\right) + E_{9,16}\left(\frac{9+16}{9\times 16}\right) \\
&+ E_{10,12}\left(\frac{10+12}{10\times 12}\right) + E_{10,16}\left(\frac{10+16}{10\times 16}\right) + E_{12,16}\left(\frac{12+16}{12\times 16}\right) \\
&= 24\left(\frac{12}{36}\right) + 12\left(\frac{15}{54}\right) + 24\left(\frac{16}{64}\right) + (24n-12)\left(\frac{20}{96}\right) + 12\left(\frac{25}{144}\right) + (12n-24)\left(\frac{22}{120}\right) \\
&+ (12n-24)\left(\frac{26}{160}\right) + (24n-24)\left(\frac{28}{192}\right) \\
NReZ_1[MON_1(n)](G) &= \frac{1265}{100}n + \frac{6395}{1250}.
\end{aligned}$$

□

Theorem 3.2. *The neighborhood version of the redefined second Zagreb index of $MON_1(n)$ with $n \geq 2$ is given by*

$$NReZ_2[MON_1(n)](G) = \frac{52384}{125}n - \frac{275566}{1250}.$$

Proof. The neighborhood version of the redefined second Zagreb index

$$\begin{aligned}
NReZ_2(G) &= \textstyle\sum_{st \in E(G)} \frac{[S_G(s) \times S_G(t)]}{[S_G(s) + S_G(t)]} \\
NReZ_2[MON_1(n)](G) &= E_{6,6}\left(\frac{6\times 6}{6+6}\right) + E_{6,9}\left(\frac{6\times 9}{6+9}\right) + E_{8,8}\left(\frac{8\times 8}{8+8}\right) + E_{8,12}\left(\frac{8\times 12}{8+12}\right)
\end{aligned}$$

$$+ E_{9,16}\left(\frac{9\times 16}{9+16}\right) + E_{10,12}\left(\frac{10\times 12}{10+12}\right) + E_{10,16}\left(\frac{10\times 16}{10+16}\right) + E_{12,16}\left(\frac{12\times 16}{12+16}\right)$$
$$= 24\left(\frac{36}{12}\right) + 12\left(\frac{54}{15}\right) + 24\left(\frac{64}{16}\right) + (24n-12)\left(\frac{96}{20}\right) + 12\left(\frac{144}{25}\right) + (12n-24)\left(\frac{120}{22}\right)$$
$$+ (12n-24)\left(\frac{160}{26}\right) + (24n-24)\left(\frac{192}{28}\right)$$
$$NReZ_2[MON_1(n)](G) = \frac{52384}{125}n - \frac{275566}{1250}.$$

□

Theorem 3.3. *The Reciprocal Sanskruti index of $MON_1(n)$ with $n \geq 2$ is given by*
$$RS[MON_1(n)](G) = \frac{1569}{5000}n + \frac{6511}{10000}.$$

Proof. The Reciprocal Sanskruti index
$$RS(G) = \sum_{st\in E(G)}\left(\frac{S_G(s)+S_G(t)-2}{S_G(s)\times S_G(t)}\right)^3$$
$$RS[MON_1(n)](G) = E_{6,6}\left(\frac{6+6-2}{6\times 6}\right)^3 + E_{6,9}\left(\frac{6+9-2}{6\times 9}\right)^3 + E_{8,8}\left(\frac{8+8-2}{8\times 8}\right)^3$$
$$+ E_{8,12}\left(\frac{8+12-2}{8\times 12}\right)^3 + E_{9,16}\left(\frac{9+16-2}{9\times 16}\right)^3 + E_{10,12}\left(\frac{10+12-2}{10\times 12}\right)^3$$
$$+ E_{10,16}\left(\frac{10+16-2}{10\times 16}\right)^3 + E_{12,16}\left(\frac{12+16-2}{12\times 16}\right)^3$$
$$= 24\left(\frac{10}{36}\right)^3 + 12\left(\frac{13}{54}\right)^3 + 24\left(\frac{14}{64}\right)^3 + (24n-12)\left(\frac{18}{96}\right)^3 + 12\left(\frac{23}{144}\right)^3$$
$$+ (12n-24)\left(\frac{20}{120}\right)^3 + (12n-24)\left(\frac{24}{160}\right)^3 + (24n-24)\left(\frac{26}{192}\right)^3$$
$$RS[MON_1(n)](G) = \frac{1569}{5000}n + \frac{6511}{10000}.$$

□

3.2. Metal-organic Network $MON_2(n)$

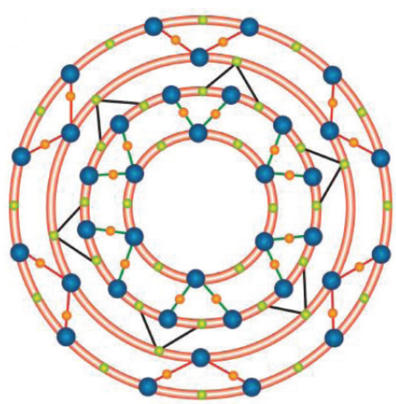


Figure 3.2: $MON_2(n)$ for n=2.

It is noticed from the Figure 3.2, that the Partition of the edges $MON_2(n)$ w.r.t neighbour degree sum of end vertices of each edge has eight partitions as below,
$$E_{6,6} = \{e = st \in E(MON_1(n)) | d_s = 6,\ \ d_t = 6\},$$

$$
\begin{aligned}
E_{6,7} &= \{e = st \in E(MON_1(n)) | d_s = 6,\ d_t = 7\},\\
E_{7,8} &= \{e = st \in E(MON_1(n)) | d_s = 7,\ d_t = 8\},\\
E_{7,12} &= \{e = st \in E(MON_1(n)) | d_s = 7,\ d_t = 12\},\\
E_{8,8} &= \{e = st \in E(MON_1(n)) | d_s = 8,\ d_t = 8\},\\
E_{8,10} &= \{e = st \in E(MON_1(n)) | d_s = 8,\ d_t = 10\},\\
E_{10,14} &= \{e = st \in E(MON_1(n)) | d_s = 10,\ d_t = 14\},\\
E_{12,14} &= \{e = st \in E(MON_1(n)) | d_s = 12,\ d_t = 14\}.
\end{aligned}
$$

such that

$$
\begin{aligned}
|E_{6,6}| &= 24,\\
|E_{6,7}| &= 12,\\
|E_{7,8}| &= 12n,\\
|E_{7,12}| &= 12(n-1),\\
|E_{8,8}| &= 12,\\
|E_{8,10}| &= 24(n-1),\\
|E_{10,14}| &= 12(n-1),\\
|E_{12,14}| &= 12(n-1)
\end{aligned}
$$

Theorem 3.4. *The neighborhood version of the redefined first Zagreb index of $MON_2(n)$ with $n \geq 2$ is given by*

$$NReZ_1[MON_2(n)](G) = \frac{15243}{1000}n + \frac{1343}{500}.$$

Proof. The neighborhood version of the redefined second Zagreb index

$$
\begin{aligned}
NReZ_1(G) &= \textstyle\sum_{st\in E(G)} \frac{[S_G(s) + S_G(t)]}{[S_G(s) \times S_G(t)]}\\
NReZ_1[MON_2(n)](G) &= E_{6,6}\left(\frac{6+6}{6\times 6}\right) + E_{6,7}\left(\frac{6+7}{6\times 7}\right) + E_{7,8}\left(\frac{7+8}{7\times 8}\right) + E_{7,12}\left(\frac{7+12}{7\times 12}\right)\\
&+ E_{8,8}\left(\frac{8+8}{8\times 8}\right) + E_{8,10}\left(\frac{8+10}{8\times 10}\right) + E_{10,14}\left(\frac{10+14}{10\times 14}\right) + E_{12,14}\left(\frac{12+14}{12\times 14}\right)\\
&= 24\left(\frac{12}{36}\right) + 12\left(\frac{13}{42}\right) + 12n\left(\frac{15}{56}\right) + (12n-12)\left(\frac{19}{84}\right) + 12\left(\frac{16}{64}\right) + (24n-24)\left(\frac{18}{80}\right)\\
&+ (12n-12)\left(\frac{24}{140}\right) + (12n-12)\left(\frac{26}{168}\right)\\
NReZ_1[MON_2(n)](G) &= \frac{15243}{1000}n + \frac{1343}{500}.
\end{aligned}
$$

□

Theorem 3.5. *The neighborhood version of the redefined second Zagreb index of $MON_2(n)$ with $n \geq 2$ is given by*

$$NReZ_2[MON_2(n)](G) = \frac{176029}{500}n - \frac{14849}{100}.$$

Proof. The neighborhood version of the redefined second Zagreb index

$$
\begin{aligned}
NReZ_2(G) &= \textstyle\sum_{st\in E(G)} \frac{[S_G(s) \times S_G(t)]}{[S_G(s) + S_G(t)]}\\
NReZ_2[MON_2(n)](G) &= E_{6,6}\left(\frac{6\times 6}{6+6}\right) + E_{6,7}\left(\frac{6\times 7}{6+7}\right) + E_{7,8}\left(\frac{7\times 8}{7+8}\right) + E_{7,12}\left(\frac{7\times 12}{7+12}\right)\\
&+ E_{8,8}\left(\frac{8\times 8}{8+8}\right) + E_{8,10}\left(\frac{8\times 10}{8+10}\right) + E_{10,14}\left(\frac{10\times 14}{10+14}\right) + E_{12,14}\left(\frac{12\times 14}{12+14}\right)\\
&= 24\left(\frac{36}{12}\right) + 12\left(\frac{42}{13}\right) + 12n\left(\frac{56}{15}\right) + (12n-12)\left(\frac{84}{19}\right) + 12\left(\frac{64}{16}\right) + (24n-24)\left(\frac{80}{18}\right)
\end{aligned}
$$

$$+ (12n-12)\left(\frac{140}{24}\right) + (12n-12)\left(\frac{168}{26}\right)$$

$$NReZ_2[MON_2(n)](G) = \frac{176029}{500}n - \frac{14849}{100}.$$

□

Theorem 3.6. *The Reciprocal Sanskruti index of $MON_2(n)$ with $n \geq 2$ is given by*

$$RS[MON_2(n)](G) = \frac{5231}{10000}n + \frac{2413}{5000}.$$

Proof. The Reciprocal Sanskruti index

$$RS(G) = \sum_{st\in E(G)} \left(\frac{S_G(s)+S_G(t)-2}{S_G(s)\times S_G(t)}\right)^3$$

$$\begin{aligned} RS[MON_2(n)](G) &= E_{6,6}\left(\frac{6+6-2}{6\times 6}\right)^3 + E_{6,7}\left(\frac{6+7-2}{6\times 7}\right)^3 + E_{7,8}\left(\frac{7+8-2}{7\times 8}\right)^3 \\ &+ E_{7,12}\left(\frac{7+12-2}{7\times 12}\right)^3 + E_{8,8}\left(\frac{8+8-2}{8\times 8}\right)^3 + E_{8,10}\left(\frac{8+10-2}{8\times 10}\right)^3 \\ &+ E_{10,14}\left(\frac{10+14-2}{10\times 14}\right)^3 + E_{12,14}\left(\frac{12+14-2}{12\times 14}\right)^3 \\ RS[MON_2(n)](G) &= \frac{5231}{10000}n + \frac{2413}{5000}. \end{aligned}$$

□

4. Numerical and graphical Comparison of indices

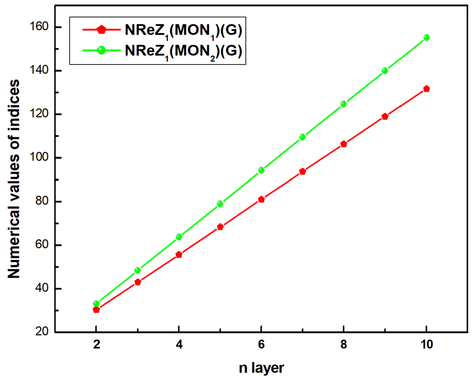
Here we calculate several indices for different values of n. It is noticed from Table 4.1 and Table 4.2, that values of topological indices increases with increase in the n value. The obtained topological indices are represented graphically for some values of n as shown in Figure 4.1.

Table 4.1: Numerical comparison of different indices of $MON_1(n)$ for n=2 to 10.

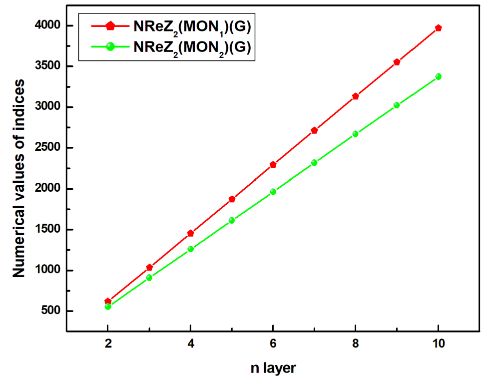
n	2	3	4	5	6	7	8	9	10
$NReZ_1(G)$	30.4160	43.066	55.716	68.366	81.016	93.666	106.316	118.966	131.616
$NReZ_2(G)$	617.6912	1036.8	1455.8	1874.9	2294	2713.1	3132.1	3551.2	3970.3
$RS(G)$	22344	41797	61250	80704	100160	119610	139060	158520	177970

Table 4.2: Numerical comparison of different indices of $MON_2(n)$ for n=2 to 10.

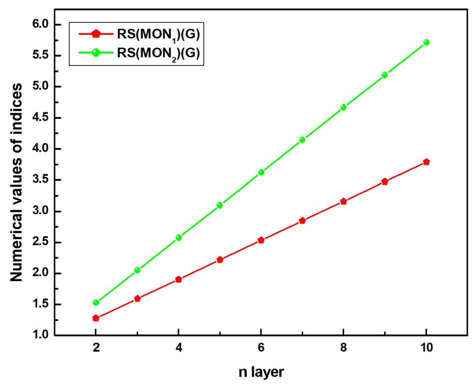
n	2	3	4	5	6	7	8	9	10
$NReZ_1(G)$	33.1720	48.4150	63.6580	78.901	94.144	109.387	124.63	139.873	155.116
$NReZ_2(G)$	555.626	907.684	1259.7	1611.8	1963.9	2315.9	2668	3020	3372.1
$RS(G)$	16509	29124	41739	54355	66970	79585	92200	104820	117430



(a)



(b)



(c)

Figure 4.1: Graphical comparison of $MON_1(n)$ and $MON_2(n)$

5. Conclusion

Topological indices are the numerical descriptors which plot molecular structure to a real number. TIs play an important role in chemistry to predict physical and chemical characteristics and bio-activities of chemical compounds. In this paper, various topological indices are computed viz., neighborhood redefined first Zagreb index, neighborhood redefined second Zagreb index and Reciprocal Sanskruti index for metal organic networks and graphical representation of both types of metal organic networks is depicted.

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ON SOME CLASSES OF MIXED GENERALIZED QUASI-EINSTEIN MANIFOLDS

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Abstract

In this paper we study mixed generalized quasi-Einstein manifold satisfying some curvature conditions like $K.Ric = 0$, $C.Ric = 0$, $N.Ric = 0$, where K, Ric, C and N denote the Reimannian curvature tensor, Ricci tensor, conformal curvature tensor and concircular curvature tensor and obtain some interesting and fruitful results on it.
2020 Mathematical Sciences Classification: 53C25, 53C35.
Keywords and Phrases: Quasi-Einstein Manifolds, generalized quasi-Einstein manifolds, mixed generalized quasi-Einstein manifolds, Riemannian curvature tensor, conformal curvature tensor, concircular curvature tensor.

1. Introduction

Let $\chi(M)$ be the set of all differentiable vector fields on the manifold M. A Riemannian manifold (M^n, g) (n $\geq$ 2) is said to be an Einstein manifold (see [9], p.148) if

$$Ric(X,Y) = \frac{r}{n} g(X,Y), \tag{1.1}$$

where Ric and r denote the Ricci tensor and scalar curvature respectively. The notion of quasi-Einstein manifolds arose during the study of exact solutions of Einstein field equations as well as during considerations of quasi umbilical hypersurfaces. A non flat Riemannian manifold (M^n, g) (n $\geq$ 2) is said to be a quasi-Einstein manifold [3] if its Ricci tensor of type (0,2) is non-zero and satisfies the following condition:

$$Ric(X,Y) = ag(X,Y) + bA(X)A(Y), \tag{1.2}$$

$\forall\ X, Y \in \chi(M)$ and a, b are scalars and A is a non-zero 1-form such that

$$g(X,\rho) = A(X), \tag{1.3}$$

for all vector field X. ρ being a unit vector field, called the generator of the manifold. Also the 1- form A is called the associated 1- form. From the above definition it follows that every Einstein manifold is a subclass of a quasi-Einstein manifold.

The study of quasi-Einstein manifold was continued by U. C. De, Gopal Chandra Ghosh [5] and many others. Several authors have generalized the notion of quasi-Einstein manifold such as generalized quasi-Einstein manifolds ([4],[6]), mixed generalized quasi-Einstein manifolds [2] and many others.

A non flat Riemannian manifold (M^n, g) $(n \geq 2)$ is said to be a generalized quasi-Einstein manifold [4] and denoted by $G(QE)_n$, if its Ricci tensor of type $(0, 2)$ is satisfies the following condition:

$$Ric(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y), \tag{1.4}$$

where a, b, c are scalars and A, B are two non-zero 1-forms. The unit vector fields ρ and σ corresponding to the 1-forms A and B respectively defined by

$$g(X,\rho) = A(X), \quad g(X,\sigma) = B(X). \tag{1.5}$$

Also,

$$g(\rho,\rho) = 1, \quad g(\sigma,\sigma) = 1, \quad g(\rho,\sigma) = 0. \tag{1.6}$$

Putting $X = Y = e_i$ in (1.4), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point on the manifold and taking summation over i, $1 \leq i \leq n$, we get

$$r = na + b + c.$$

From (1.4), (1.5) and (1.6), we have

$$\begin{aligned} &Ric(X,\rho) = (a+b)A(X), \quad Ric(X,\sigma) = (a+c)B(X), \\ &Ric(\rho,\rho) = a+b, \quad Ric(\sigma,\sigma) = a+c. \end{aligned} \tag{1.7}$$

In 2007, A. Bhattacharya and T. De [1] introduced the notion of mixed generalized quasi-Einstein manifold. A non-flat Riemannian manifold is said to be a mixed generalized quasi-Einstein manifold and denoted by $MG(QE)_n$, if its Ricci tensor is non-zero and satisfies the following condition:

$$\begin{aligned} Ric(X,Y) &= ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \\ &\quad + d[A(X)B(Y) + A(Y)B(X)], \end{aligned} \tag{1.8}$$

where a, b, c, d are scalars and A, B are two non-zero 1-forms which are defined earlier.
From (1.5), (1.6) and (1.8), we have

$$\begin{aligned} &Ric(Y,\rho) = (a+b)A(Y) + dB(Y), \quad Ric(Y,\sigma) = (a+c)B(Y) + dA(Y), \\ &Ric(\rho,\sigma) = d. \end{aligned} \tag{1.9}$$

A Riemannian manifold is said to be a manifold of mixed generalized quasi constant curvature [2] denoted by $MG(QC)_n$, if the curvature tensor $\overline{K}$ of type $(0,4)$ satisfies the condition

$$\begin{aligned} \overline{K}(X,Y,Z,W) &= a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ &\quad + b[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\ &\quad + g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)] \\ &\quad + c[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z) \\ &\quad + g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)] \\ &\quad + d[g(X,W)\{A(Y)B(Z) + B(Y)A(Z)\} \\ &\quad - g(Y,W)\{A(X)B(Z) + (BX)A(Z)\} \\ &\quad + g(Y,Z)\{A(X)B(W) + B(X)A(W)\} \\ &\quad - g(X,Z)\{A(Y)B(W) + B(Y)A(W)\}], \end{aligned} \tag{1.10}$$

where a, b, c, d are scalars and A, B are non-zero 1-forms.

The Weyl conformal curvature tensor ([7], [9]) C of type $(1,3)$ of an n-dimensional Riemannian manifold (M^n, g) $(n > 3)$ is defined by

$$\begin{aligned} C(X,Y,Z) &= K(X,Y,Z) - \frac{1}{n-2}[Ric(Y,Z)X - Ric(X,Z)Y \\ &\quad + g(Y,Z)R(X) - g(X,Z)R(Y)] \\ &\quad + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{1.11}$$

and satisfynig the following properties:
$\overline{C}(X,Y,Z,W) = -\overline{C}(Y,X,Z,W)$ and $\overline{C}(X,Y,Z,W) = -\overline{C}(X,Y,W,Z)$
$\forall\ X,Y,Z,W \in \chi(M)$, where $\overline{C}(X,Y,Z,W) = g(C(X,Y)Z,W)$ is conformal curvature tensor of type $(0,4)$ and R is the Ricci tensor of type (1,1).

The Concircular curvature tensor ([7], [9]) N of type (1,3) of an n-dimensional Riemannian manifold (M^n, g) $(n \geq 3)$ is defined by

$$N(X,Y,Z) = K(X,Y,Z) - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \tag{1.12}$$

and satisfynig the following properties:
$\overline{N}(X,Y,Z,W) = -\overline{N}(Y,X,Z,W)$ and $\overline{N}(X,Y,Z,W) = -\overline{N}(X,Y,W,Z)$
$\forall\ X,Y,Z,W \in \chi(M)$, where $\overline{N}(X,Y,Z,W) = g(N(X,Y)Z,W)$ is concircular curvature tensor of type (0,4).
From (1.12), we have

$$\textstyle\sum_{i=1}^{n} \overline{N}(e_i, Y, e_i, \rho) = -Ric(Y,\rho) + \frac{r}{n} g(Y,\rho). \tag{1.13}$$

In 1972, Pokhariyal [10] introduced the notion of W_4 curvature tensor of type (0,4) defined by

$$W_4(X,Y,Z,W) = \overline{K}(X,Y,Z,W) + \frac{1}{n-1}[g(X,Z)Ric(Y,W) - g(X,Y)Ric(Z,W)]. \tag{1.14}$$

The above relations will be used in the next sections.

2. Relation between $MG(QE)_n$ and $MG(QC)_n$

Let the manifold be conformally flat, then from (1.11), we have

$$\begin{aligned}\overline{K}(X,Y,Z,W) = \frac{1}{n-2}&[Ric(Y,Z)g(X,W) - Ric(X,Z)g(Y,W)\\ &+ g(Y,Z)Ric(X,W) - g(X,Z)Ric(Y,W)]\\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)].\end{aligned} \tag{2.1}$$

Using (1.8) in (2.1), we get

$$\begin{aligned}\overline{K}(X,Y,Z,W) = \alpha&[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\\ &+ \beta[g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z)\\ &+ g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)]\\ &+ \gamma[g(X,W)B(Y)B(Z) - g(Y,W)B(X)B(Z)\\ &+ g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W)]\\ &+ \delta[g(X,W)\{A(Y)B(Z) + B(Y)A(Z)\}\\ &- g(Y,W)\{A(X)B(Z) + (BX)A(Z)\}\\ &+ g(Y,Z)\{A(X)B(W) + B(X)A(W)\}\\ &- g(X,Z)\{A(Y)B(W) + B(Y)A(W)\}],\end{aligned} \tag{2.2}$$

where $\alpha = \frac{2a(n-1)-r}{(n-1)(n-2)}$, $\beta = \frac{b}{n-2}$, $\gamma = \frac{c}{n-1}$, $\delta = \frac{d}{n-2}$.
This leads us to the following theorem:

Theorem 2.1. *Every conformally flat mixed generalized quasi-Einstein manifold is a $MG(QC)_n$.*

Now, putting $Y = Z = e_i$ in (1.10), where $\{e_1\}$ is an orthonormal basis of the tangent space at each point on the manifold and taking summation over i, $1 \le i \le n$, we get

$$\begin{aligned}Ric(X,W) = a(n-2)g(X,W) &+ b(n-2)A(X)A(W)\\ &+ c(n-2)B(X)B(W)\\ &+ d(n-2)[A(X)B(W) + A(W)B(X)],\end{aligned} \tag{2.3}$$

which gives

$$\begin{aligned}Ric(X,Y) = \alpha g(X,Y) &+ \beta A(X)A(Y) + \gamma B(X)B(Y)\\ &+ \delta[A(X)B(Y) + A(Y)B(X)]\end{aligned} \tag{2.4}$$

where $\alpha = a(n-2)$, $\beta = b(n-2)$, $\gamma = c(n-2)$, $\delta = d(n-2)$.
This leads us to the following theorem:

Theorem 2.2. *If a Riemannian manifold is a conformally flat $MG(QE)_n$, then it is a $MG(QC)_n$.*

3. Relation between $MG(QE)_n$ and W_4-curvature tensor

Let W_4-curvature tensor be flat, then from (1.14), we have

$$\overline{K}(X,Y,Z,W) = \frac{1}{n-1}[g(X,Y)Ric(Z,W) - g(X,Z)Ric(Y,W)]. \tag{3.1}$$

Putting $Z = \rho$ and $W = \sigma$ in (3.1) and using (1.9), we get

$$\overline{K}(X,Y,\rho,\sigma) = \frac{1}{n-1}[dg(X,Y) - A(X)\{(a+c)B(Y) + dA(Y)\}]. \tag{3.2}$$

Again, putting $X = \rho$, $Y = \sigma$, $Z = X$ and $W = Y$ in (3.1) and using (1.9), we have

$$\overline{K}(\rho,\sigma,X,Y) = \frac{1}{n-1}[-A(X)\{(a+c)B(Y) + dA(Y)\}]. \tag{3.3}$$

From (3.2) and (3.3), we get,

$$dg(X,Y) = 0.$$

But $g(X,Y) \neq 0$. Therefore,

$$d = 0. \tag{3.4}$$

Using (3.4) in (1.8), we have

$$Ric(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y), \tag{3.5}$$

which is a $G(QE)_n$. This leads us to the following theorem:

Theorem 3.1. *A W_4-flat mixed generalized quasi-Einstein manifold is a generalized quasi-Einstein manifold.*

4. $MG(QE)_n$ with the condition $C.Ric = 0$

Let us consider a $MG(QE)_n$ ($n > 3$) satisfying the condition $C.Ric = 0$. Then we have

$$Ric(C(X,Y)Z,W) + Ric(Z,C(X,Y)W) = 0. \tag{4.1}$$

Using (1.8) in (4.1), we get

$$\begin{aligned} &b[g(C(X,Y)Z,W) + g(Z,C(X,Y)W)] \\ &+ c[A(C(X,Y)Z)A(W) + A(Z)A(C(X,Y)W)] \\ &+ d[A(C(X,Y)Z)B(W) + A(W)B(C(X,Y)Z) \\ &+ A(Z)B(C(X,Y)W) + B(Z)A(C(X,Y)W)] = 0. \end{aligned} \tag{4.2}$$

Putting $Z = W = \rho$ in (4.2) and using symmetric property of metric tensor,
$g(C(X,Y)Z,W) = \overline{C}(X,Y,Z,W)$ and skew symmetric property of $\overline{C}(X,Y,Z,W)$
in last two slots gives

$$d[B(C(X,Y)\rho) + B(C(X,Y)\rho)] = 0, \tag{4.3}$$

which in view of (1.5) gives

$$dC(X,Y,\rho,\sigma) = 0,$$

which shows that either $d = 0$ or $C(X,Y,\rho,\sigma) = 0$. If $d = 0$, then $MG(QE)_n$ reduces to generalized quasi-Einstein manifold.
This leads us to the following theorem.

Theorem 4.1. *Every mixed generalized quasi-Einstein manifold satisfying the condition $C.Ric = 0$ is a generalized quasi-Einstein manifold provided $C(X,Y,\rho,\sigma)$ is non-zero at each point of the manifold.*

5. $MG(QE)_n$ with the condition $K.Ric = 0$

Let the manifold be conformally flat manifold. Then from (1.11)

$$\begin{aligned} K(X,Y,Z) = {}&\frac{1}{n-2}[Ric(Y,Z)X - Ric(X,Z)Y \\ &+ g(Y,Z)R(X) - g(X,Z)R(Y)] \\ &+ \frac{r}{(n-1)(n-2)}[g(X,Z)Y - g(Y,Z)X]. \end{aligned} \tag{5.1}$$

Let the mixed generalized quasi-Einstein manifold $MG(QE)_n$ satisfies the condition $K.Ric = 0$ is either a generalized quasi-Einstein manifold or the vector fields ρ and σ are co-directional provided μ is not eigenvalue of the Ricci tensor. Then

$$Ric(K(X,Y)Z,W) + Ric(Z,K(X,Y)W) = 0. \tag{5.2}$$

By virtue of (5.1) in (5.2), we get

$$\begin{aligned} &g(Y,Z)Ric(R(X),W) - g(X,Z)Ric(R(Y),W) \\ &+ g(Y,W)Ric(R(X),Z) - g(X,W)Ric(R(Y),Z) \\ &- \frac{r}{n-1}[g(Y,Z)Ric(X,W) - g(X,Z)Ric(Y,W) \\ &+ g(Y,W)Ric(X,Z) - g(X,W)Ric(Y,Z))] = 0. \end{aligned} \tag{5.3}$$

Let μ be the eigenvalue of R corresponding to the eigenvector X. Then
$R(X) = \mu X$, that is, $Ric(X,Y) = \mu g(X,Y)$.
Therefore,

$$Ric(R(X),Y) = \mu Ric(X,Y). \tag{5.4}$$

Using (5.4) in (5.3), we get

$$\begin{aligned} &[g(Y,Z)Ric(X,W) - g(X,Z)Ric(Y,W) + g(Y,W)Ric(X,Z) \\ &- g(X,W)Ric(Y,Z)]\{\mu - \frac{r}{n-1}\} = 0. \end{aligned} \tag{5.5}$$

Since, $\mu - \frac{r}{n-1} \neq 0$, hence, we obtain

$$g(Y,Z)Ric(X,W) - g(X,Z)Ric(Y,W) + g(Y,W)Ric(X,Z) - g(X,W)Ric(Y,Z) = 0. \tag{5.6}$$

Now, using (1.8) in (5.6), we obtain

$$\begin{aligned}
& g(Y,Z)[ag(X,W) + bA(X)A(W) + cB(X)B(W) + d\{A(X)B(W) + B(X)A(W)\}] \\
& - g(X,Z)[ag(Y,W) + bA(Y)A(W) + cB(Y)B(W) + d\{A(Y)B(W) + B(Y)A(W)\}] \\
& + g(Y,W)[ag(X,Z) + bA(X)A(Z) + cB(X)B(Z) + d\{A(X)B(Z) + B(X)A(Z)\}] \\
& - g(X,W)[ag(Y,Z) + bA(Y)A(Z) + cB(Y)B(Z) + d\{A(Y)B(Z) + B(Y)A(Z)\}] \\
& = 0.
\end{aligned} \tag{5.7}$$

Putting $Z = W = \rho$ and using (1.8) in (5.7), we get

$$d[A(Y)B(X) - A(X)B(Y)] = 0 \tag{5.8}$$

which shows that either $d = 0$ or $A(Y)B(X) - A(X)B(Y) = 0$. If $d = 0$, then $MG(QE)_n$ reduces to $G(QE)_n$ and if $A(Y)B(X) - A(X)B(Y) = 0$, then the vector field X and Y are co-directional.
This leads us to the following theorem.

Theorem 5.1. *In a conformally flat with the condition $K.Ric = 0$ mixed generalized quasi-Einstein manifold is either a generalized quasi-Einstein manifold or the vector fields ρ and σ are co-directional provided μ is not eigenvalue of the Ricci tensor R.*

6. Example of $MG(QE)_n$

We define a Riemannian metric g in 4-dimensional space $\mathbb{R}^4$ by the relation [8]

$$ds^2 = g_{ij}dx^i dx^j = (1 + 2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2], \tag{6.1}$$

where x^1, x^2, x^3, x^4 are non-zero finite and $p = e^{x^1}k^{-2}$. Then the covariant and contravariant components of the metric tensor are

$$g_{11} = g_{22} = g_{33} = g_{44} = (1 + 2p), \quad g_{ij} = 0 \quad \forall \quad i \neq j \tag{6.2}$$

and

$$g^{11} = g^{22} = g^{33} = g^{44} = \frac{1}{1 + 2p}, \quad g^{ij} = 0 \quad \forall \quad i \neq j. \tag{6.3}$$

The only non-vanishing components of the Christoffel symbols are

$$\begin{aligned}
&\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = \frac{p}{1 + 2p}, \\
&\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{-p}{1 + 2p}.
\end{aligned} \tag{6.4}$$

The non-zero derivatives of (6.4), we have

$$\begin{aligned}
&\frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = \frac{p}{(1 + 2p)^2}, \\
&\frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{-p}{(1 + 2p)^2}.
\end{aligned} \tag{6.5}$$

For the Riemannian curvature tensor,

$$K^l_{ijk} = \underbrace{\begin{vmatrix} \frac{\partial}{\partial x^j} & \frac{\partial}{\partial x^k} \\ \begin{Bmatrix} l \\ ij \end{Bmatrix} & \begin{Bmatrix} l \\ ik \end{Bmatrix} \end{vmatrix}}_{=\mathrm{I}} + \underbrace{\begin{vmatrix} \begin{Bmatrix} m \\ ik \end{Bmatrix} & \begin{Bmatrix} m \\ ij \end{Bmatrix} \\ \begin{Bmatrix} l \\ mk \end{Bmatrix} & \begin{Bmatrix} l \\ mj \end{Bmatrix} \end{vmatrix}}_{=\mathrm{II}}.$$

The non-zero components of (I) are:

$$\begin{aligned}
K^1_{212} &= \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \frac{-p}{(1 + 2p)^2}, \\
K^1_{313} &= \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \frac{-p}{(1 + 2p)^2}, \\
K^1_{414} &= \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{-p}{(1 + 2p)^2}
\end{aligned}$$

and the non-zero components of (II) are:

$$K^1_{313} = \begin{Bmatrix} m \\ 33 \end{Bmatrix}\begin{Bmatrix} 1 \\ m1 \end{Bmatrix} - \begin{Bmatrix} m \\ 31 \end{Bmatrix}\begin{Bmatrix} 1 \\ m3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 33 \end{Bmatrix}\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 31 \end{Bmatrix}\begin{Bmatrix} 1 \\ 13 \end{Bmatrix} = \frac{-p^2}{(1+2p)^2},$$

$$K^1_{414} = \begin{Bmatrix} m \\ 44 \end{Bmatrix}\begin{Bmatrix} 1 \\ m1 \end{Bmatrix} - \begin{Bmatrix} m \\ 31 \end{Bmatrix}\begin{Bmatrix} 1 \\ m3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 44 \end{Bmatrix}\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 31 \end{Bmatrix}\begin{Bmatrix} 1 \\ 13 \end{Bmatrix} = \frac{-p^2}{(1+2p)^2},$$

$$K^3_{232} = \begin{Bmatrix} m \\ 22 \end{Bmatrix}\begin{Bmatrix} 3 \\ m3 \end{Bmatrix} - \begin{Bmatrix} m \\ 23 \end{Bmatrix}\begin{Bmatrix} 3 \\ m2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 22 \end{Bmatrix}\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 23 \end{Bmatrix}\begin{Bmatrix} 3 \\ 12 \end{Bmatrix} = \frac{-p^2}{(1+2p)^2},$$

$$K^4_{242} = \begin{Bmatrix} m \\ 22 \end{Bmatrix}\begin{Bmatrix} 4 \\ m4 \end{Bmatrix} - \begin{Bmatrix} m \\ 42 \end{Bmatrix}\begin{Bmatrix} 4 \\ m2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 22 \end{Bmatrix}\begin{Bmatrix} 4 \\ 14 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 42 \end{Bmatrix}\begin{Bmatrix} 4 \\ 12 \end{Bmatrix} = \frac{-p^2}{(1+2p)^2},$$

$$K^4_{343} = \begin{Bmatrix} m \\ 33 \end{Bmatrix}\begin{Bmatrix} 4 \\ m4 \end{Bmatrix} - \begin{Bmatrix} m \\ 43 \end{Bmatrix}\begin{Bmatrix} 4 \\ m3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 33 \end{Bmatrix}\begin{Bmatrix} 4 \\ 14 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 43 \end{Bmatrix}\begin{Bmatrix} 4 \\ 13 \end{Bmatrix} = \frac{-p^2}{(1+2p)^2}.$$

Adding components corresponding (I) and (II), we have

$$K^1_{212} = \frac{-p}{(1+2p)^2},\ K^1_{313} = \frac{-p-p^2}{(1+2p)^2} = K^1_{414},$$

$$K^3_{232} = K^4_{242} = K^4_{343} = \frac{-p^2}{(1+2p)^2}.$$

Thus, the non-zero components of curvature tensor, up to symmetry are,

$$R_{1212} = R_{1313} = R_{1414} = \frac{-p}{1+2p},$$

$$R_{3232} = R_{4242} = R_{4343} = \frac{-p^2}{1+2p}$$

and the Ricci tensor

$$R_{11} = g^{jh}R_{1j1h} = g^{22}R_{1212} + g^{33}R_{1313} + g^{44}R_{1414} = \frac{-3p}{(1+2p)^2},$$

$$R_{22} = g^{jh}R_{2j2h} = g^{11}R_{2121} + g^{33}R_{2323} + g^{44}R_{2424} = \frac{-p}{(1+2p)},$$

$$R_{33} = g^{jh}R_{3j3h} = g^{11}R_{3131} + g^{22}R_{3232} + g^{44}R_{3434} = \frac{-p}{(1+2p)},$$

$$R_{44} = g^{jh}R_{4j4h} = g^{11}R_{4141} + g^{22}R_{4242} + g^{33}R_{4343} = \frac{-p}{(1+2p)}.$$

The scalar curvature r is

$$r = g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} + g^{44}R_{44} = \frac{-6p(1+p)}{(1+2p)^3}.$$

Let us consider the associated scalars a, b, c, d are defined by
$a = \frac{p}{(1+2p)^2},\ b = \frac{-2p}{(1+2p)^3},\ c = \frac{-p}{(1+2p)^3},\ d = \frac{-p}{(1+2p)^2}$
and the 1-forms
$A_1 = B_1 = \sqrt{1+2p}, \quad A_i = B_i = 0 \quad \forall \quad i = 2, 3, 4$
where generators are unit vector fields, then from (1.8), we have

$$R_{11} = ag_{11} + bA_1A_1 + cB_1B_1 + d(A_1B_1 + A_1B_1), \tag{6.6}$$

$$R_{22} = ag_{22} + bA_2A_2 + cB_2B_2 + d(A_2B_2 + A_2B_2), \tag{6.7}$$

$$R_{33} = ag_{33} + bA_3A_3 + cB_3B_3 + d(A_3B_3 + A_3B_3), \tag{6.8}$$

$$R_{44} = ag_{44} + bA_4A_4 + cB_4B_4 + d(A_4B_4 + A_4B_4). \tag{6.9}$$

Now,

$$R.H.S.\ of\ (6.6) \quad = \quad ag_{11} + bA_1A_1 + cB_1B_1 + d(A_1B_1 + A_1B_1)$$

$$
\begin{aligned}
&= \frac{p(1+2p)}{1+2p} - \frac{2p(1+2p)}{(1+2p)^3} - \frac{p(1+2p)}{(1+2p)^3} - \frac{2p(1+2p)}{2(1+2p)^2} \\
&= \frac{p}{1+2p} - \frac{2p}{(1+2p)^2} - \frac{p}{(1+2p)^2} - \frac{p}{1+2p} \\
&= -\frac{3p}{(1+2p)^2} \\
&= R_{11} \\
&= L.H.S.\ of\ (6.6).
\end{aligned}
$$

By similar argument it can be shown that (6.7), (6.8) and (6.9) are also true.
Hence ($\mathbb{R}^4$,g) is a $MG(QE)_n$.

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FRACTIONAL CALCULUS OF PRODUCT OF *M*-SERIES AND *I*-FUNCTION OF TWO VARIABLES

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Abstract

The object of this paper is to develop the generalized fractional calculus formulas for the product of generalized M-series and I-function of two variables which is based on generalized fractional integration and differentiation operators of arbitrary complex order involving Appell hypergeometric function F_3 as a kernel due to Saigo and Maeda. On account of general nature of the Saigo-Maeda operators, a large number of results involving Saigo and Riemann-Liouville operetors are found as corollaries. Again due to general nature of I-function of two variables and M-series, some special cases also have been considered.

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1. Introduction

No doubt, the fractional calculus has become an important mathematical equipment to solve diverse problems of mathematics, science and engineering. In last some decades,huge amount of research work in fractional calculus and related areas is published due to its applicability in the various fields of science and engineering such as dynamical system in control theory,astrophysics, electrical circuits,mathematical biology, fluid mechanics, image processing and quantum mechanics. The fractional calculus operators with the involvement of various special functions have been successfully applied to construct relevant system in various fields of science and engineering. see [2, 3, 17, 18]. Therefore so many authors have investigated different unifications and extentions of various types of fractional calculus operators. For more detail about fractional calculus operators, we refer to the research monograph by Miller and Ross [15], Samko *et al.*[21], and Kiryakova [14].

The images for special functions of one and more variables under various type of fractional calculus operators have been obtained by so many authors such as Agarwal [1] studied and developed the generalized fractional integration of the product of $\overline{H}$-function and a general class of polynomials in Saigo operators, Kumar [9] established some new unified integral and differential formulas associated with $\overline{H}$-function applying Saigo and Maeda operator and Gupta *et al.*[6] obtained the image formulas of the product of two H functions using Saigo operators. Motivated by these results, we have established some fractional calculus formulas concerning to the product of M-series and I-function of two variables.

Goyal and Agrawal [7] introduced I-function of two variables in 1995, by means of Mellin-Barnes type integrals as

$$I^{m_1,n_1:m_2,n_2;m_3,n_3}_{p,q:p_i^{(1)},q_i^{(1)};p_i^{(2)},q_i^{(2)}:r}\left[\begin{array}{c|c} z_1 & [(e_p:E_p,E_p')]:[(a_j,\alpha_j)_{1,n_2}],[(a_{ji},\alpha_{ji})_{n_2+1,p_i^{(1)}}];[(c_j,\gamma_j)_{1,n_3}],[(c_{ji},\gamma_{ji})_{n_3+1,p_i^{(2)}}] \\ z_2 & [(f_q:F_q,F_q')]:[(b_j,\beta_j)_{1,m_2}],[(b_{ji},\beta_{ji})_{m_2+1,q_i^{(1)}}];[(d_j,\delta_j)_{1,m_3}],[(d_{ji},\delta_{ji})_{m_3+1,q_i^{(2)}}]\end{array}\right]$$

$$=\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^{\xi}\,z_2^{\eta}\,d\xi\,d\eta, \tag{1.1}$$

where $\omega=\sqrt{-1}$ and $\phi_1(\xi)$, $\phi_2(\eta)$, $\psi(\xi,\eta)$ are given by

$$\phi_1(\xi)=\frac{\prod_{j=1}^{m_2}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n_2}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left[\prod_{j=m_2+1}^{q_i^{(1)}}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=n_2+1}^{p_i^{(1)}}\Gamma(a_{ji}-\alpha_{ji}\xi)\right]}, \tag{1.2}$$

$$\phi_2(\eta)=\frac{\prod_{j=1}^{m_3}\Gamma(d_j-\delta_j\eta)\prod_{j=1}^{n_3}\Gamma(1-c_j+\gamma_j\eta)}{\sum_{i=1}^{r}\left[\prod_{j=m_3+1}^{q_i^{(2)}}\Gamma(1-d_{ji}+\delta_{ji}\eta)\prod_{j=n_3+1}^{p_i^{(2)}}\Gamma(c_{ji}-\gamma_{ji}\eta)\right]}, \tag{1.3}$$

$$\psi(\xi,\eta)=\frac{\prod_{j=1}^{m_1}\Gamma(f_j-F_j\xi-F_j^{'}\eta)\prod_{j=1}^{n_1}\Gamma(1-e_j+E_j\xi+E_j^{'}\eta)}{\prod_{j=m_1+1}^{q}\Gamma(1-f_j+F_j\xi+F_j^{'}\eta)\prod_{j=n_1+1}^{p}\Gamma(e_j-E_j\xi-E_j^{'}\eta)},\tag{1.4}$$

where an empty product is interpreted as unity. z_1, z_2 are two non zero complex variables, L_1, L_2 are two Mellin-Barnes type contour integrals and $m_1, n_1;\ m_2, n_2;\ m_3, n_3,\ p, q;\ p_i^{(1)}, q_i^{(1)};$
$, p_i^{(2)}, q_i^{(2)}$ are non-negative integers satisfying the conditions $0 \le n_1 \le p$, $0 \le n_2 \le p_i^{(1)}$, $0 \le n_3 \le p_i^{(2)}$, $0 \le m_1 \le q$, $0 \le m_2 \le q_i^{(1)}$, $0 \le m_3 \le q_i^{(2)}$ for all $i = 1, 2, 3\cdots, r$ where r is also a positive integer. $\alpha_j(j = 1,\cdots,n_2), \beta_j(j = 1,\cdots,m_2), \gamma_j(j = 1,\cdots,n_3), \delta_j(j = 1,\cdots,m_3), \alpha_{ji}(j = n_2+1,\cdots,p_i^{(1)}), \beta_{ji}(j = m_2+1,\cdots,q_i^{(1)}), \gamma_{ji}(j = n_3+1,\cdots,p_i^{(2)}), \delta_{ji}(j = m_3+1,\cdots,q_i^{(2)})$ are assumed to be positive quantities for standardization purposes. $E_j, E_j^{'}, F_j, F_j^{'}$ are also positive. $a_j(j = 1,\cdots,n_2), b_j(j = 1,\cdots,m_2), c_j(j = 1,\cdots,n_3), d_j(j = 1,\cdots,m_3), a_{ji}(j = n_2+1,\cdots,p_i^{(1)}), b_{ji}(j = m_2+1,\cdots,q_i^{(1)}), c_{ji}(j = n_3+1,\cdots,p_i^{(2)}), d_{ji}(j = m_3+1,\cdots,q_i^{(2)})$ are complex for all $i = 1, 2, 3\cdots, r$.

The contour L_1 lies in the complex ξ-plane and runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of $\Gamma(b_j-\beta_j\xi)(j = 1,\cdots,m_2), \Gamma(f_j-F_j\xi-F_j^{'}\eta)(j = 1,\cdots,m_1)$ lies to the right and the poles of $\Gamma(1-a_j+\alpha_j\xi)(j = 1,\cdots,n_2), \Gamma(1-e_j+E_j\xi+E_j^{'}\eta)(j = 1,\cdots,n_1)$ to the left of the contour L_1.The contour L_2 lies in the complex η plane and runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of $\Gamma(d_j-\delta_j\eta)(j = 1,\cdots,m_3), \Gamma(f_j-F_j\xi-F_j^{'}\eta)(j = 1,\cdots,m_1)$ lies to the right and the poles of $\Gamma(1-c_j+\gamma_j\xi)(j = 1,\cdots,n_3), \Gamma(1-e_j+E_j\xi+E_j^{'}\eta)(j = 1,\cdots,n_1)$ to the left of the contour L_2. All the poles are simple poles.

Convergence conditions are as follows:

$$|\arg z_1| < \frac{A_i\pi}{2},\ |\arg z_2| < \frac{B_i\pi}{2},\tag{1.5}$$

where

$$A_i=\sum_{j=1}^{n_1}E_j-\sum_{j=n_1+1}^{p}E_j+\sum_{j=1}^{m_1}F_j-\sum_{j=m_1+1}^{q}F_j+\sum_{j=1}^{m_2}\beta_j-\sum_{j=m_2+1}^{q_i^{(1)}}\beta_{ji}+\sum_{j=1}^{n_2}\alpha_j-\sum_{j=n_2+1}^{p_i^{(1)}}\alpha_{ji}>0,\tag{1.6}$$

and

$$B_i=\sum_{j=1}^{n_1}E_j^{'}-\sum_{j=n_1+1}^{p}E_j^{'}+\sum_{j=1}^{m_1}F_j^{'}-\sum_{j=m_1+1}^{q}F_j^{'}+\sum_{j=1}^{m_3}\delta_j-\sum_{j=m_3+1}^{q_i^{(2)}}\delta_{ji}+\sum_{j=1}^{n_3}\gamma_j-\sum_{j=n_3+1}^{p_i^{(2)}}\gamma_{ji}>0,\tag{1.7}$$

for $i = 1, ..., r$.

For the sake of brevity throughout the paper we shall use following notations:
$P = m_2, n_2; m_3, n_3$,
$Q = p_i^{(1)}, q_i^{(1)}; p_i^{(2)}, q_i^{(2)} : r$,
$U = [(a_j,\alpha_j)_{1,n_2}], [(a_{ji},\alpha_{ji})_{n_2+1,p_i^{(1)}}]; [(c_j,\gamma_j)_{1,n_3}], [(c_{ji},\gamma_{ji})_{n_3+1,p_i^{(2)}}]$,
$V = [(b_j,\beta_j)_{1,m_2}], [(b_{ji},\beta_{ji})_{m_2+1,q_i^{(1)}}]; [(d_j,\delta_j)_{1,m_3}], [(d_{ji},\delta_{ji})_{m_3+1,q_i^{(2)}}]$.

The generalized M-series [22] is defined as

$${}_pM_q^{\alpha,\beta}(z)={}_pM_q^{\alpha,\beta}\left(a_1,\cdots,a_p;b_1,\cdots,b_q;z\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k\cdots\left(a_p\right)_k}{(b_1)_k\cdots\left(b_q\right)_k}\frac{z^k}{\Gamma(\alpha k+\beta)},\tag{1.8}$$

where $\alpha,\beta \in \mathbb{C}$, $z \in \mathbb{C}$, $\Re(\alpha) > 0$, $(a_i)_k\ \left(i=\overline{1,p}\right)$ and $\left(b_j\right)_k\ \left(j=\overline{1,q}\right)$ are Pochhammer symbols. The series (1.8) is defined when none of the parameters $\left(b_j\right)_k\ \left(j=\overline{1,q}\right)$ is a negative integer or zero; if any numerator parameter a_i is a negative integer or zero, then series terminates to a polynomial in z. The series (1.8) is convergent for all z if $p \le q$; it is convergent for $|z| < \delta = \alpha^{\alpha}$ if $p = q + 1$ and divergent if $p > q + 1$. When $p = q + 1$ and $|z| = \delta$, the series is convergent on conditions depending on the parameters. The detailed account of the M-series can be found in the paper written by Sharma and Jain[22], see also, [5, 8, 11, 10, 20].

If $\alpha, \alpha', \beta, \beta', \gamma \in C$ and $x > 0$, then the generalized fractional calculus operators containing Appell hypergeometric function F_3 given by Saigo and Maeda [23], studied for generalized special functions of several variables by Chandel and Gupta [4], are defined in the following manner:

$$\left(I_{0_+}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)=\frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x t^{-\alpha'}(x-t)^{\gamma-1}F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)\,dt,\ \ \Re(\gamma)>0,\tag{1.9}$$

$$= \left(\frac{d}{dx}\right)^k \left(I_{0_+}^{\alpha,\alpha',\beta+k,\beta',\gamma+k} f\right)(x), \;\; \Re(\gamma) \le 0; k = [-\Re(\gamma)+1]. \tag{1.10}$$

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty t^{-\alpha}(t-x)^{\gamma-1} F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)\,dt, \;\; \Re(\gamma) > 0, \tag{1.11}$$

$$= (-1)^k \left(\frac{d}{dx}\right)^k \left(I_{-}^{\alpha,\alpha',\beta,\beta'+k,\gamma+k} f\right)(x), \;\; \Re(\gamma) \le 0; k = [-\Re(\gamma)+1], \tag{1.12}$$

$$\left(D_{0_+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{0_+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x), \;\; \Re(\gamma) > 0, \tag{1.13}$$

$$= \left(\frac{d}{dx}\right)^k \left(I_{0_+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k} f\right)(x), \;\; \Re(\gamma) > 0; k = [\Re(\gamma)+1]. \tag{1.14}$$

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x), \;\; \Re(\gamma) > 0, \tag{1.15}$$

$$= (-1)^k \left(\frac{d}{dx}\right)^k \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k} f\right)(x), \;\; \Re(\gamma) > 0; k = [\Re(\gamma)+1]. \tag{1.16}$$

These generalized fractional calculus operators reduce to Saigo's [24] fractional calculus operators by means of the following relationship:

$$\left(I_{0_+}^{\alpha,0,\beta,\beta',\gamma} f\right)(x) = \left(I_{0_+}^{\gamma,\alpha-\gamma,-\beta} f\right)(x), \;\; (\gamma \in C), \tag{1.17}$$

$$\left(I_{-}^{\alpha,0,\beta,\beta',\gamma} f\right)(x) = \left(I_{-}^{\gamma,\alpha-\gamma,-\beta} f\right)(x), \;\; (\gamma \in C), \tag{1.18}$$

$$\left(D_{0_+}^{0,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(D_{0_+}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f\right)(x), \;\; \Re(\gamma) > 0, \tag{1.19}$$

$$\left(D_{-}^{0,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(D_{-}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f\right)(x), \;\; \Re(\gamma) > 0. \tag{1.20}$$

Our main findings in the next section are based on the following composition formula due to Saigo-Maeda [23].

Lemma 1.1. *If* $\alpha,\alpha',\beta,\beta',\gamma \in C, \Re(\gamma) > 0$ *and* $\Re(\rho) > \max[0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')]$ *then there hold the formula*

$$\left(I_{0_+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1} \frac{\Gamma(\rho)\Gamma(\rho+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+\beta'-\alpha')}{\Gamma(\rho+\gamma-\alpha-\alpha')\Gamma(\rho+\gamma-\alpha'-\beta)\Gamma(\rho+\beta')}. \tag{1.21}$$

Lemma 1.2. *If* $\alpha,\alpha',\beta,\beta',\gamma \in C, \Re(\gamma) > 0$ *and* $\Re(\rho) < 1 + \min[\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)]$ *then there hold the formula*

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1} \frac{\Gamma(1+\alpha+\alpha'-\gamma-\rho)\Gamma(1+\alpha+\beta'-\gamma-\rho)\Gamma(1-\beta-\rho)}{\Gamma(1-\rho)\Gamma(1+\alpha+\alpha'+\beta'-\gamma-\rho)\Gamma(1+\alpha-\beta-\rho)}.$$

2. Main Results

In this section we have obtained some generalized fractional calculus formulas associated to the product of M-series and I-function of two variables with the help of Saigo-Maeda generalized fractional calculus operators. Further by specializing the parameters involved in the Saigo-Maeda fractional calculus operators, we have found some corollaries concerning to Saigo fractional calculus operators and Riemann-Liouville fractional calculus operators.The results are presented in the form of theorems stated below.

Theorem 2.1. *Let* $\alpha,\alpha',\beta,\beta',\gamma,\rho,\delta,\lambda \in C$, $z,z_1,z_2 \in C$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$, $\Re(\lambda) > 0$, $\mu,\nu \in R_+$. *Further let the constants* $m_1,n_1,p,q \in N_0$, $a_j,b_j,a_{ji},b_{ji} \in C$, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji} \in R_+ (i = 1,\cdots,p_i^{(1)};\; j = 1,\cdots,q_i^{(1)})$, $c_j,d_j,c_{ji},d_{ji} \in C$, $\gamma_j,\delta_j,\gamma_{ji},\delta_{ji} \in R_+ (i = 1,\cdots,p_i^{(2)};\; j = 1,\cdots,q_i^{(2)})$, $|\arg z_1| < \frac{A_i\pi}{2}$, $|\arg z_2| < \frac{B_i\pi}{2}$, $A_i > 0$, $B_i > 0$ *and satisfy the condition*

$$\Re(\rho) + \mu\min_{1\le j\le m_2}\Re\left(\frac{b_j}{\beta_j}\right) + \nu\min_{1\le j\le m_3}\Re\left(\frac{d_j}{\delta_j}\right) > \max\left[0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\right].$$

Then the fractional integration $I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ *of the product of* M*-series and* I*-function of two variables exists and the following relation holds*

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_uM_\nu^{\delta,\lambda}(zt) I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1 t^\mu & [(e_p:E_p,E_p')]:U \\ z_2 t^\nu & [(f_q:F_q,F_q')]:V \end{array}\right]\right\}(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha'-1}\sum_{s=0}^{\infty} f(T)x^s I^{m_1,n_1+3:P}_{p+3,q+3:Q}\left[\begin{array}{c|c} z_1x^{\mu} & X_1', [(e_p:E_p,E_p')]:U \\ z_2x^{\nu} & [(f_q:F_q,F_q')], X_2':V \end{array}\right], \tag{2.1}$$

where

$$X_1' = [(1-\rho-s:\mu,\nu)], [(1-\rho-s+\alpha+\alpha'+\beta-\gamma:\mu,\nu)], [(1-\rho-s+\alpha'-\beta':\mu,\nu)],$$
$$X_2' = [(1-\rho-s+\alpha+\alpha'-\gamma:\mu,\nu)], [(1-\rho-s+\alpha'+\beta-\gamma:\mu,\nu)], [(1-\rho-s-\beta':\mu,\nu)].$$

Proof. To prove (2.1), we first express M-series in summation form and I-function of two variables in terms of Mellin-Barnes contour integral with the help of equation (1.1) and interchanging the order of integration and summation, which is valid under the conditions stated with the Theorem 2.1, we obtain (say I_1)

$$I_1 = \sum_{s=0}^{\infty} \frac{(a_1)_s\cdots(a_u)_s}{(b_1)_s\cdots(b_v)_s}\frac{z^s}{\Gamma(\delta s+\lambda)}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^{\xi}\,z_2^{\eta}(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\,t^{\rho+s+\mu\xi+\nu\eta-1})(x)\,d\xi\,d\eta. \tag{2.2}$$

Now by applying Lemma 1.1, we arrive at

$$I_1 = x^{\rho+\gamma-\alpha-\alpha'-1}\sum_{s=0}^{\infty} f(T)x^s\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,(z_1x^{\mu})^{\xi}\,(z_2x^{\nu})^{\eta}$$
$$\times\frac{\Gamma(\rho+s+\mu\xi+\nu\eta)\Gamma(\rho+s+\mu\xi+\nu\eta+\gamma-\alpha-\alpha'-\beta)\Gamma(\rho+s+\mu\xi+\nu\eta+\beta'-\alpha')}{\Gamma(\rho+s+\mu\xi+\nu\eta+\gamma-\alpha-\alpha')\Gamma(\rho+s+\mu\xi+\nu\eta+\gamma-\alpha'-\beta)\Gamma(\rho+s+\mu\xi+\nu\eta+\beta')}\,d\xi\,d\eta,$$

where

$$f(T) = \frac{(a_1)_s\cdots(a_u)_s}{(b_1)_s\cdots(b_v)_s}\frac{z^s}{\Gamma(\delta s+\lambda)}. \tag{2.3}$$

By re-arranging the Mellin-Barnes contour integral in terms of I-function of two variables defined by (1.1) ,after little simplifications, we obtain the right hand side of (2.1). This completes proof of Theorem 2.1.
In view of the relation (1.17), we get the following corollary concerning left-sided Saigo fractional integral operator [24].

Corollary 2.1. *Let $\alpha,\beta,\gamma,\rho,\delta,\lambda \in C$, $z,z_1,z_2 \in C$, $\Re(\alpha) > 0, \Re(\delta) > 0, \Re(\lambda) > 0$, $\mu,\nu \in R_+$. Further let the constants $m_1,n_1,p,q \in N_0$, $a_j,b_j,a_{ji},b_{ji} \in C$, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji} \in R_+(i = 1,\cdots,p_i^{(1)};\ j = 1,\cdots,q_i^{(1)})$, $c_j,d_j,c_{ji},d_{ji} \in C$, $\gamma_j,\delta_j,\gamma_{ji},\delta_{ji} \in R_+(i = 1,\cdots,p_i^{(2)};\ j = 1,\cdots,q_i^{(2)})$, $|\arg z_1| < \frac{A_i\pi}{2}$, $|\arg z_2| < \frac{B_i\pi}{2}$, $A_i > 0$, $B_i > 0$ and satisfy the condition*

$$\Re(\rho) + \mu\min_{1\le j\le m_2}\Re\left(\frac{b_j}{\beta_j}\right) + \nu\min_{1\le j\le m_3}\Re\left(\frac{d_j}{\delta_j}\right) > \max\left[0,\Re(\beta-\gamma)\right].$$

Then the fractional integration $I_{0+}^{\alpha,\beta,\gamma}$ of the product of M-series and I-function of two variables exists and the following relation holds

$$\left\{I_{0+}^{\alpha,\beta,\gamma}t^{\rho-1}{}_uM_v^{\delta,\lambda}(zt)I^{m_1,n_1:P}_{p,q:Q}\left[\begin{array}{c|c} z_1t^{\mu} & [(e_p:E_p,E_p')]:U \\ z_2t^{\nu} & [(f_q:F_q,F_q')]:V \end{array}\right]\right\}(x) = x^{\rho-\beta-1}\sum_{s=0}^{\infty} f(T)x^s$$

$$\times I^{m_1,n_1+2:P}_{p+2,q+2:Q}\left[\begin{array}{c|c} z_1x^{\mu} & [(1-\rho-s:\mu,\nu)],[(1-\rho-s-\gamma+\beta:\mu,\nu)],[(e_p:E_p,E_p')]:U \\ z_2x^{\nu} & [(f_q:F_q,F_q')],[(1-\rho-s+\beta:\mu,\nu)],[(1-\rho-s-\alpha-\gamma:\mu,\nu)]:V \end{array}\right], \tag{2.4}$$

where $f(T)$ is represented by (2.3).

Now if we set $\beta = -\alpha$ in (2.4), we obtain the following result concerning left-sided Riemann-Liouville fractional integral operator [24].

Corollary 2.2. *Let $\alpha,\rho,\delta,\lambda \in C$, $z,z_1,z_2 \in C$, $\Re(\alpha) > 0, \Re(\delta) > 0, \Re(\lambda) > 0 \mu,\nu \in R_+$. Further let the constants $m_1,n_1,p,q \in N_0$, $a_j,b_j,a_{ji},b_{ji} \in C$, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji} \in R_+(i = 1,\cdots,p_i^{(1)};\ j = 1,\cdots,q_i^{(1)})$, $c_j,d_j,c_{ji},d_{ji} \in C$, $\gamma_j,\delta_j,\gamma_{ji},\delta_{ji} \in R_+(i = 1,\cdots,p_i^{(2)};\ j = 1,\cdots,q_i^{(2)})$, $|\arg z_1| < \frac{A_i\pi}{2}$, $|\arg z_2| < \frac{B_i\pi}{2}$, $A_i > 0$, $B_i > 0$ and satisfy the condition*

$$\Re(\rho) + \mu\min_{1\le j\le m_2}\Re\left(\frac{b_j}{\beta_j}\right) + \nu\min_{1\le j\le m_3}\Re\left(\frac{d_j}{\delta_j}\right) > 0.$$

Then the fractional integration I_{0+}^{α} of the product of M-series and I-function of two variables exists and the following relation holds

$$\left\{ I_{0+}^{\alpha} t^{\rho-1} {}_u M_v^{\delta,\lambda}(zt) I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c} z_1 t^{\mu} \\ z_2 t^{\nu} \end{array}\middle| \begin{array}{c} [(e_p : E_p, E_p^{'})] : U \\ [(f_q : F_q, F_q^{'})] : V \end{array}\right]\right\}(x)$$

$$= x^{\rho-\alpha-1}\sum_{s=0}^{\infty} f(T) x^s I_{p+1,q+1:Q}^{m_1,n_1+1:P}\left[\begin{array}{c} z_1 x^{\mu} \\ z_2 x^{\nu} \end{array}\middle| \begin{array}{c} [(1-\rho-s:\mu,\nu)], [(e_p : E_p, E_p^{'})] : U \\ [(f_q : F_q, F_q^{'})], [(1-\rho-s-\alpha:\mu,\nu)] : V \end{array}\right], \tag{2.5}$$

where $f(T)$ is represented by (2.3).

Theorem 2.2. *Let $\alpha, \alpha', \beta, \beta', \gamma, \rho, \delta, \lambda \in C$, $z, z_1, z_2 \in C$, $\Re(\gamma) > 0, \Re(\delta) > 0, \Re(\lambda) > 0$ $\mu, \nu \in R_+$. Further let the constants $m_1, n_1, p, q \in N_0$, $a_j, b_j, a_{ji}, b_{ji} \in C$, $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji} \in R_+ (i = 1, \cdots, p_i^{(1)}; j = 1, \cdots, q_i^{(1)})$, $c_j, d_j, c_{ji}, d_{ji} \in C$, $\gamma_j, \delta_j, \gamma_{ji}, \delta_{ji} \in R_+ (i = 1, \cdots, p_i^{(2)}; j = 1, \cdots, q_i^{(2)})$, $|\arg z_1| < \frac{A_i \pi}{2}$, $|\arg z_2| < \frac{B_i \pi}{2}$, $A_i > 0$, $B_i > 0$ and satisfy the condition*

$$\Re(\rho) + \mu \max_{1\le j\le n_2}\left[\frac{\Re(a_j)-1}{\alpha_j}\right] + \nu \max_{1\le j\le n_3}\left[\frac{\Re(c_j)-1}{\gamma_j}\right] < 1 + \min\left[\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\right].$$

Then the fractional integration $I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product of M-series and I-function of two variables exists and the following relation holds

$$\left\{ I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_u M_v^{\delta,\lambda}(zt) I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c} z_1 t^{\mu} \\ z_2 t^{\nu} \end{array}\middle| \begin{array}{c} [(e_p : E_p, E_p^{'})] : U \\ [(f_q : F_q, F_q^{'})] : V \end{array}\right]\right\}(x)$$

$$= x^{\rho-\alpha-\alpha'+\gamma-1}\sum_{s=0}^{\infty} f(T) x^s I_{p+3,q+3:Q}^{m_1+3,n_1:P}\left[\begin{array}{c} z_1 x^{\mu} \\ z_2 x^{\nu} \end{array}\middle| \begin{array}{c} [(e_p : E_p, E_p^{'})], X_3^{'} : U \\ X_4^{'}, [(f_q : F_q, F_q^{'})] : V \end{array}\right], \tag{2.6}$$

where

$$X_3^{'} = [(1-\rho-s:\mu,\nu)], [(1-\rho-s+\alpha+\alpha^{'}+\beta'-\gamma:\mu,\nu)], [(1-\rho-s+\alpha-\beta:\mu,\nu)],$$
$$X_4^{'} = [(1-\rho-s+\alpha+\alpha^{'}-\gamma:\mu,\nu)], [(1-\rho-s+\alpha+\beta'-\gamma:\mu,\nu)], [(1-\rho-s-\beta:\mu,\nu)].$$

Proof. To prove (2.6), we first express M-series in summation form and I-function of two variables in terms of Mellin-Barnes contour integral with the help of equation (1.1) and interchanging the order of integration and summation, which is valid under the conditions stated with the Theorem 2.2, we obtain (say I_2)

$$I_2 = \sum_{s=0}^{\infty}\frac{(a_1)_s\cdots(a_u)_s}{(b_1)_s\cdots(b_v)_s}\frac{z^s}{\Gamma(\delta s+\lambda)}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^{\xi}\,z_2^{\eta}(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\,t^{\rho+s+\mu\xi+\nu\eta-1})(x)\,d\xi\,d\eta. \tag{2.7}$$

Now by applying Lemma 1.2, we arrive at

$$\begin{aligned} I_2 = {} & x^{\rho-\alpha-\alpha'+\gamma-1}\sum_{s=0}^{\infty} f(T) x^s \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,(z_1x^{\mu})^{\xi}\,(z_2x^{\nu})^{\eta} \\ & \times \frac{\Gamma(1-\rho-s+\alpha+\alpha'-\gamma-\mu\xi-\nu\eta)}{\Gamma(1-\rho-s-\mu\xi-\nu\eta)} \\ & \times \frac{\Gamma(1-\rho-s+\alpha+\beta'-\gamma-\mu\xi-\nu\eta)\Gamma(1-\rho-s-\beta-\mu\xi-\nu\eta)}{\Gamma(1-\rho-s+\alpha+\alpha'+\beta'-\gamma-\mu\xi-\nu\eta)\Gamma(1-\rho-s+\alpha-\beta-\mu\xi-\nu\eta)}\,d\xi\,d\eta \end{aligned}$$

where $f(T)$ is represented by (2.3).

By re-arranging the Mellin-Barnes contour integral in terms of I-function of two variables defined by (1.1), after little simplifications, we obtain the right hand side of (2.6). This completes proof of Theorem 2.2.

In view of the relation (1.18), we get following corollary concerning right-sided Saigo fractional integral operator [24].

Corollary 2.3. *Let $\alpha, \beta, \gamma, \rho, \delta, \lambda \in C$, $z, z_1, z_2 \in C$, $\Re(\alpha) > 0, \Re(\delta) > 0, \Re(\lambda) > 0$ $\mu, \nu \in R_+$. Further let the constants $m_1, n_1, p, q \in N_0$, $a_j, b_j, a_{ji}, b_{ji} \in C$, $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji} \in R_+ (i = 1, \cdots, p_i^{(1)}; j = 1, \cdots, q_i^{(1)})$, $c_j, d_j, c_{ji}, d_{ji} \in$*

C, $\gamma_j,\delta_j,\gamma_{ji},\delta_{ji}\in R_+(i=1,\cdots,p_i^{(2)};\ j=1,\cdots,q_i^{(2)})$, $|\arg z_1|<\frac{A_i\pi}{2}$, $|\arg z_2|<\frac{B_i\pi}{2}$, $A_i>0$, $B_i>0$ and satisfy the condition

$$\Re(\rho)+\mu\max_{1\leq j\leq n_2}\left[\frac{\Re(a_j)-1}{\alpha_j}\right]+\nu\max_{1\leq j\leq n_3}\left[\frac{\Re(c_j)-1}{\gamma_j}\right]<1+\min\left[\Re(\beta),\Re(\gamma)\right].$$

Then the fractional integration $I_-^{\alpha,\beta,\gamma}$ of the product of M-sereis and I-function of two variables exists and the following relation holds

$$\left\{I_-^{\alpha,\beta,\gamma}t^{\rho-1}{}_uM_\nu^{\delta,\lambda}(zt)I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1t^\mu & [(e_p:E_p,E_p^{'})]:U \\ z_2t^\nu & [(f_q:F_q,F_q^{'})]:V\end{array}\right]\right\}(x)$$

$$=x^{\rho-\beta-1}\sum_{s=0}^{\infty}f(T)x^sI_{p+2,q+2:Q}^{m_1+2,n_1:P}\left[\begin{array}{c|c} z_1x^\mu & [(e_p:E_p,E_p^{'})],[(1-\rho-s:\mu,\nu)],[(1-\rho-s+\gamma:\mu,\nu)],:U \\ z_2x^\nu & [(1-\rho-s+\beta:\mu,\nu)],[(1-\rho-s+\gamma:\mu,\nu)][(f_q:F_q,F_q^{'})]:V\end{array}\right],\tag{2.8}$$

where $f(T)$ is represented by (2.3).

Further, if we set $\beta=-\alpha$ in (2.8), we get following corollary concerning right-sided Riemann Liouville fractional integral operator [24].

Corollary 2.4. *Let $\alpha,\rho,\delta,\lambda\in C$, $z,z_1,z_2\in C$, $\Re(\alpha)>0$, $\Re(\delta)>0$, $\Re(\lambda)>0$, $\mu,\nu\in R_+$. Further let the constants $m_1,n_1,p,q\in N_0$, $a_j,b_j,a_{ji},b_{ji}\in C$, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}\in R_+(i=1,\cdots,p_i^{(1)};\ j=1,\cdots,q_i^{(1)})$, $c_j,d_j,c_{ji},d_{ji}\in C$, $\gamma_j,\delta_j,\gamma_{ji},\delta_{ji}\in R_+(i=1,\cdots,p_i^{(2)};\ j=1,\cdots,q_i^{(2)})$, $|\arg z_1|<\frac{A_i\pi}{2}$, $|\arg z_2|<\frac{B_i\pi}{2}$, $A_i>0$, $B_i>0$ and satisfy the condition*

$$\Re(\alpha)+\Re(\rho)+\mu\max_{1\leq j\leq n_2}\left[\frac{\Re(a_j)-1}{\alpha_j}\right]+\nu\max_{1\leq j\leq n_3}\left[\frac{\Re(c_j)-1}{\gamma_j}\right]<1.$$

Then the fractional integration I_-^α of the product of M- series and I-function of two variables exists and the following relation holds

$$\left\{I_-^{\alpha}t^{\rho-1}{}_uM_\nu^{\delta,\lambda}(zt)I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1t^\mu & [(e_p:E_p,E_p^{'})]:U \\ z_2t^\nu & [(f_q:F_q,F_q^{'})]:V\end{array}\right]\right\}(x)$$

$$=x^{\rho-\alpha-1}\sum_{s=0}^{\infty}f(T)x^sI_{p+1,q+1:Q}^{m_1+1,n_1:P}\left[\begin{array}{c|c} z_1x^\mu & [(e_p:E_p,E_p^{'})],[(1-\rho-s:\mu,\nu)]:U \\ z_2x^\nu & [(1-\rho-s-\alpha:\mu,\nu)],[(f_q:F_q,F_q^{'})]:V\end{array}\right],\tag{2.9}$$

where $f(T)$ is represented by (2.3).

Theorem 2.3. *Let $\alpha,\alpha',\beta,\beta',\gamma,\rho,\delta,\lambda\in C$, $z,z_1,z_2\in C$, $\Re(\gamma)>0$, $\Re(\delta)>0$ $\Re(\lambda)>0$, $\mu,\nu\in R_+$. Further let the constants $m_1,n_1,p,q\in N_0$, $a_j,b_j,a_{ji},b_{ji}\in C$, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}\in R_+(i=1,\cdots,p_i^{(1)};\ j=1,\cdots,q_i^{(1)})$, $c_j,d_j,c_{ji},d_{ji}\in C$, $\gamma_j,\delta_j,\gamma_{ji},\delta_{ji}\in R_+(i=1,\cdots,p_i^{(2)};\ j=1,\cdots,q_i^{(2)})$, $|\arg z_1|<\frac{A_i\pi}{2}$, $|\arg z_2|<\frac{B_i\pi}{2}$, $A_i>0$, $B_i>0$ and satisfy the condition*

$$\Re(\rho)+\mu\min_{1\leq j\leq m_2}\Re\left(\frac{b_j}{\beta_j}\right)+\nu\min_{1\leq j\leq m_3}\Re\left(\frac{d_j}{\delta_j}\right)>\max\left[0,\Re(-\alpha-\alpha'-\beta'+\gamma),\Re(\beta-\alpha)\right].$$

Then the fractional derivative $D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}$ of the product of M-series and I-function of two variables exists and the following relation holds

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}{}_uM_\nu^{\delta,\lambda}(zt)I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1t^\mu & [(e_p:E_p,E_p^{'})]:U \\ z_2t^\nu & [(f_q:F_q,F_q^{'})]:V\end{array}\right]\right\}(x)$$

$$=x^{\rho+\alpha+\alpha^{'}-\gamma-1}\sum_{s=0}^{\infty}f(T)x^sI_{p+3,q+3:Q}^{m_1,n_1+3:P}\left[\begin{array}{c|c} z_1x^\mu & X_5^{'},[(e_p:E_p,E_p^{'})]:U \\ z_2x^\nu & [(f_q:F_q,F_q^{'})],X_6^{'}:V\end{array}\right],\tag{2.10}$$

where

$$X_5^{'}=[(1-\rho-s:\mu,\nu)],[(1-\rho-s-\alpha-\alpha^{'}-\beta'+\gamma:\mu,\nu)],[(1-\rho-s-\alpha+\beta:\mu,\nu)],$$
$$X_6^{'}=[(1-\rho-s-\alpha-\beta'+\gamma:\mu,\nu)],[(1-\rho-s+\beta:\mu,\nu)],[(1-\rho-s-\alpha-\alpha'+\gamma:\mu,\nu)].$$

Proof. To prove the fractional differential formula (2.10) we express M-series in summation form and I-function of two variables in terms of double Mellin-Barnes contour integral with the help of equations (1.1) and interchanging the order of integration and summation, we obtain the following form after little simplification:

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}{}_uM_v^{\delta,\lambda}(zt) I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1t^{\mu} & [(e_p:E_p,E_p')]:U \\ z_2t^{\nu} & [(f_q:F_q,F_q')]:V \end{array}\right]\right\}(x)$$
$$= \frac{d^k}{dx^k}\left\{I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k} t^{\rho-1}{}_uM_v^{\delta,\lambda}(zt) I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1t^{\mu} & [(e_p:E_p,E_p')]:U \\ z_2t^{\nu} & [(f_q:F_q,F_q')]:V \end{array}\right]\right\}(x)$$

$$= \sum_{s=0}^{\infty}\frac{(a_1)_s\cdots(a_u)_s}{(b_1)_s\cdots(b_v)_s}\frac{z^s}{\Gamma(\delta s+\lambda)}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^{\xi}\,z_2^{\eta}$$
$$\frac{d^k}{dx^k}\left(I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k}\,t^{\rho+s+\mu\xi+\nu\eta-1}\right)(x)\,d\xi\,d\eta. \tag{2.11}$$

where $k=[Re(\gamma)+1]$.

Applying Lemma 1.1 to (2.11),we obtain
L.H.S. of (2.11)

$$= \sum_{s=0}^{\infty}\frac{(a_1)_s\cdots(a_u)_s}{(b_1)_s\cdots(b_v)_s}\frac{z^s}{\Gamma(\delta s+\lambda)}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,z_1^{\xi}\,z_2^{\eta}$$
$$\frac{\Gamma(\rho+s+\mu\xi+\nu\eta)\Gamma(\rho+s+\mu\xi+\nu\eta-\gamma+\alpha'+\alpha+\beta')\Gamma(\rho+s+\mu\xi+\nu\eta-\beta+\alpha)}{\Gamma(\rho+s+\mu\xi+\nu\eta+\alpha'+\alpha-\gamma+k)\Gamma(\rho+s+\mu\xi+\nu\eta-\gamma+\alpha+\beta')\Gamma(\rho+s+\mu\xi+\nu\eta-\beta)}$$
$$\frac{d^k}{dx^k}x^{\rho+s+\mu\xi+\nu\eta+\alpha'+\alpha-\gamma+k-1}\,d\xi\,d\eta.$$

Using $\frac{d^n}{dx^n}x^m=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}x^{m-n}$ where $m\geq n$ in the above expression, we obtain
L.H.S. of (2.11)

$$= x^{\rho+\alpha+\alpha'-\gamma-1}\sum_{s=0}^{\infty}f(T)x^s\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi_1(\xi)\,\phi_2(\eta)\,\psi(\xi,\eta)\,(z_1x)^{\xi}\,(z_2x)^{\eta}$$
$$\frac{\Gamma(\rho+s+\mu\xi+\nu\eta)\Gamma(\rho+s+\mu\xi+\nu\eta-\gamma+\alpha'+\alpha+\beta')\Gamma(\rho+s+\mu\xi+\nu\eta-\beta+\alpha)}{\Gamma(\rho+s+\mu\xi+\nu\eta-\gamma+\alpha+\beta')\Gamma(\rho+s+\mu\xi+\nu\eta-\beta)\Gamma(\rho+s+\mu\xi+\nu\eta+\alpha'+\alpha-\gamma)}\,d\xi\,d\eta,$$

where $f(T)$ is represented by (2.3).

Now re-arranging the Mellin-Barnes contour integral in terms of I-function of two variables defined by (1.1), after little simplifications, we obtain the right hand side of (2.10). This completes proof of Theorem 2.3.
In view of the relation(1.19), we get following corollary concerning left-sided Saigo fractional derivative operator [24].

Corollary 2.5. *Let* $\alpha,\beta,\gamma,\rho,\delta,\lambda\in C$, $z,z_1,z_2\in C$, $\Re(\alpha)>0,\Re(\delta)>0,\Re(\lambda)>0$, $\mu,\nu\in R_+$. *Further let the constants* $m_1,n_1,p,q\in N_0$, $a_j,b_j,a_{ji},b_{ji}\in C$, $\alpha_j,\beta_j,\alpha_{ji},\beta_{ji}\in R_+(i=1,\cdots,p_i^{(1)};\ j=1,\cdots,q_i^{(1)})$, $c_j,d_j,c_{ji},d_{ji}\in C$, $\gamma_j,\delta_j,\gamma_{ji},\delta_{ji}\in R_+(i=1,\cdots,p_i^{(2)};\ j=1,\cdots,q_i^{(2)})$, $|\arg z_1|<\frac{A_i\pi}{2}$, $|\arg z_2|<\frac{B_i\pi}{2}$, $A_i>0$, $B_i>0$ *and satisfy the condition*

$$\Re(\rho)+\mu\min_{1\leq j\leq m_2}\Re\left(\frac{b_j}{\beta_j}\right)+\nu\min_{1\leq j\leq m_3}\Re\left(\frac{d_j}{\delta_j}\right)>\max\left[0,\Re(-\alpha-\beta-\gamma)\right].$$

Then the fractional derivative $D_{0+}^{\alpha,\beta,\gamma}$ *of the product of* M*-series and* I*-function of two variables exists and the following relation holds*

$$\left\{D_{0+}^{\alpha,\beta,\gamma} t^{\rho-1}{}_uM_v^{\delta,\lambda}(zt) I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1t^{\mu} & [(e_p:E_p,E_p')]:U \\ z_2t^{\nu} & [(f_q:F_q,F_q')]:V \end{array}\right]\right\}(x)$$

$$= x^{\rho+\beta-1}\sum_{s=0}^{\infty}f(T)x^sI_{p+2,q+2:Q}^{m_1,n_1+2:P}\left[\begin{array}{c|c} z_1x^{\mu} & [(1-\rho-s:\mu,\nu)],[(1-\rho-s-\alpha-\beta-\gamma:\mu,\nu)],[(e_p:E_p,E_p')]:U \\ z_2x^{\nu} & [(f_q:F_q,F_q')],[(1-\rho-s-\gamma:\mu,\nu)],[(1-\rho-s-\beta:\mu,\nu)]:V \end{array}\right], \tag{2.12}$$

where $f(T)$ *is represented by (2.3).*

Next, if we set $\beta = -\alpha$ in the above result, we obtain following result concerning left-sided Riemann-Liouville fractional derivative operator[24].

Corollary 2.6. *Let* $\alpha, \rho, \delta, \lambda \in C$, $z, z_1, z_2 \in C$, $\Re(\alpha) > 0, \Re(\delta) > 0, \Re(\lambda) > 0$, $\mu, \nu \in R_+$. *Further let the constants* $m_1, n_1, p, q \in N_0$, $a_j, b_j, a_{ji}, b_{ji} \in C$, $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji} \in R_+ (i = 1, \cdots, p_i^{(1)};\ j = 1, \cdots, q_i^{(1)})$, $c_j, d_j, c_{ji}, d_{ji} \in C$, $\gamma_j, \delta_j, \gamma_{ji}, \delta_{ji} \in R_+ (i = 1, \cdots, p_i^{(2)};\ j = 1, \cdots, q_i^{(2)})$, $|\arg z_1| < \frac{A_i \pi}{2}$, $|\arg z_2| < \frac{B_i \pi}{2}$, $A_i > 0$, $B_i > 0$ *and satisfy the condition*

$$\Re(\rho) + \mu \min_{1 \le j \le m_2} \Re\left(\frac{b_j}{\beta_j}\right) + \nu \min_{1 \le j \le m_3} \Re\left(\frac{d_j}{\delta_j}\right) > 0.$$

Then the fractional derivative D_{0+}^{α} *of the product of* M*-series and* I*-function of two variables exists and the following relation holds:*

$$\left\{ D_{0+}^{\alpha} t^{\rho-1} {}_uM_v^{\delta,\lambda}(zt) I_{p,q:Q}^{m_1,n_1:P} \left[\begin{array}{c} z_1 t^{\mu} \\ z_2 t^{\nu} \end{array} \middle| \begin{array}{c} [(e_p : E_p, E_p')] : U \\ [(f_q : F_q, F_q')] : V \end{array} \right] \right\}(x)$$

$$= x^{\rho+\alpha-1} \sum_{s=0}^{\infty} f(T) x^s I_{p+1,q+1:Q}^{m_1,n_1+1:P} \left[\begin{array}{c} z_1 x^{\mu} \\ z_2 x^{\nu} \end{array} \middle| \begin{array}{c} [(1-\rho-s:\mu,\nu)], [(e_p : E_p, E_p')] : U \\ [(f_q : F_q, F_q')], [(1-\rho-s-\alpha:\mu,\nu)] : V \end{array} \right], \tag{2.13}$$

where $f(T)$ *is represented by* (2.3).

Theorem 2.4. *Let* $\alpha, \alpha', \beta, \beta', \gamma, \rho, \delta, \lambda \in C$, $z, z_1, z_2 \in C$, $\Re(\gamma) > 0, \Re(\delta) > 0, \Re(\lambda) > 0$, $\mu, \nu \in R_+$. *Further let the constants* $m_1, n_1, p, q \in N_0$, $a_j, b_j, a_{ji}, b_{ji} \in C$, $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji} \in R_+ (i = 1, \cdots, p_i^{(1)};\ j = 1, \cdots, q_i^{(1)})$, $c_j, d_j, c_{ji}, d_{ji} \in C$, $\gamma_j, \delta_j, \gamma_{ji}, \delta_{ji} \in R_+ (i = 1, \cdots, p_i^{(2)};\ j = 1, \cdots, q_i^{(2)})$, $|\arg z_1| < \frac{A_i \pi}{2}$, $|\arg z_2| < \frac{B_i \pi}{2}$, $A_i > 0$, $B_i > 0$ *and satisfy the condition*

$$\Re(\rho) + \mu \max_{1 \le j \le n_2} \left[\frac{\Re(a_j) - 1}{\alpha_j} \right] + \nu \max_{1 \le j \le n_3} \left[\frac{\Re(c_j) - 1}{\gamma_j} \right] < 1 + \min\left[\Re(\beta'), \Re(-\alpha - \alpha' + \gamma), \Re(-\alpha' - \beta + \gamma) \right].$$

Then the fractional derivative $D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}$ *of the product of* M*-series and* I*-function of two variables exists and the following relation holds*

$$\left\{ D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_uM_v^{\delta,\lambda}(zt) I_{p,q:Q}^{m_1,n_1:P} \left[\begin{array}{c} z_1 t^{\mu} \\ z_2 t^{\nu} \end{array} \middle| \begin{array}{c} [(e_p : E_p, E_p')] : U \\ [(f_q : F_q, F_q')] : V \end{array} \right] \right\}(x)$$

$$= x^{\rho+\alpha+\alpha'-\gamma-1} \sum_{s=0}^{\infty} f(T) x^s I_{p+3,q+3:Q}^{m_1+3,n_1:P} \left[\begin{array}{c} z_1 x^{\mu} \\ z_2 x^{\nu} \end{array} \middle| \begin{array}{c} [(e_p : E_p, E_p')], X_7' : U \\ X_8', [(f_q : F_q, F_q')] : V \end{array} \right], \tag{2.14}$$

where

$$X_7' = [(1-\rho-s:\mu,\nu)], [(1-\rho-s-\alpha-\alpha'-\beta+\gamma:\mu,\nu)], [(1-\rho-s-\alpha'+\beta':\mu,\nu)],$$
$$X_8' = [(1-\rho-s-\alpha'-\beta+\gamma:\mu,\nu)], [(1-\rho-s+\beta':\mu,\nu)], [(1-\rho-s-\alpha-\alpha'+\gamma:\mu,\nu)].$$

Proof. To prove the fractional differential formula (2.14) we express M-series in summation form and I-function of two variables in terms of double Mellin-Barnes contour integral with the help of equations (1.1) and interchanging the order of integration and summation, we obtain the following form after little simplification:

$$\left\{ D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_uM_v^{\delta,\lambda}(zt) I_{p,q:Q}^{m_1,n_1:P} \left[\begin{array}{c} z_1 t^{\mu} \\ z_2 t^{\nu} \end{array} \middle| \begin{array}{c} [(e_p : E_p, E_p')] : U \\ [(f_q : F_q, F_q')] : V \end{array} \right] \right\}(x)$$

$$= (-1)^k \frac{d^k}{dx^k} \left\{ I_{-}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k} t^{\rho-1} {}_uM_v^{\delta,\lambda}(zt) I_{p,q:Q}^{m_1,n_1:P} \left[\begin{array}{c} z_1 t^{\mu} \\ z_2 t^{\nu} \end{array} \middle| \begin{array}{c} [(e_p : E_p, E_p')] : U \\ [(f_q : F_q, F_q')] : V \end{array} \right] \right\}(x)$$
$$= \sum_{s=0}^{\infty} \frac{(a_1)_s \cdots (a_u)_s}{(b_1)_s \cdots (b_v)_s} \frac{z^s}{\Gamma(\delta s + \lambda)} (-1)^k \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi_1(\xi)\, \phi_2(\eta)\, \psi(\xi, \eta)\, z_1^{\xi}\, z_2^{\eta}$$
$$\times \frac{d^k}{dx^k} (I_{-}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k}\, t^{\rho+s+\mu\xi+\nu\eta-1})(x)\, d\xi\, d\eta, \tag{2.15}$$

where $k = [Re(\gamma) + 1]$.

Applying Lemma 1.2 to (2.15),we obtain

$$= \sum_{s=0}^{\infty} \frac{(a_1)_s \cdots (a_u)_s}{(b_1)_s \cdots (b_v)_s} \frac{z^s}{\Gamma(\delta s + \lambda)} \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi_1(\xi)\, \phi_2(\eta)\, \psi(\xi, \eta)\, z_1^{\xi}\, z_2^{\eta}$$

$$\times \frac{\Gamma(1 - \rho - s - \alpha - \alpha' + \gamma - k - \mu\xi - \nu\eta)}{\Gamma(1 - \rho - s - \mu\xi - \nu\eta)}$$

$$\times \frac{\Gamma(1 - \rho - s - \alpha' - \beta + \gamma - \mu\xi - \nu\eta)\Gamma(1 - \rho - s - \beta' - \mu\xi - \nu\eta)}{\Gamma(1 - \rho - s - \alpha - \alpha' - \beta + \gamma - \mu\xi - \nu\eta)\Gamma(1 - \rho - s - \alpha' + \beta' - \mu\xi - \nu\eta)}$$

$$\times(-1)^k \frac{d^k}{dx^k} x^{\rho+s+\mu\xi+\nu\eta+\alpha'+\alpha-\gamma+k-1}\, d\xi\, d\eta$$

$$= \sum_{s=0}^{\infty} \frac{(a_1)_s \cdots (a_u)_s}{(b_1)_s \cdots (b_v)_s} \frac{z^s}{\Gamma(\delta s + \lambda)} \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi_1(\xi)\, \phi_2(\eta)\, \psi(\xi, \eta)\, z_1^{\xi}\, z_2^{\eta}$$

$$\times \frac{\Gamma(1 - \rho - s - \alpha - \alpha' + \gamma - k - \mu\xi - \nu\eta)}{\Gamma(1 - \rho - s - \mu\xi - \nu\eta)}$$

$$\times \frac{\Gamma(1 - \rho - s - \alpha' - \beta + \gamma - \mu\xi - \nu\eta)\Gamma(1 - \rho - s + \beta' - \mu\xi - \nu\eta)}{\Gamma(1 - \rho - s - \alpha - \alpha' - \beta + \gamma - \mu\xi - \nu\eta)\Gamma(1 - \rho - s - \alpha' + \beta' - \mu\xi - \nu\eta)}$$

$$\times(1 - \rho - s - \alpha - \alpha' + \gamma - k - \mu\xi - \nu\eta)_k\, x^{\rho+s+\mu\xi+\nu\eta+\alpha'+\alpha-\gamma-1}\, d\xi\, d\eta$$

$$= \sum_{s=0}^{\infty} f(T) x^s \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi_1(\xi)\, \phi_2(\eta)\, \psi(\xi, \eta)\, z_1^{\xi}\, z_2^{\eta}$$

$$\times \frac{\Gamma(1 - \rho - s - \alpha' - \beta + \gamma - \mu\xi - \nu\eta)}{\Gamma(1 - \rho - s - \mu\xi - \nu\eta)}$$

$$\times \frac{\Gamma(1 - \rho - s + \beta' - \mu\xi - \nu\eta)\Gamma(1 - \rho - s - \alpha - \alpha' + \gamma - \mu\xi - \nu\eta)}{\Gamma(1 - \rho - s - \alpha - \alpha' - \beta + \gamma - \mu\xi - \nu\eta)\Gamma(1 - \rho - s - \alpha' + \beta' - \mu\xi - \nu\eta)}$$

$$\times x^{\rho+\mu\xi+\nu\eta+\alpha'+\alpha-\gamma-1}\, d\xi\, d\eta,$$

where $f(T)$ is represented by (2.3).

Now re-arranging the Mellin-Barnes contour integral in terms of I-function of two variables defined by (1.1), after little simplifications, we obtain the right hand side of (2.14). This completes proof of Theorem 2.4.

In view of the relation (1.20), we get following corollary concerning right-sided Saigo fractional derivative operator [24].

Corollary 2.7. *Let $\alpha, \beta, \gamma, \rho, \delta, \lambda \in C$, $z, z_1, z_2 \in C$, $\Re(\alpha) > 0, \Re(\delta) > 0, \Re(\lambda) > 0$, $\mu, \nu \in R_+$. Further let the constants $m_1, n_1, p, q \in N_0$, $a_j, b_j, a_{ji}, b_{ji} \in C$, $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji} \in R_+(i = 1, \cdots, p_i^{(1)};\ j = 1, \cdots, q_i^{(1)})$, $c_j, d_j, c_{ji}, d_{ji} \in C$, $\gamma_j, \delta_j, \gamma_{ji}, \delta_{ji} \in R_+(i = 1, \cdots, p_i^{(2)};\ j = 1, \cdots, q_i^{(2)})$, $|\arg z_1| < \frac{A_i\pi}{2}$, $|\arg z_2| < \frac{B_i\pi}{2}$, $A_i > 0$, $B_i > 0$ and satisfy the condition*

$$\Re(\rho) + \mu\max_{1\le j\le n_2}\left[\frac{\Re(a_j)-1}{\alpha_j}\right] + \nu\max_{1\le j\le n_3}\left[\frac{\Re(c_j)-1}{\gamma_j}\right] < 1 + \min\left[\Re(-\beta), \Re(\alpha+\gamma)\right].$$

Then the fractional derivative $D_{-}^{\alpha,\beta,\gamma}$ of the product of M-series and I-function of two variables exists and the following relation holds

$$\left\{D_{-}^{\alpha,\beta,\gamma} t^{\rho-1}\, {}_uM_v^{\delta,\lambda}(zt) I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c|c} z_1 t^{\mu} & [(e_p : E_p, E_p^{'})] : U \\ z_2 t^{\nu} & [(f_q : F_q, F_q^{'})] : V \end{array}\right]\right\}(x)$$

$$=x^{\rho+\beta-1} \sum_{s=0}^{\infty} f(T) x^s$$

$$\times I_{p+2,q+2:Q}^{m_1+2,n_1:P}\left[\begin{array}{c|c} z_1 x^{\mu} & [(e_p : E_p, E_p^{'})], [(1-\rho-s:\mu,\nu)], [(1-\rho-s-\beta+\gamma:\mu,\nu)] : U \\ z_2 x^{\nu} & [(1-\rho-s+\alpha+\gamma:\mu,\nu)], [(1-\rho-s-\beta':\mu,\nu)], [(f_q : F_q, F_q^{'})] : V \end{array}\right], \tag{2.16}$$

where $f(T)$ is represented by (2.3).

Further, if we set $\beta = -\alpha$ in (2.16), we obtain following corollary concerning right-sided Riemann-Liouville derivative operator [24].

Corollary 2.8. *Let* $\alpha,\rho,\delta,\lambda \in C$, $z, z_1, z_2 \in C$, $\Re(\alpha) > 0, \Re(\delta) > 0, \Re(\lambda) > 0$, $\mu,\nu \in R_+$. *Further let the constants* $m_1, n_1, p, q \in N_0$, $a_j, b_j, a_{ji}, b_{ji} \in C$, $\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji} \in R_+ (i = 1,\cdots,p_i^{(1)};\ j = 1,\cdots,q_i^{(1)})$, $c_j, d_j, c_{ji}, d_{ji} \in C$, $\gamma_j, \delta_j, \gamma_{ji}, \delta_{ji} \in R_+ (i = 1,\cdots,p_i^{(2)};\ j = 1,\cdots,q_i^{(2)})$, $|\arg z_1| < \frac{A_i\pi}{2}$, $|\arg z_2| < \frac{B_i\pi}{2}$, $A_i > 0$, $B_i > 0$ *and satisfy the condition*

$$\Re(\rho) + \Re(\alpha) + \mu\max_{1\le j\le n_2}\left[\frac{\Re(a_j)-1}{\alpha_j}\right] + \nu\max_{1\le j\le n_3}\left[\frac{\Re(c_j)-1}{\gamma_j}\right] < 0.$$

Then the fractional derivative D_-^{α} *of the product of* M*-series and* I*-function of two variables exists and the following relation holds*

$$\left\{D_-^{\alpha} t^{\rho-1}{}_uM_v^{\delta,\lambda}(zt) I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c} z_1t^{\mu} \\ z_2t^{\nu}\end{array}\middle|\begin{array}{c}[(e_p:E_p,E_p^{'})]:U \\ [(f_q:F_q,F_q^{'})]:V\end{array}\right]\right\}(x)$$

$$= x^{\rho+\alpha-1}\sum_{s=0}^{\infty} f(T)x^s I_{p+1,q+1:Q}^{m_1+1,n_1:P}\left[\begin{array}{c} z_1x^{\mu} \\ z_2x^{\nu}\end{array}\middle|\begin{array}{c}[(e_p:E_p,E_p^{'})],[(1-\rho-s:\mu,\nu)]:U \\ [(1-\rho-s-\alpha:\mu,\nu)],[(f_q:F_q,F_q^{'})]:V\end{array}\right], \tag{2.17}$$

where $f(T)$ is represented by (2.3).

3. Special Cases

The I-function of two variables is a most generalized form of special functions, consequently it can be reduced in a large number of special functions (or product of such functions) by suitably specializing the parameters involved in the function. M-series also reduces to generalized hypergeometric function and generalized Mittag-Leffler function by suitably specializing the parameters.Here we provide a few special cases of our main results.

(i) If we set $\delta = 1$ and $\lambda = 1$ in Theorem 2.1, we get generalized fractional integration of the product of I-function of two variables and hypergeometric function

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}{}_uF_v(zt) I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c} z_1t^{\mu} \\ z_2t^{\nu}\end{array}\middle|\begin{array}{c}[(e_p:E_p,E_p^{'})]:U \\ [(f_q:F_q,F_q^{'})]:V\end{array}\right]\right\}(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha'-1}\sum_{s=0}^{\infty} f(T')x^s I_{p+3,q+3:Q}^{m_1,n_1+3:P}\left[\begin{array}{c} z_1x^{\mu} \\ z_2x^{\nu}\end{array}\middle|\begin{array}{c}X_1^{'},[(e_p:E_p,E_p^{'})]:U \\ [(f_q:F_q,F_q^{'})],X_2^{'}:V\end{array}\right], \tag{3.1}$$

where

$$f(T') = \frac{(a_1)_s\cdots(a_u)_s}{(b_1)_s\cdots(b_v)_s}\frac{z^s}{s!} \tag{3.2}$$

and $X_1^{'}$ and $X_2^{'}$ are same as given in Theorem 2.1. The conditions of validity of the above result easily follow from Theorem 2.1.

(ii) If we set $u = 0$ and $v = 0$ in Theorem 2.1, we get generalized fractional integration of the product of I-function of two variables and Mittag-leffler function

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}E_{\delta,\lambda}(zt) I_{p,q:Q}^{m_1,n_1:P}\left[\begin{array}{c} z_1t^{\mu} \\ z_2t^{\nu}\end{array}\middle|\begin{array}{c}[(e_p:E_p,E_p^{'})]:U \\ [(f_q:F_q,F_q^{'})]:V\end{array}\right]\right\}(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha'-1}\sum_{s=0}^{\infty} \frac{x^s}{\Gamma(\delta s+\lambda)} I_{p+3,q+3:Q}^{m_1,n_1+3:P}\left[\begin{array}{c} z_1x^{\mu} \\ z_2x^{\nu}\end{array}\middle|\begin{array}{c}X_1^{'},[(e_p:E_p,E_p^{'})]:U \\ [(f_q:F_q,F_q^{'})],X_2^{'}:V\end{array}\right], \tag{3.3}$$

where $X_1^{'}$ and $X_2^{'}$ are same as given in Theorem 2.1. The conditions of validity of the above result easily follow from Theorem 2.1.

(iii) If we set $m_1 = n_1 = p = q = 0$ in Theorem 2.1 then we have following result in terms of product of I-function of one variable introduced by Saxena [26]

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}{}_uM_v^{\delta,\lambda}(zt) I_{p_i^{(1)},q_i^{(1)}:r}^{m_2,n_2}\left[z_1t^{\mu}\middle|\begin{array}{c}(a_j,\alpha_j)_{1,n_2},(a_{ji},\alpha_{ji})_{n_2+1,p_i^{(1)}} \\ (b_j,\beta_j)_{1,m_2},(b_{ji},\beta_{ji})_{m_2+1,q_i^{(1)}}\end{array}\right]\right.$$

$$\left.I_{p_i^{(2)},q_i^{(2)}:r}^{m_3,n_3}\left[z_2t^{\nu}\middle|\begin{array}{c}(c_j,\gamma_j)_{1,n_3},(c_{ji},\gamma_{ji})_{n_3+1,p_i^{(2)}} \\ (d_j,\delta_j)_{1,m_3},(d_{ji},\delta_{ji})_{m_3+1,q_i^{(2)}}\end{array}\right]\right\}(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha'-1}\sum_{s=0}^{\infty} f(T)x^s I_{3,3:p_i^{(1)},q_i^{(1)};p_i^{(2)},q_i^{(2)}:r}^{0,3:m_2,n_2;m_3,n_3}\left[\begin{array}{c} z_1x^{\mu} \\ z_2x^{\nu}\end{array}\middle|\begin{array}{c}X_1^{'},\ldots:U \\ X_2^{'},\ldots:V\end{array}\right], \tag{3.4}$$

where $f(T)$ is represented by (2.3), $X_1^{'}$ and $X_2^{'}$ are same as given in Theorem 2.1. The conditions of validity of the above result easily follow from Theorem 2.1.
(iv) If we set $m_1 = 0$ and $r = 1$ in Theorem 2.1, the I-function of two variables occurring in L.H.S. reduces into H-function of two variables [25] then we have following result

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_uM_{\nu}^{\delta,\lambda}(zt) H^{0,n_1:m_2,n_2;m_3,n_3}_{p,q:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c|c} z_1t^{\mu} & [(e_p:E_p,E_p^{'})]:T_1 \\ z_2t^{\nu} & [(f_q:F_q,F_q^{'})]:T_2 \end{array}\right]\right\}(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha^{'}-1} \sum_{s=0}^{\infty} f(T) x^s H^{0,n_1+3:m_2,n_2;m_3,n_3}_{p+3,q+3:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c|c} z_1x^{\mu} & X_1^{'},[(e_p:E_p,E_p^{'})]:T_1 \\ z_2x^{\nu} & [(f_q:F_q,F_q^{'})],X_2^{'}:T_2 \end{array}\right], \tag{3.5}$$

where $f(T)$ is represented by (2.3) and

$$T_1 = [(a_j,\alpha_j)_{1,p_1^{(1)}}];[(c_j,\gamma_j)_{1,p_1^{(2)}}], \quad T_2 = [(b_j,\beta_j)_{1,q_1^{(1)}}];[(d_j,\delta_j)_{1,q_1^{(2)}}],$$

also $X_1^{'}$ and $X_2^{'}$ are same as given in Theorem 2.1.The conditions of validity of the above result easily follow from Theorem 2.1.
(v) If we set $m_1 = n_1 = p = q = 0$ and $r = 1$ in Theorem 2.1, then we have following result in terms of product of H-functions

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_uM_{\nu}^{\delta,\lambda}(zt) H^{m_2,n_2}_{p_1^{(1)},q_1^{(1)}}\left[z_1t^{\mu}\left|\begin{array}{c} (a_j,\alpha_j)_{1,p_1^{(1)}} \\ (b_j,\beta_j)_{1,q_1^{(1)}} \end{array}\right.\right] \times H^{m_3,n_3}_{p_1^{(2)},q_1^{(2)}}\left[z_2t^{\nu}\left|\begin{array}{c} (c_j,\gamma_j)_{1,p_1^{(2)}} \\ (d_j,\delta_j)_{1,q_1^{(2)}} \end{array}\right.\right]\right\}(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha^{'}-1} \sum_{s=0}^{\infty} f(T) x^s H^{0,3:m_2,n_2;m_3,n_3}_{3,3:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c|c} z_1x^{\mu} & X_1^{'},\ldots:T_1 \\ z_2x^{\nu} & X_2^{'},\ldots:T_2 \end{array}\right], \tag{3.6}$$

where $f(T)$ is represented by (2.3). $X_1^{'}$ and $X_2^{'}$ are same as given in Theorem 2.1, T_1 and T_2 are also same as given in (3.5). The conditions of validity of the above result easily follow from Theorem 2.1.
(vi) On putting $m_1 = n_1 = p = q = 0, r = 1, \mu = 1, p_1^{(1)} = 0, m_2 = q_1^{(1)} = 1, b_1 = 0$ and $\beta_1 = 1$ in Theorem 2.1 then by virtue of the relation $H^{1,0}_{0,1}\left[z_1t\left|\begin{array}{c} \\ (0,1) \end{array}\right.\right] = e^{-z_1t}$ we have following result

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} e^{-z_1t} {}_uM_{\nu}^{\delta,\lambda}(zt) H^{m_3,n_3}_{p_1^{(2)},q_1^{(2)}}\left[z_2t^{\nu}\left|\begin{array}{c} (c_j,\gamma_j)_{1,p_1^{(2)}} \\ (d_j,\delta_j)_{1,q_1^{(2)}} \end{array}\right.\right]\right\}(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha^{'}-1} \sum_{s=0}^{\infty} f(T) x^s H^{0,3:1,0;m_3,n_3}_{3,3:0,1;p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c|c} z_1x & X_9^{'},\ldots:-;(c_j,\gamma_j)_{1,p_1^{(2)}} \\ z_2x^{\nu} & X_{10}^{'},\ldots:(0,1);(d_j,\delta_j)_{1,q_1^{(2)}} \end{array}\right], \tag{3.7}$$

where $f(T)$ is represented by (2.3) and

$$X_9^{'} = [(1-\rho-s:1,\nu)],[(1-\rho-s+\alpha+\alpha^{'}+\beta-\gamma:1,\nu)],[(1-\rho-s+\alpha^{'}-\beta^{'}:1,\nu)],$$
$$X_{10}^{'} = [(1-\rho-s+\alpha+\alpha^{'}-\gamma:1,\nu)],[(1-\rho-s+\alpha^{'}+\beta-\gamma:1,\nu)],[(1-\rho-s-\beta^{'}:1,\nu)].$$

The conditions of validity of the above result easily follow from Theorem 2.1.
(vii) If we set $m_1 = n_1 = p = q = 0$, $\delta = 1$ and $\lambda = 1$ in Theorem 2.1 then we have following result in terms of product of I-function of one variable introduced by Saxena [26]

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_uF_{\nu}(zt) I^{m_2,n_2}_{p_i^{(1)},q_i^{(1)}:r}\left[z_1t^{\mu}\left|\begin{array}{c} (a_j,\alpha_j)_{1,n_2},(a_{ji},\alpha_{ji})_{n_2+1,p_i^{(1)}} \\ (b_j,\beta_j)_{1,m_2},(b_{ji},\beta_{ji})_{m_2+1,q_i^{(1)}} \end{array}\right.\right]\right.$$

$$\left. I^{m_3,n_3}_{p_i^{(2)},q_i^{(2)}:r}\left[z_2t^{\nu}\left|\begin{array}{c} (c_j,\gamma_j)_{1,n_3},(c_{ji},\gamma_{ji})_{n_3+1,p_i^{(2)}} \\ (d_j,\delta_j)_{1,m_3},(d_{ji},\delta_{ji})_{m_3+1,q_i^{(2)}} \end{array}\right.\right]\right\}(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha^{'}-1} \sum_{s=0}^{\infty} f(T') x^s I^{0,3:m_2,n_2;m_3,n_3}_{3,3:p_i^{(1)},q_i^{(1)};p_i^{(2)},q_i^{(2)}:r}\left[\begin{array}{c|c} z_1x^{\mu} & X_1^{'},\ldots:U \\ z_2x^{\nu} & X_2^{'},\ldots:V \end{array}\right]. \tag{3.8}$$

where $f(T')$ is represented by (3.2) and X_1' and X_2' are same as given in Theorem 2.1. The conditions of validity of the above result easily follow from Theorem 2.1.
(viii) If we set $m_1 = 0$, $r = 1$, $\delta = 1$, and $\lambda = 1$ in Theorem 2.1, the I-function of two variables occurring in L.H.S. reduces into H-function of two variables [25] then we have following result

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_uF_v(zt) H^{0,n_1:m_2,n_2;m_3,n_3}_{p,q:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c} z_1t^\mu \\ z_2t^\nu \end{array}\middle| \begin{array}{c} [(e_p:E_p,E_p')]:T_1 \\ [(f_q:F_q,F_q')]:T_2 \end{array}\right]\right\}(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha'-1}\sum_{s=0}^{\infty} f(T')x^s H^{0,n_1+3:m_2,n_2;m_3,n_3}_{p+3,q+3:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c} z_1x^\mu \\ z_2x^\nu \end{array}\middle| \begin{array}{c} X_1',[(e_p:E_p,E_p')]:T_1 \\ [(f_q:F_q,F_q')],X_2':T_2 \end{array}\right], \tag{3.9}$$

where $f(T')$ is represented by (3.2), X_1' and X_2' are same as given in Theorem 2.1, T_1 and T_2 are also same as given in (3.5). The conditions of validity of the above result easily follow from Theorem 2.1.
(ix) If we set $m_1 = n_1 = p = q = 0$, $r = 1$, $\delta = 1$ and $\lambda = 1$ in Theorem 2.1, then we have following result in terms of product of H-functions

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} {}_uF_v(zt) H^{m_2,n_2}_{p_1^{(1)},q_1^{(1)}}\left[z_1t^\mu\middle| \begin{array}{c} (a_j,\alpha_j)_{1,p_1^{(1)}} \\ (b_j,\beta_j)_{1,q_1^{(1)}} \end{array}\right] \times H^{m_3,n_3}_{p_1^{(2)},q_1^{(2)}}\left[z_2t^\nu\middle| \begin{array}{c} (c_j,\gamma_j)_{1,p_1^{(2)}} \\ (d_j,\delta_j)_{1,q_1^{(2)}} \end{array}\right]\right\}(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha'-1}\sum_{s=0}^{\infty} f(T')x^s H^{0,3:m_2,n_2;m_3,n_3}_{3,3:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c} z_1x^\mu \\ z_2x^\nu \end{array}\middle| \begin{array}{c} X_1',\ldots:T_1 \\ X_2',\ldots:T_2 \end{array}\right], \tag{3.10}$$

where $f(T')$ is represented by (3.2), X_1' and X_2' are same as given in Theorem 2.1,T_1 and T_2 are also same as given in (3.5). The conditions of validity of the above result easily follow from Theorem 2.1.
(x) On putting $m_1 = n_1 = p = q = 0, r = 1, \mu = 1, p_1^{(1)} = 0, m_2 = q_1^{(1)} = 1, b_1 = 0, \beta_1 = 1, \delta = 1$ and $\lambda = 1$ in Theorem 2.1 then by virtue of the relation $H^{1,0}_{0,1}\left[z_1t\middle| \begin{array}{c} \\ (0,1) \end{array}\right] = e^{-z_1t}$ we have following result

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} e^{-z_1t} {}_uF_v(zt) H^{m_3,n_3}_{p_1^{(2)},q_1^{(2)}}\left[z_2t^\nu\middle| \begin{array}{c} (c_j,\gamma_j)_{1,p_1^{(2)}} \\ (d_j,\delta_j)_{1,q_1^{(2)}} \end{array}\right]\right\}(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha'-1}\sum_{s=0}^{\infty} f(T')x^s H^{0,3:1,0;m_3,n_3}_{3,3:0,1;p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c} z_1x \\ z_2x^\nu \end{array}\middle| \begin{array}{c} X_9'\ldots:-;(c_j,\gamma_j)_{1,p_1^{(2)}} \\ X_{10}'\ldots:(0,1);(d_j,\delta_j)_{1,q_1^{(2)}} \end{array}\right], \tag{3.11}$$

where $f(T')$ is represented by (3.2), X_9' and X_{10}' are same as given in (3.7). The conditions of validity of the above result easily follow from Theorem 2.1.
(xi) If we set $m_1 = n_1 = p = q = 0$, $u = 0$ and $v = 0$ in Theorem 2.1 then we have following result in terms of product of I-function of one variable introduced by Saxena [26]

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} E_{\delta,\lambda}(zt) I^{m_2,n_2}_{p_i^{(1)},q_i^{(1)}:r}\left[z_1t^\mu\middle| \begin{array}{c} (a_j,\alpha_j)_{1,n_2},(a_{ji},\alpha_{ji})_{n_2+1,p_i^{(1)}} \\ (b_j,\beta_j)_{1,m_2},(b_{ji},\beta_{ji})_{m_2+1,q_i^{(1)}} \end{array}\right]\right.$$

$$\left.\times I^{m_3,n_3}_{p_i^{(2)},q_i^{(2)}:r}\left[z_2t^\nu\middle| \begin{array}{c} (c_j,\gamma_j)_{1,n_3},(c_{ji},\gamma_{ji})_{n_3+1,p_i^{(2)}} \\ (d_j,\delta_j)_{1,m_3},(d_{ji},\delta_{ji})_{m_3+1,q_i^{(2)}} \end{array}\right]\right\}(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha'-1}\sum_{s=0}^{\infty} \frac{x^s}{\Gamma(\delta s+\lambda)} I^{0,3:m_2,n_2;m_3,n_3}_{3,3:p_i^{(1)},q_i^{(1)};p_i^{(2)},q_i^{(2)}:r}\left[\begin{array}{c} z_1x^\mu \\ z_2x^\nu \end{array}\middle| \begin{array}{c} X_1',\ldots:U \\ X_2',\ldots:V \end{array}\right], \tag{3.12}$$

where X_1' and X_2' are same as given in Theorem 2.1. The conditions of validity of the above result easily follow from Theorem 2.1.
(xii) If we set $m_1 = 0$, $r = 1$, $u = 0$, and $v = 0$ in Theorem 2.1, the I-function of two variables occurring in L.H.S. reduces into H-function of two variables [25] then we have following result

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} E_{\delta,\lambda}(zt) H^{0,n_1:m_2,n_2;m_3,n_3}_{p,q:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c} z_1t^\mu \\ z_2t^\nu \end{array}\middle| \begin{array}{c} [(e_p:E_p,E_p')]:T_1 \\ [(f_q:F_q,F_q')]:T_2 \end{array}\right]\right\}(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha'-1}\sum_{s=0}^{\infty}\frac{x^s}{\Gamma(\delta s+\lambda)}H^{0,n_1+3:m_2,n_2;m_3,n_3}_{p+3,q+3:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c|c} z_1x^{\mu} & X_1', [(e_p:E_p,E_p')]:T_1 \\ z_2x^{\nu} & [(f_q:F_q,F_q')], X_2':T_2 \end{array}\right], \tag{3.13}$$

where X_1' and X_2' are same as given in Theorem 2.1, T_1 and T_2 are also same as given in (3.5). The conditions of validity of the above result easily follow from Theorem 2.1.

(xiii) If we set $m_1 = n_1 = p = q = 0$, $r = 1$, $u = 0$ and $v = 0$ in Theorem 2.1, then we have following result in terms of product of H-functions

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}E_{\delta,\lambda}(zt)H^{m_2,n_2}_{p_1^{(1)},q_1^{(1)}}\left[z_1t^{\mu}\left|\begin{array}{c}(a_j,\alpha_j)_{1,p_1^{(1)}}\\(b_j,\beta_j)_{1,q_1^{(1)}}\end{array}\right.\right]\times H^{m_3,n_3}_{p_1^{(2)},q_1^{(2)}}\left[z_2t^{\nu}\left|\begin{array}{c}(c_j,\gamma_j)_{1,p_1^{(2)}}\\(d_j,\delta_j)_{1,q_1^{(2)}}\end{array}\right.\right]\right\}(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha'-1}\sum_{s=0}^{\infty}\frac{x^s}{\Gamma(\delta s+\lambda)}H^{0,3:m_2,n_2;m_3,n_3}_{3,3:p_1^{(1)},q_1^{(1)};p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c|c} z_1x^{\mu} & X_1',\ldots:T_1 \\ z_2x^{\nu} & X_2',\ldots:T_2 \end{array}\right], \tag{3.14}$$

where X_1' and X_2' are same as given in Theorem 2.1,T_1 and T_2 are also same as given in (3.5). The conditions of validity of the above result easily follow from Theorem 2.1.

(xiv) On putting $m_1 = n_1 = p = q = 0, r = 1, \mu = 1, p_1^{(1)} = 0, m_2 = q_1^{(1)} = 1, b_1 = 0, \beta_1 = 1$, $u = 0$ and $v = 0$ in Theorem 2.1 then by virtue of the relation$H^{1,0}_{0,1}\left[z_1t\left|\begin{array}{c}\\(0,1)\end{array}\right.\right] = e^{-z_1t}$ we have following result

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}e^{-z_1t}E_{\delta,\lambda}(zt)H^{m_3,n_3}_{p_1^{(2)},q_1^{(2)}}\left[z_2t^{\nu}\left|\begin{array}{c}(c_j,\gamma_j)_{1,p_1^{(2)}}\\(d_j,\delta_j)_{1,q_1^{(2)}}\end{array}\right.\right]\right\}(x)$$

$$= x^{\rho+\gamma-\alpha-\alpha'-1}\sum_{s=0}^{\infty}\frac{x^s}{\Gamma(\delta s+\lambda)}H^{0,3:1,0;m_3,n_3}_{3,3:0,1;p_1^{(2)},q_1^{(2)}}\left[\begin{array}{c|c} z_1x & X_9'\ldots:-;(c_j,\gamma_j)_{1,p_1^{(2)}} \\ z_2x^{\nu} & X_{10}',\ldots:(0,1);(d_j,\delta_j)_{1,q_1^{(2)}} \end{array}\right], \tag{3.15}$$

where X_9' and X_{10}' are same as given in (3.7). The conditions of validity of the above result easily follow from Theorem 2.1.

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ENCRYPTION BASED ON CONFERENCE MATRIX

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Abstract

In this article, an encryption scheme based on conference matrix has been developed. An easier algorithm of formation of encryption/decryption keys have been discussed. The decryption key comprising of fixed number of positive integers with prime power yields a high level security of message. Some popular attacks have been discussed in the context of cryptoanalysis and observed that it is robust against the popular cipher attack and the security of the information does not compromise.

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1. Introduction

Cryptography is the study of techniques of secured communications i.e. a study of techniques which ensure that communicated information cannot be understood by anyone except the intended receiver.

In 1929 Lester [10] introduced the Hill cipher in which an invertible matrix is used as a private key and inverse of that matrix is used to decrypt the message.

Koukouvinos and Simos [8] also developed an encryption scheme using circulant Hadamard core in which the first row of the Hadamard core required to be transmitted as a private key to the intended receiver.

There are several methods of constructions of Hadamard matrices have been developed [2, 4, 12, 14]. In this article we propose a private symmetric key encryption scheme based on conference matrices of order n such that $n = p^r(p+2)^{r'}$: where $r = 1,\ r' = 1$ or $r \geq 1,\ r' = 0$, and both $p,\ p+2$ are odd primes with modular base q, a positive integer. In this encryption scheme message has been encrypted by a conference matrix of order n and decrypted by its transpose with the identity matrix I_n and J_n (a square matrix of order n with all entries 1). The involvement of the transpose of conference matrix and two standard matrices make easy to construct the decryption key for intended receiver. This scheme requires transmission of numbers $(p,\ r,\ r',\ d,\ q)$ as a private key. In addition to these numbers the primitive polynomial $P(p^r)$ of $GF(p^r)$ which has been used to construct the conference matrix of order n is required to be transmitted in case of $n = p^r,\ r > 1$ [5, 7, 9, 13].

It is not easy to find the conference matrix of order n unless such n and $P(p^r)$ (either case) are given. Subsequently it may raise the difficulties for the intruder by involvement of a positive integer modular base and order of conference matrix n formed by a large prime.

The main goal of the proposed technique includes the following:

1. Require the private key which is shared by the sender and receiver only once.
2. Easy transmission of private key
3. Computation of encryption and decryption are fast.
4. Difficult to guess the key for intruder.
5. Robust to cryptographic attack.

The structure of this paper is as follows. In Section 2, the required definitions and information as well as algorithm of formation of the key matrix have been discussed as preliminaries. In Section 3 we have discussed the encryption/decryption algorithms and its mechanism which ensure that the encrypted message is determined uniquely. Furthermore in Section 4 we have done security analysis of the scheme with different cryptographic attack. In Section 5 we have given an example and finally in Section 6 we have concluded the results explaining its limitations and benefits.

2. Preliminaries

In this section some basic terminologies are defined which have been used to design the cryptographic algorithm.

Definition 2.1 ([5, 11]). **Quadratic Residue**

An element α of $GF(n)$ is said to be quadratic residue if it is a perfect square in $GF(n)$ otherwise α is a quadratic non-residue

Definition 2.2 ([11]). **Extended Quadratic character**

Let $n = p^r$, where p is an odd prime, r is any positive integer and $GF(n) = \{0 = \alpha_0, \alpha_1, \alpha_2, ..., \alpha_{n-1}\}$. Then the Extended quadratic character is a map χ defined on $GF(n)$ as

$$\chi(\alpha_i) = \begin{cases} 1; & \text{if } \alpha_i \text{ is quadratic residue in } GF(n) \\ 0; & \text{if } \alpha_i = 0 \\ -1; & \text{if } \alpha_i \text{ is quadratic nonresidue in } GF(n). \end{cases}$$

Definition 2.3. Legendre Symbol

Consider an odd prime p and suppose $a \in GF(p)$. The Legendre symbol of a modulo p is defined as

$$\left(\frac{a}{p}\right) = \begin{cases} 1; & \text{if } a \text{ is a quadratic residue mod } p \\ 0; & \text{if } a = 0 \\ -1; & \text{otherwise.} \end{cases}$$

Definition 2.4 ([8]). **Encryption scheme**

An Encryption is the process in which we encode a message or information in such a manner so that only intended person can access it. There are three sets in the encryption scheme: a message set or plaintext M, a ciphertext (encrypted message) C, and a key set K together with the following three algorithms.

1. *A key set K which generates the valid encryption key $k \in K$ and a valid key $k^{-1} \in K$ to decrypt the message.*
2. *An encryption algorithm in which message $m \in M$ and key $k \in K$ together produce an element $c \in C$ which is defined as $c = E_k(m)$.*
3. *A decryption algorithm in which an element $c \in C$ with decryption key $k^{-1} \in K$ return back an element of message $m \in M$ with $m = D_{k^{-1}}(c)$.*

Note that $D_{k^{-1}}(E_k(m)) = m$.

Definition 2.5 ([8]). **O-notation**

This notation is used to describe the complexity or performance of an algorithm. Basically "big O" defines an upper bound of an algorithm. Formally, If $f(n)$ and $g(n)$ are two functions, we denote $O(g(n))$ the set of functions and defined as $O(g(n)) = \{f(n)$: there exist positive constant c and n_0 such that $0 \leq f(n) \leq cg(n)$ for all $n \geq n_0\}$

Definition 2.6 ([11]). **Conference matrix**

A square matrix Q of order n whose off diagonal entries are from $\{-1, 1\}$ and diagonal entries are 0 is known as conference matrix if

$$Q^T Q = QQ^T = nI_n - J_n.$$

Definition 2.7 ([6]). **Primitive Polynomial**

An irreducible polynomial $f(x)$ is said to be primitive polynomial if the root of $f(x) = 0$ in $GF(p^r)$ is the generator of the cyclic group of non-zero field elements of the finite field $GF(p^r)$.
Generally primitive polynomial of the field $GF(p^r)$ is denoted as $P(p^r)$.

Definition 2.8. Circulant Matrix

Let $C = [c_{ij}]$ be a square matrix of order n with first row $c_0,\ c_1,\ \cdots,\ c_{n-1}$. Then matrix C is called Circulant matrix if

$$c_{ij} = c_{(j-i) \bmod n}, \qquad \text{for } 1 \leq i,\ j \leq n.$$

2.1. Formation of key matrix

There are some methods which ensure the existence of conference matrices of order n which are used to construct the key matrices. Difference sets are also used to construct key matrices in some cases.

Method I. Let $\mathbb{F}$ be a field of order $n = p^r$, where p is an odd prime and r is positive integer. Suppose $0 = \alpha_0,\ \alpha_1,\ \cdots,\ \alpha_{n-1}$ are the elements of field $\mathbb{F}$. Then the square matrix $Q = [q_{ij}]$ is defined as

$$q_{ij} = \begin{cases} \chi\{(\alpha_j - \alpha_i)(\text{mod } p)\} & \text{if } r = 1, \\ \chi\{(\alpha_j - \alpha_i)(\text{mod } P(p^r))\} & \text{if } r > 1, \end{cases} \tag{2.1}$$

where $P(p^r)$ is the primitive polynomial of $\mathbb{F}$.

It can be observe that the matrix Q based on finite field using quadratic residue and non-residue is a circulant matrix.

Case I. For $p^r \equiv 3(\text{mod } 4)$, the square matrix A of order $n = p^r$ is defined as

$$A = Q + I, \tag{2.2}$$

considered as a key matrix.

Case II. For $p^r \equiv 1 (\text{mod } 4)$, the square matrix A of order $n = 2p^r$ is defined as

$$A = \begin{bmatrix} Q+I & -Q+I \\ -Q+I & -Q-I \end{bmatrix}_{n\times n}, \tag{2.3}$$

considered as a key matrix [9].

Remark 2.1. However for $r = 1$ the leading elements of the matrix A of order n are $0,\ 1,\ 2, \cdots,\ n-1$ and A is circulant. For $r > 1$ the leading elements of A are $0,\ \lambda^1,\ \lambda^2, \cdots,\ \lambda^{n-1}$ where λ is a root of primitive polynomial $P(p^r)$ of $GF(p^r)$.

Lemma 2.1 ([11])**.** *Let $\mathbb{F}$ be a field then $\sum_b \chi(b)\chi(b+c) = -1$ if $c \neq 0$, and $b, c \in \mathbb{F}$.*

Lemma 2.2 ([11])**.** *The matrix $Q = [q_{ij}] = \chi(\alpha_j - \alpha_i)$ of order $n\times n$, where $\alpha_i, \alpha_j \in GF(n)$, has the following properties*
(a) Q is symmetric if $n \equiv 1$ (mod 4) and skew symmetric if $n \equiv 3$ (mod 4).
(b) $QQ^T = nI_n - J_n$ i.e. Q is a conference matrix.

Method II. When n is a product of twin primes i.e. $n = p(p+2)$, where both p and $p+2$ are primes, the key matrix A of order n is formed using difference sets [7, 11].

Let $a_1,\ a_2,\ \cdots,\ a_m$ are those $m = \frac{(p-1)(p+1)}{2}$ elements of $\mathbb{Z}_n$ for which $(\frac{a_i}{p}) = (\frac{a_i}{p+2})$ and $a_{m+1},\ a_{m+2},\ \cdots,\ a_{m+p}$ are integers $0,\ (p+2),\ 2(p+2), \cdots,\ (p-1)(p+2)$ respectively (here $(\frac{a_i}{p})$ denotes the Legendre symbol of a_i modulo p). Then $D = \{a_1,\ a_2,\ \cdots,\ a_k\}$ forms a difference set where $k = m + p = \frac{p(p+2)-1}{2}$. The key matrix A of order n is a circulant matrix with leading elements $0,\ 1,\ 2,\ \cdots,\ (p(p+2)-1)$ whose first row is formed by assigning 1 to the elements of difference set D and -1 to rest of the elements.

2.2. Examples

2.2.1. Construction of key matrix A using method I

Suppose $n = 7$, then $\mathbb{F} = \{0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6\}$
In this field $\mathbb{F}$, there are three quadratic residues 1, 2 and 4.
So,

$$\chi(\alpha_i) = \begin{cases} 1; & \text{if } \alpha_i \in \{1,\ 2,\ 4\} \\ 0; & \text{if } \alpha_i = 0 \\ -1; & \text{if } \alpha_i \in \{3,\ 5,\ 6\}. \end{cases}$$

Using the leading element $0, 1, 2, \cdots, 6$ first row is formed by assigning 1 to $\{1, 2, 4\}$ and -1 to rest of the element. Then matrix Q can be obtained by circulating the first row

$$\begin{bmatrix} 0 & 1 & 1 & -1 & 1 & -1 & -1 \end{bmatrix}.$$

⋆ For $p^r \equiv 1(\text{mod } 4)$, similar method can be used to construct the square matrix Q.
Key matrix A can be obtained by using equation (2.2) or (2.3) on the basis of types of p^r.

2.2.2. Construction of square matrix Q by using method II

Let $n = 15 = 3 \times 5$ i.e. product of twin primes.
So, $\mathbb{Z}_{15} = \{0,\ 1,\ 2,\ 3,\ 4,\ , \cdots,\ 14\}$.
To form the difference set, we need to calculate $(\frac{\alpha_i}{p}) = (\frac{\alpha_i}{q})$ for $\alpha_i \in \mathbb{Z}_{15}$.
In this case the set $D = \{1,\ 2,\ 4,\ 8\}$ provides equal Legendre symbol. The set D together with $\{0,\ 5,\ 10\}$ form a difference set, at which we assign $+1$ and -1 to rest of the elements. Here we can see the first row of the matrix $[A]_{15\times 15}$

$$\begin{bmatrix} 1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 \end{bmatrix}.$$

2.2.3. Algorithm to construct the key matrix

The algorithm to construct the key matrix A is as follows:-

Algorithm 2.1 Formation of key matrix A

Require: $p,\ r,\ r'$. where p is an odd prime
Ensure: $r \geq 1,\ r' \in \{0, 1\}$
if $r' = 1,\ r = 1$ **then**
  if both $p,\ p+2$ are odd primes **then**
    use method-II to construct the key matrix A.
    $n \leftarrow p(p+2)$
  else
    no need to choose this method
  end if
else $\{r' = 0,\ r \geq 1\}$
  construct Q of order p^r using case-I and case-II of method-I
  if $p^r \equiv 3(\text{mod } 4)$ **then**
    $A \leftarrow Q + I,$
    $n \leftarrow p^r$
  else $\{p^r \equiv 1(\text{mod } 4)\}$
    $A \leftarrow \begin{bmatrix} Q+I & -Q+I \\ -Q+I & -Q-I \end{bmatrix}$
    $n \leftarrow 2p^r$
  end if
end if

3. Results

3.1. Design of Cryptographic Algorithm

Let there are q distinct characters in the language in which the message or information is written. We convert the message to be transmitted into its corresponding numeric plain text (ASCII code) in modulo q. In order to block cipher we divide the plain text into blocks of each size n and each block represented as a column vector. We add "space" in the last block to make it of size n if needed.

The encrypted message to be transmitted over a communication channel of a column vector M is

$$C \equiv (AM + de_n)(\text{mod } q), \tag{3.1}$$

where d is any constant, $e_n = (1,\ 1,\ \cdots,\ 1)^T$, A is a key matrix of order n and q is a positive integer modular base with $gcd(n+1,\ q) = 1$, in case of $n = p^r \equiv 3(\text{mod } 4)$ or $n = p(p+2)$ and that of $gcd(n+2,\ q) = 1$ where $n = 2p^r,\ p^r \equiv 1(\text{mod } 4)$.

According to Hill cipher it requires A^{-1} to decrypt the message. However in this scheme $C - de_n$ is pre-multiplied by A^T by the intended receiver to disposed off the calculation of A^{-1}. Now to get the original message receiver has to decrypt the message using the transformation

$$M \equiv (A^T A)^{-1} A^T (C - de_n)(\text{mod } q). \tag{3.2}$$

In general it is quite difficult to obtain $(A^T A)^{-1}$ for large value of n. But, since matrix A is formed by using conference matrix Q of order n and difference set of $\mathbb{Z}_n$ so $(A^T A)^{-1}$ may be obtained using I and J with simple calculations.

For $r' = 1,\ r = 1$ consider $n = p(p+2)$ and for $r' = 0,\ r \geq 1$ with $p^r \equiv 3(\text{mod } 4)$ consider $n = p^r$. In both the cases we have $A = Q + I$.

$$\begin{aligned} \Rightarrow AA^T &= [Q+I][Q+I]^T \\ &= [Q+I][Q^T+I] \\ &= [QQ^T + Q + Q^T + I] \end{aligned}$$

Since $p^r \equiv 3(\text{mod } 4)$, so Q is a skew symmetric matrix of order p^r. i.e. $Q^T = -Q$, So,

$$\begin{aligned} AA^T &= QQ^T + I = nI_n - J_n + I_n, \\ A^T A &= (n+1)I_n - J_n. \end{aligned} \tag{3.3}$$

and

$$(A^T A)^{-1} = \frac{1}{n+1}(I_n + J_n). \tag{3.4}$$

We can see that

$$
\begin{aligned}
(A^TA)(A^TA)^{-1} &= \{(n+1)I_n - J_n\}\{\frac{1}{n+1}(I_n+J_n)\}\\
&= (n+1)I_n\frac{1}{n+1}I_n + (n+1)I_n\frac{1}{n+1}J_n - J_n\frac{1}{n+1}I_n - J_n\frac{1}{n+1}J_n\\
&= I_n + (n+1)J_n\frac{1}{n+1} - \frac{J_n}{n+1} - \frac{nJ_n}{n+1}\\
&= I_n,
\end{aligned}
$$

so equation (3.2) reduced to

$$M \equiv \frac{1}{n+1}(I_n+J_n)A^T(C-de_n)(mod\ q), \tag{3.5}$$

which is quite easy to form the decryption key for intended receiver as n is known.

For $r'=0,\ r\geq 1$ with $p^r \equiv 1(mod\ 4)$ consider $n=2p^r$ and we have

$$A = \begin{bmatrix} Q+I & -Q+I \\ -Q+I & -Q-I \end{bmatrix}.$$

So,

$$
\begin{aligned}
A^TA &= \begin{bmatrix} Q+I & -Q+I \\ -Q+I & -Q-I \end{bmatrix}\begin{bmatrix} Q+I & -Q+I \\ -Q+I & -Q-I \end{bmatrix}^T\\
&= \begin{bmatrix} Q+I & -Q+I \\ -Q+I & -Q-I \end{bmatrix}\begin{bmatrix} Q^T+I & -Q^T+I \\ -Q^T+I & -Q^T-I \end{bmatrix}\\
&= \begin{bmatrix} QQ^T+Q+Q^T+I+QQ^T-Q-Q^T+I & -QQ^T+Q-Q^T+I+QQ^T+Q-Q^T-I \\ -QQ^T-Q+Q^T+I+QQ^T-Q+Q^T-I & QQ^T-Q-Q^T+I+QQ^T+Q+Q^T+I \end{bmatrix}
\end{aligned}
$$

since order of Q is $p^r \equiv 1(\text{mod } 4)$, so $Q^T = Q$. Hence

$$
\begin{aligned}
A^TA &= \begin{bmatrix} 2(QQ^T+I) & \mathbf{0} \\ \mathbf{0} & 2(QQ^T+I) \end{bmatrix}\\
&= \begin{bmatrix} 2(p^rI_{p^r} - J_{p^r} + I_{p^r}) & \mathbf{0} \\ \mathbf{0} & 2(p^rI_{p^r} - J_{p^r} + I_{p^r}) \end{bmatrix}\\
&= \begin{bmatrix} (2p^r+2)I_{p^r} - 2J_{p^r} & \mathbf{0} \\ \mathbf{0} & (2p^r+2)I_{p^r} - 2J_{p^r} \end{bmatrix}\\
&= \begin{bmatrix} (n+2)I_{\frac{n}{2}} - 2J_{\frac{n}{2}} & \mathbf{0} \\ \mathbf{0} & (n+2)I_{\frac{n}{2}} - 2J_{\frac{n}{2}} \end{bmatrix}\\
&= \{(n+2)I_{\frac{n}{2}} - 2J_{\frac{n}{2}}\} \otimes I_2,
\end{aligned} \tag{3.6}
$$

and,

$$(A^TA)^{-1} = \frac{1}{(n+2)}\{(I_{\frac{n}{2}} + J_{\frac{n}{2}}) \otimes I_2\}. \tag{3.7}$$

We can see that

$$
\begin{aligned}
(A^TA)(A^TA)^{-1} &= \{\{(n+2)I_{\frac{n}{2}} - 2J_{\frac{n}{2}}\} \otimes I_2\}\frac{1}{(n+2)}\{(I_{\frac{n}{2}} + J_{\frac{n}{2}}) \otimes I_2\}\\
&= \{(n+2)I_{\frac{n}{2}} \otimes I_2 - 2J_{\frac{n}{2}} \otimes I_2\}\{\frac{1}{n+2}I_{\frac{n}{2}} \otimes I_2 + \frac{1}{n+2}J_{\frac{n}{2}} \otimes I_2\}\\
&= \{(n+2)I_n - 2J_{\frac{n}{2}} \otimes I_2\}\{\frac{1}{n+2}I_n + \frac{1}{n+2}J_{\frac{n}{2}} \otimes I_2\}\\
&= \frac{n+2}{n+2}I_n + \frac{n+2}{n+2}I_n(J_{\frac{n}{2}} \otimes I_2) - \frac{2}{n+2}(J_{\frac{n}{2}} \otimes I_2)I_n - \frac{2}{n+2}(J_{\frac{n}{2}} \otimes I_2)(J_{\frac{n}{2}} \otimes I_2)\\
&= I_n + \frac{n+2}{n+2}(J_{\frac{n}{2}} \otimes I_2) - \frac{2}{n+2}(J_{\frac{n}{2}} \otimes I_2) - \frac{2}{n+2}(J_{\frac{n}{2}}J_{\frac{n}{2}} \otimes I_2I_2)\\
&= I_n + \frac{n+2}{n+2}(J_{\frac{n}{2}} \otimes I_2) - \frac{2}{n+2}(J_{\frac{n}{2}} \otimes I_2) - \frac{2}{n+2}(\frac{n}{2}J_{\frac{n}{2}} \otimes I_2)
\end{aligned}
$$

$$= I_n + \frac{1}{n+2}(n+2-2-n)(J_{\frac{n}{2}} \otimes I_2)$$
$$= I_n,$$

and equation (3.2) reduces to

$$M \equiv \frac{1}{n+2}\{(I_{\frac{n}{2}} + J_{\frac{n}{2}}) \otimes I_2\}A^T(C - de_n)(\text{mod } q). \tag{3.8}$$

The cryptographic algorithm for encryption is given by

Algorithm 3.1 Encryption Algorithm

Require: Require msg to encrypt
select $p,\ r,\ r',\ d,\ q,\ P(p^r)$
if $r = 1,\ r' \in \{0, 1\}$ **then**
  $T \leftarrow (p,\ r,\ r',\ d,\ q)$
else $\{r > 1,\ r' = 0\}$
  $T \leftarrow (p,\ r,\ d,\ q,\ P(p^r))$
end if
$M \leftarrow$ convert(msg) //* convert message into its corresponding numeric value.
construct the key matrix A //* construct matrix A using ***Algorithm*** **2.1**.
$C \leftarrow (AM + de_n)(\text{mod } q)$ //* Encrypted message is C.
Transmit(C, T)

In order to fulfill the objectives of the cryptography the encrypted message C has to be decrypted uniquely.

Theorem 3.1. *If C is the encrypted message which is transmitted with the encryption algorithm then*

1. *For $r' = 1,\ r = 1$ consider $n = p(p+2)$ and for $r' = 0,\ r \geq 1$ with $p^r \equiv 3(\text{mod } 4)$ consider $n = p^r$ the decrypted message*
$$D \equiv (I_n + J_n)A^T(C - de_n)t(mod\ q)$$
is uniquely determined and is equal to M, where t is solution of $(n+1)x \equiv 1(\text{mod } q)$.
2. *For $r' = 0,\ r \geq 1$ with $p^r \equiv 1(\text{mod } 4)$, consider $n = 2p^r$, the decrypted message*
$$D \equiv \{(I_{\frac{n}{2}} + J_{\frac{n}{2}}) \otimes I_2\}A^T(C - de_n)t(\text{mod } q)$$
is uniquely determined and is equal to M, where t is a solution of $(n+2)x \equiv 1(\text{mod } q)$.

Proof. 1. Since $gcd(n+1,\ q) = 1$ so $(n+1)x \equiv 1(\text{mod } q)$ has unique solution t. As C is an encrypted message with respect to the encryption algorithm (3.1). So
$$C \equiv (AM + de_n)(\text{mod } q)$$
$$\Rightarrow C - de_n \equiv AM(\text{mod } q).$$
Since,
$$\begin{aligned}
D &\equiv (I_n + J_n)A^T(C - de_n)t(\text{mod } q) \\
&\equiv (I_n + J_n)A^T AMt(\text{mod } q) && (as\ C - de_n = AM(\text{mod } q)) \\
&\equiv (I_n + J_n)((n+1)I_n - J_n)Mt(\text{mod } q) && [\ by\ (3.3)] \\
&\equiv \{(n+1)I_n - J_n + (n+1)J_n - nJ_n\}Mt(\text{mod } q) \\
&\equiv (n+1)I_n Mt(\text{mod } q) \\
&\equiv M(n+1)t(\text{mod } q) && as\ t\ is\ a\ solution\ of\ (n+1)x \equiv 1(\text{mod } q) \\
&\equiv M(\text{mod } q).
\end{aligned}$$

2. Let $gcd(n+2,\ q) = 1$ so $(n+2)x \equiv 1(\text{mod } q)$ has unique solution t. Since
$$\begin{aligned}
D &\equiv \{(I_{\frac{n}{2}} + J_{\frac{n}{2}}) \otimes I_2\}A^T(C - de_n)t(\text{mod } q) \\
&\equiv \{(I_{\frac{n}{2}} + J_{\frac{n}{2}}) \otimes I_2\}A^T AMt(\text{mod } q) \\
&\equiv \{(I_{\frac{n}{2}} + J_{\frac{n}{2}}) \otimes I_2\}\{((n+2)I_{\frac{n}{2}} - 2J_{\frac{n}{2}}) \otimes I_2\}Mt(\text{mod } q) && by\ (3.6)
\end{aligned}$$

$$\equiv \{(n+2)I_{\frac{n}{2}} - 2J_{\frac{n}{2}} + (n+2)J_{\frac{n}{2}} - \frac{2n}{2}J_{\frac{n}{2}}\} \otimes I_2 Mt (\text{mod } q)$$
$$\equiv (n+2)I_{\frac{n}{2}} \otimes I_2 Mt (\text{mod } q)$$
$$\equiv (n+2)I_n Mt (\text{mod } q)$$
$$\equiv (n+2)tM (\text{mod } q) \qquad \textit{as t is a solution of } (n+2)x \equiv 1 (\text{mod } q)$$
$$\equiv M (\text{mod } q).$$

So, $D = M$ i.e. message is uniquely decrypted.

□

The decryption algorithm to obtain the plain text is given by

Algorithm 3.2 Decryption Algorithm

Require: Require received cipher text C and T

$T = p, r, r', d, q, P(p^r)$
construct A^T *// using ***Algorithm* 2.1**
if $r = 1, r' = 1$ **then**
  $n \leftarrow p(p+2)$
  find t (t is a solution of $(n+1)x \equiv 1(\text{mod } q)$)
else $\{r \geq 1, r' = 0\}$
  if $p^r \equiv 3(\text{mod } 4)$ **then**
    $n \leftarrow p^r$
    find t (t is a solution of $(n+1)x \equiv 1(\text{mod } q)$
    $k \leftarrow (I_n + J_n)A^T t$ *//set private key
  else $\{p^r \equiv 1(\text{mod } 4)\}$
    $n \leftarrow 2p^r$
    find t (t is a solution of $(n+2)x \equiv 1(\text{mod } q)$)
    $k \leftarrow \{(I_{\frac{n}{2}} + J_{\frac{n}{2}}) \otimes I_2\}A^T t$
  end if
end if
$M \leftarrow (k(C - de_n))(\text{mod } q)$
msg $\leftarrow$ convert(M) //* assign corresponding character to the numeric value

3.2. Analysis of time complexity of algorithm

In the above mentioned encryption scheme sender transmits numbers (p, r, r', d, q) and $P(p^r)$ (in either case) as a private key. To get the original message intended receiver has to use the transformation

$$M \equiv \begin{cases} \{(I_{\frac{n}{2}} + J_{\frac{n}{2}}) \otimes I_2\}A^T(C - de_n)t(\text{mod } q) & \text{if } p^r \equiv 1(\text{mod } 4) \\ (I_n + J_n)A^T(C - de_n)t(\text{mod } q) & \text{otherwise} \end{cases}$$

It means intended receiver has to find out A^T and t only. The matrix A^T and integer t may be obtained by using Algorithm 2.1 and Euclidean algorithm respectively. The time complexity to find A^T and integer t is $O(n)$ and $O(\log n)$ respectively. We see that the time complexity of the decryption algorithm depends on matrix multiplications of the matrices $(I_n + J_n), A^T$ and the column vector $C - de_n$ which is $O(n^3)$.

4. Security analysis of the method

In cryptography, the main aim is to protect the information about the key, plain text and cypher text from the intruder. But intruder always tries to attack a cipher or cryptographic system so that they can get a lead to break it fully or only partially [3, 15].

The types of main attacks are follows:

- Brute force attack
- Known Plain text attack
- Ciphertext-only attack

4.1. Cryptanalysis of Brute force attack

In Brute force attack, the intruder tries all possible character combination to find the keys and checks which one of them returns the plain-text.

In order to break above discussed encryption scheme using brute force attack, intruder has to find the key matrix A and t. It is difficult to find A and t unless n, q and $P(p^r)$ in either case are known. Since the set of numbers $(p,\ r,\ r',\ d,\ q)$ and $P(p^r)$ is a private key comprising a large prime p and primality testing is NP. However there is an algorithm of primality testing for a given integer is in P [1]. In either case to obtain the suitable primitive polynomial of $GF(p^r)$ increases the difficulty level.

It is difficult to guess the size of A exactly. For known order of the key matrix A consisting of entries $(-1,\ 1)$ only the size of the key space, $K(A)$, is $|K(A)| = 2^{n^2}$. Thus the computational complexity to find A is $O(2^{n^2})$.

Its complexity increases exponentially. Thus it seems that the above discussed encryption scheme is robust against the Brute force attack.

4.2. Cryptanalysis of known Plaintext attack

The known plaintext attack is the one where intruder has an access to the quantity of plaintext as well as its corresponding cipher text. In this type of attack the main goal is to guess the private key or to develop an algorithm so that they can decrypt any further message. In the above discussed encryption scheme, we have

$$C \equiv (AM + de_n)(\text{mod } q).$$

So, basically to find the encryption scheme they have to solve the large n-dimensional non-homogeneous system of linear equation which is very difficult.

Proposition 4.1 ([8]). *All encryption scheme using Hadamard matrices (conference matrix) with circulant cores are secure against known-plain text attacks under the assumption that the adversary has knowledge of less than n messages of length n of the plain text and the corresponding cipher text.*

4.3. Cryptanalysis of ciphertext-only attacks

The ciphertext-only attack is the one where intruder has access to the number of encrypted message. They have no idea about exact plain text and private key. In this type of attack the main goal is to deduce the private key or plain text. Mainly they focus on finding the private key so that they can use that to decrypt the further encrypted message.

So, to design the encryption algorithm it is particularly important to protect them against the cipher text only attack. As we can say this attack is the starting point of cryptanalyst.

When we use conference matrix in encryption scheme two same letters of the plain text M corresponds to different values of the encrypted text C. So, an attacker cannot observe the plain text or any information regarding the private key after seeing the encrypted message.

Proposition 4.2 ([8]). *All encryption scheme using Hadamard matrices are secure against ciphertext-only attack.*

5. Example

Consider a message *HELLO* which has to be transmit using the encryption scheme discussed above.
Message in *ASCII* code is

$$M = \begin{bmatrix} 72 & 69 & 76 & 76 & 79 \end{bmatrix}^T.$$

Suppose $p = 19 \equiv 3(mod\ 4),\ r = 1,\ r' = 0$ so size of the conference matrix is $n = 19$.
Since M contains 5 letters and $n = 19$ so to make it equal "space" is added 14 times in M. The ASCII code of "space" is 32. Thus

$$M = \begin{bmatrix} 72 & 69 & 76 & 76 & 79 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 \end{bmatrix}^T.$$

So for modular base we can take $q = 81$ as $gcd(20,\ 81) = 1$ So, $n = 19$, and suppose $d = 2$. Thus encrypted message $C = (AM + de_n)(\text{mod } 81)$.
where the first row of the circulant conference matrix A of order 19 obtained by using construction defined in subsection (2.1) is given by

$$A = \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

Therefore

$$C = \begin{bmatrix} 70 & 65 & 78 & 77 & 78 & 79 & 4 & 78 & 64 & 58 & 71 & 65 & 3 & 64 & 77 & 71 & 4 & 11 & 3 \end{bmatrix}^T.$$

Generally $0-31$ and 127 are not printable and it is indicated with "NA". But here for our convenience we use

$$0 \to 0*$$
$$1 \to 1*$$
$$2 \to 2*$$

and so on. In this case intended receiver has to understand that when $n*$ is included in encrypted message its numeric value will be n, where n is non printable character. After converting the ASCII code of encrypted message into its corresponding printable character
C= {F A N M N O 4* N @ : G A 3* @ M G 4* 11* 3* }.
Sender need to send the private key $(19, 1, 0, 2, 81)$ along with the encrypted message C. Now intended receiver get plain text using transformation

$$M = (I_n + J_n)A^T(C - de_n)t(\text{mod } 81)$$

Matrices I_n and J_n are well known. A^T is obtained by ***Algorithm* 2.1** and using Euclidean algorithm t may be obtained. Thus receiver decrypt the message and get the column vector

$$M = \begin{bmatrix} 72 & 69 & 76 & 76 & 79 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 & 32 \end{bmatrix}^T.$$

6. Conclusion

In this article we have developed an encryption scheme using conference matrix. The sender shares only the numbers $(p,\ r,\ r',\ d,\ q)$ and $P(p^r)$ (in either case) as a private key to the intended receiver. Private key comprising of limited numbers makes easy transmission. The theoretical development of decryption key makes easy to decrypt the message for intended receiver. However for intruder it is very difficult to guess the private key as finding prime p is an NP problem. Also obtaining the suitable primitive polynomial increases the difficulty level in case of involvement of prime power. It has been observed that the encryption scheme is robust against the cipher attack.

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ON A CERTAIN CLASS OF ANALYTIC UNIVALENT FUNCTIONS WITH POSITIVE COEFFICIENTS DEFINED BY q-CALCULUS

By

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Abstract

The main object of this paper is to introduce a new subclass of analytic univalent functions by using q-calculus. We obtain results regarding coefficient estimates extreme points, distortion bounds, convolution condition and convex combination for this class. Finally, we discuss a class preserving integral operator for this class. Relevant connections of the results presented herewith various well-known results are briefly indicated.

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1. Introduction

Let $\boldsymbol{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. As usual, we denote by S the subclass of $\boldsymbol{A}$ consisting of functions $f(z)$ of the form (1.1) which are univalent in U.

A function $f \in S$ is said to be starlike of order α, $0 \le \alpha < 1$, if it satisfies the following analytic criteria

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha.$$

Similarly, a function $f \in S$ is said to be convex of order α, $0 \le \alpha < 1$, if it satisfies the condition

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad z \in U.$$

The classes of all starlike and convex functions of order α are denoted by $S^*(\alpha)$ and $K(\alpha)$, respectively, introduced and studied by Robertson [14]. These classes with negative coefficients extensively studied by Silverman [16].

In 1994, Uralegaddi et al. [17] introduced the analogues classes of starlike and convex functions of order β with positive coefficients and opened up a new and interesting direction of research in geometric function theory. They introduced the classes $M(\beta)$, $L(\beta)$ and $R(\beta)$ in the following way.

A function $f(z)$ of the form (1.1) is said to be in the class $M(\beta)$, if it satisfy the following condition

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} < \beta, \quad z \in U,$$

where $1 < \beta \le 4/3$.

A function $f(z)$ of the form (1.1) is said to be in the class $L(\beta)$, if it satisfy the condition

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \beta, \quad z \in U,$$

where $1 < \beta \le 3/2$.

Similarly, a function $f(z)$ of the form (1.1) is said to be in the class $R(\beta)$ if it satisfy the condition

$$\Re\{f'(z)\} < \beta, \quad z \in U,$$

where $1 < \beta \le 2$.

Let S_j denote the class of functions of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k, \tag{1.2}$$

where $j \in N = \{1, 2, 3, ...\}$ and $z \in U$, which are analytic and univalent in the open unit disk U. It is interesting to note that for $j = 1$, the class S_j reduces to the class S of analytic univalent functions.

Further, we let V_j be the subclass S_j consisting of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} |a_k| z^k. \tag{1.3}$$

Next, we let the operator $D^n (n \in N_0 = NU\{0\})$ be defined for a function $f \in S_k$ by

$$D^0 f(z) = f(z)$$

$$D^1 f(z) = zf'(z)$$

$$\dots \dots$$

$$D^n f(z) = D(D^{n-1} f(z)).$$

The operator D^n is known as Salagean operator introduced by Salagean [15] in 1983.

Recently, we have come to know that the concept of q-calculus is widely used in geometric function theory. The concept of q-calculus were initially introduced by Aral et al. [3]. They defined the q-number for $k \in N$ in the following way

$$[k]_q = \frac{1-q^k}{1-q}, \quad 0 \leq q < 1.$$

It is worthy to note that $[k]_q$ can be represented as a geometric series as follows

$$[k]_q = \sum_{i=0}^{k-1} q^i. \tag{1.4}$$

From the definition of $[k]_q$ it is obvious that

$$\lim_{k\to\infty} [k]_q = \frac{1}{1-q} \text{ and } \lim_{q\to 1} [k]_q = k.$$

The q-derivative for a function f is defined as

$$D_q(f(z)) = \frac{f(qz) - f(z)}{(q-1)z}, \quad q \neq 1,\ z \neq 0$$

and $D_q(f(0)) = f'(0)$.

If we take the function $h(z) = z^k$ then the q-derivative of $h(z)$ is defined as

$$\begin{aligned} D_q(h(z)) &= D_q(z^k) \\ &= \frac{(1-q^k)}{1-q} z^{k-1} \\ &= [k]_q z^{k-1}. \end{aligned}$$

Then

$$\begin{aligned} \lim_{q\to 1} D_q(h(z)) &= \lim_{q\to 1} [k]_q z^{k-1} \\ &= kz^{k-1} \\ &= h'(z), \end{aligned}$$

where h' is the ordinary derivative.

Now, for $m \in N,\ n \in N_0,\ m > n,\ 1 < \beta \leq 4/3,\ j \in N, 0 \leq t \leq 1, 0 \leq q < 1$, we define the subclass $S_j(m, n, q, t, \beta)$ consisting of functions $f(z)$ of the form (1.2) satisfying the condition

$$\Re \left\{ \frac{z(D_q(D^m f(z)))}{D^n f_t(z)} \right\} < \beta, \tag{1.5}$$

where $f_t(z) = (1-t)z + tf(z)$.

Further, we define

$$V_j(m, n, q, t, \beta) \equiv S_j(m, n, q, t, \beta) \cap V_j.$$

By specializing the parameters in subclasses $S_j(m, n, q, t, \beta)$ and $V_j(m, n, q, t, \beta)$ studied earlier by various researchers.

1. $S_j(n+1, n, 0, 1, \beta) \equiv S_j(n, \beta)$ and $V_j(n+1, n, 0, 1, \beta) \equiv V_j(n, \beta)$ studied by Dixit and Chandra [4].
2. $S_j(n+p, n, 0, 1, \beta) \equiv S_j(n, p, \beta)$ and $V_j(n+1, n, 0, 1, \beta) \equiv V_j(n, p, \beta)$ studied by Dixit et al. [7].
3. $S_1(1, 0, 0, 1, \beta) \equiv M(\beta)$ and $V_j(1, 0, 0, 1, \beta) \equiv V(\beta)$ studied by Uralegaddi et al. [17].
4. $V_1(2, 1, 0, 1, \beta) \equiv L(\beta)$ and $V_j(2, 1, 0, 1, \beta) \equiv U(\beta)$ studied by Uralegaddi et al. [17].
5. $V_1(1, 0, 0, 0, \beta) \equiv R(\beta)$ studied by Uralegaddi et al. [18].

Motivating with the above mentioned work and by work of Atshan and Abid Zaid [1], Dixit and Pathak [5, 6], Dixit et al. [8], El-Ashwah et al. [9], Kanas and Srivastava [10], Porwal and Dixit [12] and Porwal et al. [13], we obtain coefficient estimates, distortion bounds, covering results, extreme points, convolution condition, convex combination. Finally, we discuss an integral operator and q-Jackson type integral operator for this class.

Our results generalized the results of Dixit and Chandra [4], Dixit et al. [7] and Uralegaddi et al. [17].

2. Main Results

In our first theorem we give a necessary and sufficient condition for the class $V_j(m,n,q,t,\beta)$.

Theorem 2.1. *The function $f \in V_j(m,n,q,t,\beta)$, if and only if*

$$\sum_{k=j+1}^{\infty}(k^m[k]_q - \beta t k^n)|a_k| \le \beta - 1, \tag{2.1}$$

where $m \in N,\ n \in N_0,\ 0 \le q < 1,\ 0 \le t \le 1,\ 1 < \beta \le 4/3,\ j \in N.$

The result is sharp.

Proof. First we suppose that the inequality (2.1) holds. To prove $f \in V_j(m,n,q,t,\beta)$ it suffices to show that

$$\left|\frac{\frac{zD_q(D^m f(z))}{D^n f_t(z)} - 1}{\frac{zD_q(D^m f(z))}{D^n f_t(z)} - (2\beta - 1)}\right| < 1, \quad z \in U.$$

Now, we have

$$\left|\frac{\frac{zD_q(D^m f(z))}{D^n f_t(z)} - 1}{\frac{zD_q(D^m f(z))}{D^n f_t(z)} - (2\beta - 1)}\right|$$

$$= \left|\frac{\frac{z+\sum_{k=j+1}^{\infty}[k]_q k^m |a_k| z^k}{z+\sum_{k=j+1}^{\infty}k^n t|a_k|z^k} - 1}{\frac{z+\sum_{k=j+1}^{\infty}[k]_q k^m |a_k| z^k}{z+\sum_{k=j+1}^{\infty}k^n t|a_k|z^k} - (2\beta - 1)}\right|$$

$$= \left|\frac{\sum_{k=j+1}^{\infty}([k]_q k^m - k^n t)|a_k|z^k}{\sum_{k=j+1}^{\infty}([k]_q k^m - (2\beta-1)k^n t)|a_k|z^k - 2(\beta-1)z}\right|$$

$$\le \frac{\sum_{k=j+1}^{\infty}([k]_q k^m - k^n t)|a_k||z|^k}{2(\beta-1)|z| - \sum_{k=j+1}^{\infty}([k]_q k^m - (2\beta-1)k^n t)|a_k|\ |z|^k}$$

$$< \frac{\sum_{k=j+1}^{\infty}([k]_q k^m - k^n t)|a_k|}{2(\beta-1) - \sum_{k=j+1}^{\infty}([k]_q k^m - (2\beta-1)k^n t)|a_k|}.$$

The last expression is bounded above by 1, if

$$\sum_{k=j+1}^{\infty}([k]_q k^m - k^n t)|a_k| \le 2(\beta-1) - \sum_{k=j+1}^{\infty}([k]_q k^m - (2\beta-1)k^n t)|a_k|,$$
$$\Rightarrow \sum_{k=j+1}^{\infty}([k]_q k^m - \beta k^n t)|a_k| \le \beta - 1,$$

which is true by hypothesis.

Thus, we have $f \in V_j(m,n,q,t,\beta)$.

To prove the converse part, we assume that $f(z)$ is defined by (1.3) and in the class $V_j(m,n,q,t,\beta)$ so that condition (1.5) reduces as

$$\Re\left\{\frac{z\{D_q(D^m f(z))\}}{D^n f_t(z)}\right\} < \beta,$$

$$\Re\left\{\frac{z+\sum_{k=j+1}^{\infty}[k]_q k^m |a_k| z^k}{z+\sum_{k=j+1}^{\infty}k^n t|a_k|z^k}\right\} < \beta.$$

The above condition must hold for all values of $z; |z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \le z = r < 1$, we have

$$\frac{r+\sum_{k=j+1}^{\infty}[k_q]k^m|a_k|r^k}{r+\sum_{k=j+1}^{\infty}k^n t|a_k|r^k} < \beta$$

or

$$\frac{(\beta-1) - \sum_{k=j+1}^{\infty}([k]_q k^m - \beta k^n t)|a_k|r^{k-1}}{1+\sum_{k=j+1}^{\infty}k^n t|a_k|r^{k-1}} > 0. \tag{2.2}$$

If the condition (2.1) does that not hold then the numerator in (2.2) is negative for r sufficiently close to 1. Thus there exists a $z_0 = r_0$ in (0,1) for which the quotient in (2.2) is negative. This contradicts the required condition for $f \in V_j(m,n,q,t,\beta)$ and so the proof is complete. □

Corollary 2.1. *Let the function $f(z)$ defined by* (1.3) *belong to the class $V_j(m,n,q,t,\beta)$. Then*

$$|a_k| \le \frac{\beta-1}{[k]_q k^m - \beta t k^n}, \quad (k \ge j+1).$$

The following results are some easy consequences of definition of class $V_j(m,n,q,t,\beta)$ and Theorem 2.1. Therefore, we only state the results.

Theorem 2.2. *Let $1 < \beta_1 \le \beta_2 \le 4/3$. Then $V_j(m,n,q,t,\beta_1) \subseteq V_j(m,n,q,t,\beta_2)$.*

Theorem 2.3. *Let $m_1 \le m_2$. Then $V_j(m_1,n,q,t,\beta) \supseteq V_j(m_2,n,q,t,\beta)$.*

Theorem 2.4. *Let $j_1 \le j_2$. Then $V_{j_1}(m,n,q,t,\beta) \subseteq V_{j_2}(m,n,q,t,\beta)$.*

Theorem 2.5. *Let $q_1 \le q_2$. Then $V_j(m,n,q_1,t,\beta) \supseteq V_j(m,n,q_2,t,\beta)$.*

Next, we determine the extreme points of closed convex hulls of $V_j(m,n,q,t,\beta)$ denoted by clco $V_j(m,n,q,t,\beta)$.

Theorem 2.6. *Let $f_j(z) = z$ and $f_k(z) = z + \frac{\beta-1}{[k]_q k^m - \beta t k^n} z^k$, $(k = j+1,\ j+2,...)$. Then $f \in V_j(m,n,q,t,\beta)$, if and only if, it can be expressed in the form*

$$f(z) = \textstyle\sum_{k=j}^{\infty} \lambda_k f_k(z), \tag{2.3}$$

where $\lambda_k \ge 0$ and $\sum_{k=j+1}^{\infty} \lambda_k = 1$.

In particular the extreme points of $V_j(m,n,q,t,\beta)$ is $\{f_k\}$.

Proof. Let $f(z) = \sum_{k=j}^{\infty} \lambda_k f_k(z)$

$$f(z) = \lambda_j z + \textstyle\sum_{k=j+1}^{\infty} \lambda_k \left(z + \dfrac{\beta-1}{[k]_q k^m - \beta t k^n} z^k \right)$$

$$f(z) = z + \textstyle\sum_{k=j+1}^{\infty} \lambda_k \dfrac{(\beta-1)}{[k]_q k^m \beta t k^n} z^k.$$

Now

$$\textstyle\sum_{k=j+1}^{\infty} \left\{ [k]_q k^m - \beta t k^n \right\} \dfrac{(\beta-1)\lambda_k}{[k]_q k^m - \beta t k^n}$$
$$= (\beta-1) \textstyle\sum_{k=j+1}^{\infty} \lambda_k$$
$$= (\beta-1)(1-\lambda_j)$$
$$\le \beta - 1.$$

Therefore, by Theorem 2.1 we conclude that $f \in V_j(m,n,q,t,\beta)$.

Conversely, suppose that the function class $V_j(m,n,q,t,\beta)$, then

$$|a_k| \le \frac{\beta-1}{[k]_q k^m - \beta t k^n}, \quad (k = j+1,\ j+2,...).$$

Setting

$$\lambda_k = \frac{[k]_q k^m - \beta t k^n}{\beta - 1} |a_k|, \quad (k = j+1,\ j+2,...)$$

and

$$\lambda_j = 1 - \textstyle\sum_{k=j+1}^{\infty} \lambda_j.$$

Thus, the proof of Theorem 2.6 is complete. □

In our next result, we obtain the bounds for $f \in V_j(m,n,q,t,\beta)$.

Theorem 2.7. *If $f \in V_j(m,n,q,t,\beta)$, then*

$$r - \frac{(\beta-1)\, r^{j+1}}{[j+1]_q (j+1)^m - \beta t (j+1)^n} \le |f(z)| \le r + \frac{(\beta-1)\, r^{j+1}}{[j+1]_q (j+1)^m - \beta t (j+1)^n}.$$

Furthermore

$$r - \frac{(\beta-1)\, r^{j+1}}{[j+1]_q (j+1)^{m-n} - \beta t} \le |D^n f(z)| \le r + \frac{(\beta-1)\, r^{j+1}}{[j+1]_q (j+1)^{m-n} - \beta t}.$$

Proof. Since

$$f(z) = z + \sum_{k=j+1}^{\infty}|a_k|z^k$$

$$|f(z)| \leq |z| + \sum_{k=j+1}^{\infty}|a_k|\,|z|^k$$

$$\leq r + r^{j+1}\sum_{k=j+1}^{\infty}|a_k|$$

$$= r + r^{j+1}\sum_{k=j+1}^{\infty}\frac{1}{[k]_1k^m - \beta tk^n}([k]_qk^m - \beta tk^n)|a_k|$$

$$= r + r^{j+1}\frac{1}{[j+1]_q(j+1)^m - \beta t(j+1)^n}\sum_{k=j+1}^{\infty}([k]_qk^m - \beta tk^n)|a_k|$$

$$\leq r + \frac{(\beta-1)r^{j+1}}{[j+1]_q(j+1)^m - \beta t(j+1)^m}.$$

Similarly, we prove that

$$|f(z)| \geq r - \frac{(\beta-1)\ r^{j+1}}{[j+1]_q(j+1)^m - \beta t(j+1)^n}.$$

Thus

$$|D^nf(z)| \leq r + \sum_{k=j+1}^{\infty}k^n|a_k|r^k$$

$$\leq r + r^{j+1}\sum_{k=j+1}^{\infty}\frac{1}{[k]_qk^{m-n} - \beta t}\left([k]_qk^m - \beta tk^n\right)|a_k|$$

$$\leq r + r^{j+1}\frac{1}{[j+1]_q(j+1)^{m-n} - \beta t}.$$

Similarly,

$$|D^nf(z)| \geq r - \frac{(\beta-1)r^{j+1}}{[j+1]_q(j+1)^{m-n} - \beta t}.$$

The following covering result yields from left hand inequality of Theorem 2.7. □

Theorem 2.8. *The disk* $|z| < 1$ *is mapped onto a domain that contain the disk*

$$|w| < \frac{[j+1]_q(j+1)^m - \beta t(j+1)^n - (\beta-1)}{[j+1]_q(j+1)^m - \beta t(j+1)^n},$$

by any $f \in V_j(m,n,q,t,\beta)$*. The result is sharp for the extremal function*

$$f(z) = z + \frac{(\beta-1)z^{j+1}}{[j+1]_q(j+1)^m - \beta t(j+1)^n}.$$

Proof. Making $r \to 1$ in left hand inequality of Theorem 2.7, we obtain the required result. □

3. Convolution and Convex Combination

The convolution or (Hadamard product) of two function $f(z)$ be defined by (1.3) and let the function $g(z)$ be defined by

$$g(z) = z + \sum_{k=j+1}^{\infty}|b_k|z^k, \tag{3.1}$$

is defined as $(f * g)(z) = z + \sum_{k=j+1}^{\infty}|a_k|\,|b_k|z^k$.

Theorem 3.1. *Let the function* $f(z)$ *be defined by* (1.3) *and* $g(z)$ *be defined by* (3.1) *are in the classes* $V_j(m_1,n_1,q,t,\beta_1)$ *and* $V_j(m_2,n_2,q,t,\beta_2)$ *respectively. Then the Hadamard product* $(f*g)(z)$ *belongs to the classes* $V_j(m_1+n_2,n_1+n_2,q,t,\beta_1)$*, where* $1 < \beta_1 \leq \beta_2 \leq 4/3$.

Proof. Since $f(z) \in V_j(m_1, n_1, q, t, \beta_1)$, then by Theorem 2.1, we have
$$\sum_{k=j+1}^{\infty} \frac{[k]_q k^{m_1} - \beta_1 t k^{n_1}}{\beta_1 - 1} |a_k| \le 1. \tag{3.2}$$
Let $g(z) \in V_j(m_2, n_2, q, t, \beta_2)$, we have
$$\sum_{k=j+1}^{\infty} \frac{[k]_q k^{m_2} - \beta_2 t k^{n_2}}{\beta_2 - 1} |b_k| \le 1.$$
$$\Rightarrow \sum_{k=j+1}^{\infty} k^{n_2} \left[\frac{[k]_q k^{m_2 - n_2} - \beta_2 t}{\beta_2 - 1} \right] |b_k| \le 1.$$
$$\Rightarrow k^{n_2} |b_k| \le 1 \quad (k = j+1,\ j+2, ...).$$
To prove $(f * g)(z) \in V_j(m_1 + m_2, n_1 + n_2, q, t, \beta_1)$ it is sufficient to prove that
$$\sum_{k=j+1}^{\infty} \frac{[k]_q k^{m_1+n_2} - \beta_1 t k^{n_1+n_2}}{\beta_1 - 1} |a_k|\, |b_k| \le 1.$$
Now
$$\sum_{k=j+1}^{\infty} \frac{[k]_q k^{m_1+n_2} - \beta_1 t k^{n_1+n_2}}{\beta_1 - 1} |a_k|\, |b_k| \le \sum_{k=j+1}^{\infty} \frac{[k]_q k^{m_1} - \beta_1 t k^{n_1}}{\beta_1 - 1} |a_k|$$
$$\le 1, \qquad \text{(by (3.2))}.$$
Therefore, $(f * g)(z) \in V_j(m_1 + n_2, n_1 + n_2, q, t, \beta)$.

In our next theorem, we show that the class $V_j(m, n, q, t, \beta)$ is closed under convex combination. □

Theorem 3.2. *Let the function $f_i(z)$ be defined by*
$$f_i(z) = z + \sum_{k=j+1}^{\infty} |a_{k,i}| z^k, \quad (i = 1, 2, ...)$$
be in the class $V_j(m, n, q, t, \beta)$. Then the function $\sum_{i=1}^{\infty} t_i f_i(z)$ is in the class $V_j(m, n, q, t, \beta)$, where $\sum_{i=1}^{\infty} t_i = 1$.

Proof. For $i = 1, 2, 3...$ let $f_i(z) \in V_j(m, n, q, t, \beta)$, wehre $f_i(z)$ is of the form
$$f_i(z) = z + \sum_{k=j+1}^{\infty} |a_{k,i}| z^k, \quad (i = 1, 2, ...).$$
Then from Theorem 2.1, we have
$$\sum_{k=j+1}^{\infty} ([k]_q k^m - \beta t k^n) |a_{k,i}| \le \beta - 1.$$
For $\sum_{i=1}^{\infty} t_i = 1,\ 0 \le t_i \le 1$, the convex combination of f_i may be written as
$$\sum_{i=1}^{\infty} t_i f_i(z) = z + \sum_{k=j+1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{k,i}| \right) z^k.$$
Then by Theorem 2.1, we have
$$\sum_{k=j+1}^{\infty} \left([k]_q k^m - \beta t k^n \right) \left(\sum_{i=1}^{\infty} t_i |a_{k,i}| \right)$$
$$= \sum_{i=1}^{\infty} t_i \left\{ \sum_{k=j+1}^{\infty} \left\{ [k]_q k^m - \beta t k^n \right\} \right\} |a_{k,i}|$$
$$\le \sum_{i=1}^{\infty} t_i (\beta - 1) = \beta - 1.$$
Then by Theorem 2.1, we have $\sum_{i=1}^{\infty} t_i f_i(z) \in V_j(m, n, q, t, \beta)$. □

Theorem 3.3. *Let the function $f_1(z), f_2(z), ... f_m(z)$ defined by*
$$f_i(z) = z + \sum_{k=j+1}^{\infty} |a_{k,i}| z^k, \quad (i = 1, 2, ..., m)$$
be in the class $V_j(m, n, q, t, \beta)$. Then the function $h(z) = \frac{1}{m} \sum_{i=1}^{m} f_i(z)$ is also in the class $V_j(m, n, q, t, \beta)$.

Proof. By the definition of $h(z)$, we have
$$h(z) = z + \sum_{i=1}^{m} \left[\frac{1}{m} \sum_{k=j+1}^{\infty} |a_{k,i}| \right] z^k.$$
Since $f_i(z) \in V_j(m, n, q, t, \beta)$, therefore
$$\sum_{k=j+1}^{\infty} [k]_q k^m - \beta t k^n |a_{k,i}| \le \beta - 1.$$
Now
$$\sum_{k=j+1}^{\infty} ([k]_q k^m - \beta t k^n) \left(\sum_{i=1}^{m} \frac{1}{m} |a_{k,i}| \right)$$
$$= \sum_{i=1}^{m} \frac{1}{m} \left(\sum_{k=j+1}^{\infty} \left([k]_q k^m - \beta t k^m \right) \right) |a_{k,i}|$$
$$\le \sum_{i=1}^{m} \frac{1}{m} (\beta - 1)$$
$$= \beta - 1.$$
This completes the proof of theorem. □

4. A Family of Class Preserving Integral Operator

Definition 4.1. *Let $f(z)$ be defined by the relation* (1.1) *then we define the integral operator $F(z)$ defined by the relation*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt, \quad (c > -1). \tag{4.1}$$

Theorem 4.1. *Let $f(z)$ be defined by* (1.3) *and $f(z) \in V_j(m,n,q,t,\beta)$. Then $F(z)$ defined by the relation* (4.1) *is also in the class $V_j(m,n,q,t,\beta)$.*

Proof. From the representation of $F(z)$ given by (1.1), we may express $F(z)$ as follows

$$F(z) = z + \sum_{k=j+1}^{\infty} \frac{c+1}{c+k}|a_k|z^k. \tag{4.2}$$

Since it is given that

$$f(z) \in V_j(m,n,q,t,\beta).$$

From Theorem 2.1, we have

$$\sum_{k=j+1}^{\infty}([k]_q k^m - \beta t k^n)|a_k| \leq \beta - 1. \tag{4.3}$$

Now

$$\begin{aligned}&\sum_{k=j+1}^{\infty}([k]_q k^m - \beta t k^n)\frac{c+1}{c+k}|a_k| \\ &\leq \sum_{k=j+1}^{\infty}([k]_q k^m - \beta t k^n)|a_k| \\ &\leq \beta - 1, \quad by (4.3).\end{aligned}$$

Thus

$$F(z) \in V_j(m,n,q,t,\beta).$$

□

Definition 4.2. *Let $f = h + \overline{g}$ be defined by* (1.1). *Then the q-Jackson integral operator $F_q : A \to A$ is defined by the relation*

$$F_q(z) = \frac{[c]_q}{z^{c+1}}\int_0^z t^c h(t) d_q t, \tag{4.4}$$

where $[c]_q$ is the q-number defined by (1.4) *and $\int_0^z f(t)d_q t$ is defined as*

$$\int_0^z f(t)d_q t = (1-q)z\sum_{i=0}^{\infty} f(zq^i)q^i, \quad z \in C.$$

For detailed study one may refer to [3].

Theorem 4.2. *Let $f(z)$ be defined by* (1.3) *and $f(z) \in V_j(m,n,q,t,\beta)$. Then $F_q(z)$ defined by* (4.4) *is in the class $V_j(m,n,q,t,\beta)$.*

Proof. From the representation of (4.4), we have

$$F_q(z) = z + \sum_{k=j+1}^{\infty}\frac{[c]_q}{[k+c+1]_q}|a_k|z^k. \tag{4.5}$$

Since

$$\begin{aligned}&[k+c+1]_q - [c]_q \\ &= \sum_{i=0}^{k+c} q^i - \sum_{i=0}^{c-1} q^i \\ &= \sum_{i=0}^{k+c} q^i > 0 \quad [\because q > 0] \\ &\Rightarrow [k+c+1]_q > [c]_q\end{aligned}$$

Now

$$\begin{aligned}&\sum_{k=j+1}^{\infty}([k]_q k^m - \beta t k^n)\frac{[c]_q}{[k+c+1]_q}|a_k| \\ &\leq \sum_{k=j+1}^{\infty}([k]_q k^m - \beta t k^n)|a_k| \\ &\leq \beta - 1, \quad (\because f(z) \in V_j(m,n,q,t,\beta)).\end{aligned}$$

Thus, $F_q(z) \in V_j(m,n,q,t,\beta)$. □

Remark 4.1.

1. If we put $m = n+1,\ q = 0,\ t = 1$ then we obtain the corresponding results of Dixit and Chandra [4].

2 If we put $m = n+p,\ q = 0,\ t = 1$ then we obtain the corresponding results of Dixit et al. [7].

3 If we put $m = 1,\ n = 0,\ q = 0,\ t = 1,\ j = 1$ then we obtain the corresponding results of Uralegaddi et al. [17].

4 If we put $m = 2,\ n = 1,\ q = 0,\ t = 1,\ j = 1$ then we obtain the corresponding results of Uralegaddi et al. [17].

5 If we put $m = 1,\ n = 0,\ q = 0,\ t = 0,\ j = 1$ then we obtain the corresponding results of Uralegaddi et al. [18].

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COMMON FIXED POINT THEOREM FOR COMPATIBLE MAPPINGS OF TYPE (A-1) IN INTUITIONISTIC MENGER SPACE

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Abstract

The present paper aims to prove some new common fixed point theorems in intutionistic Menger spaces. In this paper some common fixed point results for two pairs of compatible mappings of type (A-1) satisfying contractive condition on intuitionistic menger space are also established. The concept of compatible mappings of type (A-1) was given by Khan et al. [6]. Our results substantially generalize and improve a multitude of relevant common fixed point theorems of the existing literature in metric as well as intutionistic menger spaces.

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Keywords and Phrases: Common fixed point, Intuitionistic menger space, Compatible mappings, Compatible mappings of type (A-1).

1. Introduction

In 1922, Banach proved the mile stone in the fixed point theory and its applications. Several authors addressed a new class of fixed point problems in metric spaces. They proved fixed point theorem for mappings satisfying certain inequalities involving the altering distances function. There have been a number of generalizations of metric spaces. One such generalization is Menger space introduced in 1942 by Menger [7] who used distribution functions instead of nonnegative real numbers as values of the metric. This space was expanded rapidly with the pioneering works of Schweizer and Sklar [13,14]. Modifying the idea of Kramosil and Michalek [4], George and Veeramani [1] introduced fuzzy metric spaces which are very similar to that of Menger space. Park [10] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms.

Kutukcu et al. [5] introduced the notion of intuitionistic Menger Spaces with the help of t-norms and t-conorms as a generalization of Menger space due to Menger [7]. Further they introduced the notion of Cauchy sequences and found a necessary and sufficient condition for an intuitionistic Menger Space to be complete. Sessa [15] initiated the tradition of improving commutativity in fixed point theorems by introducing the notion of weakly commuting maps in metric spaces. Jungck [3] soon enlarged this concept to compatible maps. The notion of compatible mapping in a Menger space has been introduced by Mishra [8].

The concept of type A-compatible and S-compatible was given by Pathak and Khan [9]. Khan et al. [6] renamed A-compatible and A-compatible as compatible mappings of type (A-1) and compatible mappings of type (A-2) respectively.

Singh et al. [16,17] proved fixed point theorems in fuzzy metric space and menger space using the concept of semicompatibility, weak compatibility and compatibility of type (β) respectively.

Gupta et al. [2] obtained some coupled fixed point results on modified intuitionistic fuzzy metric spaces and application to integral type contractions. Also Pant et al. [11] established new fixed point theorems in partial metric spaces with applications. Recently Shukla et al. [12] derived fixed point results for non linear contractions with application to integral equations. Very recently Wasfi et al. [18] established new results on modified intuitionistic generalized fuzzy metric spaces by employing $E.\ A.$ property and common $E.\ A.$ property for coupled maps.

These observations motivated us to prove a common fixed point theorem in intuitionistic Menger spaces. In this paper, we prove some new common fixed point theorems in intuitionistic Menger spaces. While proving our results, we utilize the idea of compatibility of type (A-1). Consequently, our results improve and develop many known common fixed point theorems available in the existing literature of intutionistic menger fixed point theory.

2. Preliminaries

Definition 2.1 ([13])**.** *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is a t-norm if* $*$ *satisfies the following conditions:*

1. $*$ *is commutative and associative,*
2. $*$ *is continuous,*
3. $a * 1 = a$*, for all* $a \in [0,1]$*,*
4. $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$*, for all* $a, b, c, d \in [0,1]$*.*

Definition 2.2 ([14]). *A binary operation* $\diamond : [0,1]\times[0,1] \to [0,1]$ *is a t-conorm if* $\diamond$ *satisfies the following conditions:*
1. $\diamond$ *is commutative and associative,*
2. $\diamond$ *is continuous,*
3. $a\diamond 0 = a$, *for all* $a \in [0,1]$,
4. $a\diamond b \leq c\diamond d$ *whenever* $a \leq c$ *and* $b \leq d$, *for all* $a, b, c, d \in [0,1]$.

Remark 2.1. The concept of triangular norms (t-norms) and triangular conforms (t-conorms) are known as the axiomatic skeletons that we use for characterizing fuzzy intersection and union respectively. These concepts were originally introduced by Menger [7] in his study of statistical metric spaces.

Definition 2.3 ([5]). *A distance distribution function is a function* $F : \mathbb{R} \to \mathbb{R}^+$ *which is non-decreasing, left continuous on* $\mathbb{R}$ *and* $\inf\{F(t) : t \in \mathbb{R}\} = 0$ *and* $\sup\{F(t) : t \in \mathbb{R}\} = 1$. *We will denote by D the family of all distance distribution functions while H will always denote the specific distribution function defined by*

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases}$$

If X is a non-empty set, $F : X \times X \to D$ *is called a probabilistic distance on X and* $F(x, y)$ *is usually denoted by the* $F_{x,y}$.

Definition 2.4 ([5]). *A non-distance distribution function is a function* $L : \mathbb{R} \to \mathbb{R}^+$ *which is non-increasing, right continuous on* $\mathbb{R}$ *and* $\inf\{L(t) : t \in \mathbb{R}\} = 1$ *and* $\sup\{L(t) : t \in \mathbb{R}\} = 0$. *We will denote by E the family of all non-distance distribution functions while G will always denote the specific distribution function defined by*

$$G(t) = \begin{cases} 1, & t \leq 0 \\ 0, & t > 0. \end{cases}$$

If X is a non-empty set, $L : X \times X \to E$ *is called a probabilistic non-distance on X and* $L(x, y)$ *is usually denoted by the* $L_{x,y}$.

Definition 2.5 ([5]). *A 5-tuple* $(X, F, L, *, \diamond)$ *is said to be an intuitionistic Menger space if X is an arbitrary set,* $*$ *is a continuous t-norm,* $\diamond$ *is continuous t-conorm, F is a probabilistic distance and L is a probabilistic non-distance on X satisfying the following conditions: for all* $x, y, z \in X$ *and* $t, s \geq 0$
1 $F_{x,y}(t) + L_{x,y}(t) \leq 1$,
2 $F_{x,y}(0) = 0$,
3 $F_{x,y}(t) = H(t)$ *if and only if* $x = y$,
4 $F_{x,y}(t) = F_{y,x}(t)$
5 *if* $F_{x,y}(t) = 1$ *and* $F_{y,z}(s) = 1$, *then* $F_{x,z}(t + s) = 1$,
6 $F_{x,z}(t + s) \geq F_{x,y}(t) * F_{y,z}(s)$,
7 $L_{x,y}(0) = 1$,
8 $L_{x,y}(t) = G(t)$ *if and only if* $x = y$,
9 $L_{x,y}(t) = L_{y,x}(t)$,
10 *if* $L_{x,y}(t) = 0$ *and* $L_{y,z}(s) = 0$, *then* $L_{x,z}(t + s) = 0$,
11 $L_{x,z}(t + s) \leq L_{x,y}(t)\diamond L_{y,z}(s)$.
The function $F_{x,y}(t)$ *and* $L_{x,y}(t)$ *denote the degree of nearness and degree of non-nearness between x and y with respect to t, respectively.*

Remark 2.2. Every Menger space $(X, F, *)$ is intuitionistic Menger space of the form $(X, F, 1-F, *, \diamond)$ such that t-norm $*$ and t-conorm $\diamond$ are associated, that is $x\diamond y = 1 - (1 - x) * (1 - y)$ for any $x, y \in X$.

Example 2.1. Let (X, d) be a metric space. Then the metric d induces a distance distribution function F defined by $F_{x,y}(t) = H(t - d(x, y))$ and a non-distance function L defined by $L_{x,y}(t) = G(td(x, y))$ for all $x, y \in X$ and $t \geq 0$. Then (X, F, L) is an intuitionistic probabilistic metric space. We call this intuitionistic probabilistic metric space induced by a metric d the induced intuitionistic probabilistic metric space. If t-norm $*$ is $a * b = \min\{a, b\}$ and t-conorm $\diamond$ is $a\diamond b = \min\{1, a + b\}$ for all $a, b \in [0,1]$ then $(X, F, L, *, \diamond)$ is an intuitionistic Menger space.

Remark 2.3. Note that the above example holds even with the t-norm $a * b = \min\{a, b\}$ and t-conorm $a\diamond b = \max\{a, b\}$ and hence $(X, F, L, *, \diamond)$ is an intuitionistic Menger space with respect to any t-norm and t-conorm. Also note t-norm $*$ and t-conorm $\diamond$ are not associated.

Definition 2.6 ([5]). *Let $(X, F, L, *, \diamond)$ be an intuitionistic Menger space with $t * t \geq t$ and $(1t)\diamond(1-t) \leq (1t)$. Then:*
1. *A sequence $\{x_n\}$ in X is said to be convergent to x in X if, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists positive integer N such that $F_{x_n,x}(\varepsilon) > 1-\lambda$ and $L_{x_n,x}(\varepsilon) < \lambda$ whenever $n \geq N$.*
2. *A sequence $\{x_n\}$ in X is called Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists positive interger N such that $F_{x_n,x_m}(\varepsilon) > 1-\lambda$ and $L_{x_n,x_m}(\varepsilon) < \lambda$ whenever $n, m \geq N$.*
3. *An intuitionistic Menger space $(X, F, L, *, \diamond)$ is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.*

The proof of the following Lemmas is on the lines of Mishra [8].

Lemma 2.1. *Let $(X, F, L, *, \diamond)$ be an intuitionistic Menger space with $t * t \geq t$ and $(1-t)\diamond(1-t) \leq (1-t)$ and $\{y_n\}$ be a sequence in X. If there exists a number $k \in (0,1)$ such that:*
1 $F_{yn+2,yn+1}(kt) \geq F_{yn+1,yn}(t)$,
2 $L_{yn+2,yn+1}(kt) \leq L_{yn+1,yn}(t)$ *for all $t > 0$ and $n = 1,2,3,4,...$ Then $\{y_n\}$ is a Cauchy sequence in X.*

Proof. By simple induction with the condition *(1)*, we have for all $t > 0$ and $n = 1,2,3,...$, $F_{yn+1,yn+2}(t) \geq F_{y1,y2}(t/k^n)$, $L_{yn+1,yn+2}(t) \leq L_{y1,y2}(t/k^n)$. Thus by Definition 2.5 (6) and (11), for any positive integer $m \geq n$ and number $t > 0$, we have

$$\begin{aligned} F_{yn,ym}(t) &\geq F_{yn,yn+1}\Big(\frac{t}{m-n}\Big) * F_{yn+1,yn+2}\Big(\frac{t}{m-n}\Big) * \cdots * F_{ym-1,ym}\Big(\frac{t}{m-n}\Big) \\ &\geq \overbrace{(1-\lambda) * (1-\lambda) * \cdots * (1-\lambda)} > (1-\lambda), \end{aligned}$$

and

$$\begin{aligned} L_{yn,ym}(t) &\leq L_{yn,yn+1}\Big(\frac{t}{m-n}\Big)\diamond L_{yn+1,yn+2}\Big(\frac{t}{m-n}\Big)\diamond \cdots \diamond L_{ym-1,ym}\Big(\frac{t}{m-n}\Big) \\ &\leq \overbrace{\lambda\diamond\lambda\diamond\cdots\diamond\lambda} < \lambda, \end{aligned}$$

which implies that $\{y_n\}$ is a Cauchy sequence in X. This completes the proof. □

Lemma 2.2. *Let $(X, F, L, *, \diamond)$ be an intuitionistic Menger space with $t * t \geq t$ and $(1-t)\diamond(1-t) \leq (1-t)$ and for all $x, y \in X$, $t > 0$ and if for a number $k \in (0,1)$*

$$F_{x,y}(kt) \geq F_{x,y}(t) \ \text{ and } \ L_{x,y}(kt) \leq L_{x,y}(t), \tag{2.1}$$

then $x = y$.

Proof. Since $t > 0$ and $k \in (0,1)$ we get $t > kt$. In intuitionistic Menger space $(X, F, L, *, \diamond)$, $F_{x,y}$ is non decreasing and $L_{x,y}$ is non-increasing for all $x, y \in X$, then we have

$$F_{x,y}(t) \geq F_{x,y}(kt) \ \text{ and } \ L_{x,y}(t) \geq L_{x,y}(kt).$$

Using (2.1) and the definition of intuitionistic Menger space, we have $x = y$. □

Definition 2.7 ([9]). *The self-maps A and B of an intuitionistic Menger space $(X, F, L, *, \diamond)$ are said to be compatible if for all $t > 0$,*

$$\lim_{n\to\infty} F_{ABx_n,BAx_n}(t) = 1 \ \text{ and } \ L_{ABx_n,BAx_n}(t) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 2.8 ([9]). *The self-maps A and B of an intuitionistic Menger space $(X, F, L, *, \diamond)$ are said to be compatible of type (A) if for all $t > 0$,*

$$\lim_{n\to\infty} F_{ABx_n,BBx_n}(t) = \lim_{n\to\infty} F_{BAx_n,AAx_n}(t) = 1 \ \text{ and } \ \lim_{n\to\infty} L_{ABx_n,BBx_n}(t) = \lim_{n\to\infty} L_{BAx_n,AAx_n}(t) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 2.9 ([6]). *The self-maps A and B of an intuitionistic Menger space $(X, F, L, *, \diamond)$ are said to be compatible of type (A-1) if for all $t > 0$,*

$$\lim_{n\to\infty} F_{BAx_n,AAx_n}(t) = 1 \ \text{ and } \ \lim_{n\to\infty} L_{BAx_n,AAx_n}(t) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$.

Proposition 2.1 ([6]). *Let S and T be self maps of an intuitionistic Menger space $(X, F, L, *, \diamond)$. If the pair (S, T) are compatible of type (A-1) and $Sz = Tz$ for some z in X then $STz = TTz$.*

Proposition 2.2 ([6]). *Let S and T be self maps of an intuitionistic Menger space $(X, F, L, *, \diamond)$ with $t * t > t$ and $(1-t)\diamond(1-t) \leq (1-t)$ for all t in $[0, 1]$. If the pair (S, T) are compatible of type (A-1) and $Sx_n, Tx_n \to z$ for some z in X and a sequence $\{x_n\}$ in X then $TTx_n \to Sz$ if S is continuous at z.*

Proposition 2.3 ([6]). *Let S and T be self maps of an intuitionistic Menger space $(X, F, L, *, \diamond)$. If the pair (S, T) are compatible of type (A-1) and $Sz = Tz$ for some z in X then $TSz = SSz$.*

3. Common Fixed Point Theorems

In this section, we establish common fixed point theorems for compatible mappings of type (A-1).

Theorem 3.1. *Let $(X, F, L, *, \diamond)$ be a complete intuitionistic Menger space with $t * t \geq t$ and $(1-t)\diamond(1-t) \leq (1-t)$. Let A, B, S and T be selfmappings of X such that the following conditions are satisfied :*
(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$,
(ii) *S and T are continuous,*
(iii) *There exists $k \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$,*

$$F_{Ax,By}(kt) \geq \Big\{F_{Sx,Ty}(t) * F_{Ax,Sx}(t) * F_{By,Ty}(t) * F_{Ax,Ty}(t)\Big\}. \tag{3.1}$$

and

$$L_{Ax,By}(kt) \leq \Big\{L_{Sx,Ty}(t)\diamond L_{Ax,Sx}(t)\diamond L_{By,Ty}(t)\diamond L_{Ax,Ty}(t)\Big\}. \tag{3.2}$$

If the pair (A, S) and (B, T) are compatible mappings of type (A-1), then A, B, S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, there exist $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$ and $Bx_1 = Sx_2$. Inductively, we construct the sequences $\{y_n\}$ and $\{x_n\}$ in X such that

$$y_{2n+1} = Ax_{2n} = Tx_{2n+1}, \quad y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$$

for $n = 0, 1, 2, ...$
Now putting in (3.1) and (3.2) $x = x_{2n}, y = x_{2n+1}$, we obtain

$$F_{Ax_{2n},Bx_{2n+1}}(kt) \geq \Big\{F_{Sx_{2n},Tx_{2n+1}}(t) * F_{Ax_{2n},Sx_{2n}}(t) * F_{Bx_{2n+1},Tx_{2n+1}}(t) * F_{Ax_{2n},Tx_{2n+1}}(t)\Big\}$$

that is

$$\begin{aligned} F_{y_{2n+1},y_{2n+2}}(kt) &\geq \Big\{F_{y_{2n},y_{2n+1}}(t) * F_{y_{2n+1},y_{2n}}(t) * F_{y_{2n+2},y_{2n+1}}(t) * F_{y_{2n+1},y_{2n+1}}(t)\Big\} \\ &\geq \Big\{F_{y_{2n},y_{2n+1}}(t) * F_{y_{2n+1},y_{2n+2}}(t)\Big\} \\ &\geq F_{y_{2n},y_{2n+1}}(t). \end{aligned}$$

Also

$$L_{Ax_{2n},Bx_{2n+1}}(kt) \leq \Big\{L_{Sx_{2n},Tx_{2n+1}}(t)\diamond L_{Ax_{2n},Sx_{2n}}(t)\diamond L_{Bx_{2n+1},Tx_{2n+1}}(t)\diamond L_{Ax_{2n},Tx_{2n+1}}(t)\Big\}$$

that is

$$\begin{aligned} L_{y_{2n+1},y_{2n+2}}(kt) &\leq \Big\{L_{y_{2n},y_{2n+1}}(t)\diamond L_{y_{2n+1},y_{2n}}(t)\diamond L_{y_{2n+2},y_{2n+1}}(t)\diamond L_{y_{2n+1},y_{2n+1}}(t)\Big\} \\ &\leq \Big\{L_{y_{2n},y_{2n+1}}(t)\diamond L_{y_{2n+1},y_{2n+2}}(t)\Big\} \\ &\leq L_{y_{2n},y_{2n+1}}(t). \end{aligned}$$

Similarly,

$$F_{y_{2n+2},y_{2n+3}}(kt) \geq F_{y_{2n+1},y_{2n+2}}(t) \text{ and } L_{y_{2n+2},y_{2n+2}}(kt) \leq L_{y_{2n+1},y_{2n+2}}(t).$$

Thus, we have

$$F_{y_{n+1},y_{n+2}}(kt) \geq F_{y_n,y_{n+1}}(t) \text{ and } L_{y_{n+1},y_{n+2}}(kt) \leq L_{y_n,y_{n+1}}(t) \text{ for } n = 1, 2, 3, ...$$

Therefore, we have

$$F_{y_n,y_{n+1}}(t) \geq F_{y_n,y_{n+1}}\Big(\frac{t}{q}\Big) \geq F_{y_{n-1},y_n}\Big(\frac{t}{q^2}\Big) \geq \cdots \geq F_{y_1,y_2}\Big(\frac{t}{q^n}\Big) \to 1$$

and

$$L_{y_n,y_{n+1}}(t) \leq L_{y_n,y_{n+1}}\Big(\frac{t}{q}\Big) \leq L_{y_{n-1},y_n}\Big(\frac{t}{q^2}\Big) \leq \cdots \leq L_{y_1,y_2}\Big(\frac{t}{q^n}\Big) \to 0 \text{ when } n \to \infty.$$

For each $\varepsilon > 0$ and $t > 0$, we can choose $n_0 \in N$ such that $F_{y_n,y_{n+1}}(t) > 1\varepsilon$ and $L_{y_n,y_{n+1}}(t) < \varepsilon$ for each $n \geq n_0$.

For $m, n \in N$, we suppose $m \geq n$. Then, we have

$$\begin{aligned} F_{y_n,y_m}(t) &\geq F_{y_n,y_{n+1}}\Big(\frac{t}{m-n}\Big) * F_{y_{n+1},y_{n+2}}\Big(\frac{t}{m-n}\Big) * \cdots * F_{y_{m-1},y_m}\Big(\frac{t}{m-n}\Big) \\ &> \Big((1\varepsilon) * (1\varepsilon) * ...(mn) \text{ times } ... * (1\varepsilon)\Big) \\ &\geq (1\varepsilon), \end{aligned}$$

and

$$\begin{aligned} L_{y_n,y_m}(t) &\leq L_{y_n,y_{n+1}}\Big(\frac{t}{m-n}\Big) \diamond L_{y_{n+1},y_{n+2}}\Big(\frac{t}{m-n}\Big) \diamond \cdots \diamond L_{y_{m-1},y_m}\Big(\frac{t}{m-n}\Big) \\ &< \Big((\varepsilon) \diamond (\varepsilon) \diamond \cdots (mn) \text{ times} \cdots \diamond (\varepsilon)\Big) \\ &\leq (\varepsilon). \end{aligned}$$

$$F_{y_n,y_m}(t) > (1\varepsilon), \ L_{y_n,y_m}(t) < \varepsilon.$$

Hence $\{y_n\}$ is a Cauchy sequence in X. As X is complete, $\{y_n\}$ converges to some point $z \in X$. Also, its subsequences converges to this point $z \in X$, i.e. $\{Bx_{2n+1}\} \to z, \{Sx_{2n}\} \to z, \{Ax_{2n}\} \to z, \{Tx_{2n+1}\} \to z$.

Since the pair (A, S) and (B, T) are compatible mappings of type $(A$-$1)$, then from Proposition 2.2, we have

$$AAx_{2n} \to Sz \quad \text{and} \quad BBx_{2n+1} \to Tz.$$

By (3.1) for $x = Ax_{2n}$ and $y = Bx_{2n+1}$, we have

$$F_{AAx_{2n},BBx_{2n+1}}(kt) \geq \Big\{F_{SAx_{2n},TBx_{2n+1}}(t) * F_{AAx_{2n},SAx_{2n}}(t) * F_{BBx_{2n+1},TBx_{2n+1}}(t) * F_{AAx_{2n},TBx_{2n+1}}(t)\Big\}.$$

Taking $\lim_{n\to\infty}$, using (3.3) and Proposition 2.1, we get

$$\begin{aligned} F_{Sz,Tz}(kt) &\geq \Big\{F_{Sz,Tz}(t) * F_{Sz,Sz}(t) * F_{Tz,Tz}(t) * F_{Sz,Tz}(t)\Big\} \\ F_{Sz,Tz}(kt) &\geq F_{Sz,Tz}(t). \end{aligned}$$

By (3.2) for $x = Ax_{2n}$ and $y = Bx_{2n+1}$, we have

$$L_{AAx_{2n},BBx_{2n+1}}(kt) \leq \Big\{L_{SAx_{2n},TBx_{2n+1}}(t) \diamond L_{AAx_{2n},SAx_{2n}}(t) \diamond L_{BBx_{2n+1},TBx_{2n+1}}(t) \diamond L_{AAx_{2n},TBx_{2n+1}}(t)\Big\}.$$

Taking $\lim_{n\to\infty}$, using (3.3) and Proposition 2.1, we get

$$L_{Sz,Tz}(kt) \leq \Big\{L_{Sz,Tz}(t) \diamond L_{Sz,Sz}(t) \diamond L_{Tz,Tz}(t) \diamond L_{Sz,Tz}(t)\Big\} L_{Sz,Tz}(kt) \qquad \leq L_{Sz,Tz}(t).$$

By Lemma 2.2,

$$Sz = Tz. \tag{3.4}$$

Again by inequality (3.1), for $x = z$ and $y = Bx_{2n+1}$, we have

$$F_{Az,BBx_{2n+1}}(kt) \geq \Big\{F_{Sz,TBx_{2n+1}}(t) * F_{Az,Sz}(t) * F_{BBx_{2n+1},TBx_{2n+1}}(t) * F_{Az,TBx_{2n+1}}(t)\Big\}.$$

Taking $\lim_{n\to\infty}$ and using (3.3), (3.4), we get

$$\begin{aligned} F_{Az,Tz}(kt) &\geq \Big\{F_{Sz,Tz}(t) * F_{Az,Sz}(t) * F_{Tz,Tz}(t) * F_{Az,Tz}(t)\Big\} \\ &\geq \Big\{F_{Sz,Sz}(t) * F_{Az,Sz}(t) * F_{Tz,Tz}(t) * F_{Az,Sz}(t)\Big\} \\ &\geq F_{Az,Sz}(t). \end{aligned}$$

Again by inequality (3.2), for $x = z$ and $y = Bx_{2n+1}$, we have

$$L_{Az,BBx_{2n+1}}(kt) \leq \Big\{L_{Sz,TBx_{2n+1}}(t) \diamond L_{Az,Sz}(t) \diamond L_{BBx_{2n+1},TBx_{2n+1}}(t) \diamond L_{Az,TBx_{2n+1}}(t)\Big\}.$$

Taking $\lim_{n\to\infty}$ and using (3.3), (3.4) we get

$$\begin{aligned} L_{Az,Tz}(kt) &\leq \Big\{L_{Sz,Tz}(t) \diamond L_{Az,Sz}(t) \diamond L_{Tz,Tz}(t) \diamond L_{Az,Tz}(t)\Big\} \\ &\leq \Big\{L_{Sz,Sz}(t) \diamond L_{Az,Sz}(t) \diamond L_{Tz,Tz}(t) \diamond L_{Az,Sz}(t)\Big\} \\ &\leq L_{Az,Sz}(t). \end{aligned}$$

By Lemma 2.2,

$$Az = Sz. \tag{3.5}$$

Again by inequality (3.1), for $x = z$ and $y = z$, we have

$$F_{Az,Bz}(kt) \geq \Big\{F_{Sz,Tz}(t) * F_{Az,Sz}(t) * F_{Bz,Tz}(t) * F_{Az,Tz}(t)\Big\}.$$

Using (3.4) and (3.5)

$$F_{Az,Bz}(kt) \geq \left\{F_{Sz,Sz}(t) * F_{Sz,Sz}(t) * F_{Bz,Az}(t) * F_{Tz,Tz}(t)\right\}$$
$$\geq F_{Bz,Az}(t).$$

Again by inequality (3.2), for $x = z$ and $y = z$, we have

$$L_{Az,Bz}(kt) \leq \left\{L_{Sz,Tz}(t) \diamond L_{Az,Sz}(t) \diamond L_{Bz,Tz}(t) \diamond L_{Az,Tz}(t)\right\}.$$

Using (3.4) and (3.5)

$$L_{Az,Bz}(kt) \leq \left\{L_{Sz,Sz}(t) \diamond L_{Sz,Sz}(t) \diamond L_{Bz,Az}(t) \diamond L_{Tz,Tz}(t)\right\},$$
$$L_{Az,Bz}(kt) \leq L_{Bz,Az}(t).$$

By Lemma 2.2,

$$Az = Bz. \tag{3.6}$$

Thus from (3.4), (3.5) and (3.6), we get

$$Az = Bz = Sz = Tz. \tag{3.7}$$

Now we shall prove that $Az = z$.

By inequality (3.1), putting $x = z$ and $y = x_{2n+1}$,

$$F_{Az,Bx_{2n+1}}(kt) \geq \left\{F_{Sz,Tx_{2n+1}}(t) * F_{Az,Sz}(t) * F_{Bx_{2n+1},Tx_{2n+1}}(t) * F_{Az,Tx_{2n+1}}(t)\right\}.$$

Taking $\lim_{n\to\infty}$ and using (3.7), we get

$$F_{Az,z}(kt) \geq \left\{F_{Sz,z}(t) * F_{Az,Sz}(t) * F_{z,z}(t) * F_{Az,z}(t)\right\}$$
$$\geq F_{Az,z}(t).$$

By inequality (3.2), putting $x = z$ and $y = x_{2n+1}$,

$$L_{Az,Bx_{2n+1}}(kt) \geq \left\{L_{Sz,Tx_{2n+1}}(t) * L_{Az,Sz}(t) * L_{Bx_{2n+1},Tx_{2n+1}}(t) * L_{Az,Tx_{2n+1}}(t)\right\}.$$

Taking $\lim_{n\to\infty}$ and using (3.7), we get

$$L_{Az,z}(kt) \leq \left\{L_{Sz,z}(t) \diamond L_{Az,Sz}(t) \diamond L_{z,z}(t) \diamond L_{Az,z}(t)\right\}$$
$$\leq L_{Az,z}(t).$$

By Lemma 2.2,

$$Az = z.$$

Combining all results, we get $z = Az = Bz = Sz = Tz$.

From this we conclude that z is a common fixed point of A, B, S and T. □

Uniqueness. Let z_1 be another common fixed point of A, B, S and T. Then

$$z_1 = Az_1 = Bz_1 = Sz_1 = Tz_1$$
$$\text{and} \qquad z = Az = Bz = Sz = Tz.$$

Using inequality (3.1), putting $x = z$ and $y = z_1$, we get

$$F_{Az,Bz_1}(kt) \geq \left\{F_{Sz,Tz_1}(t) * F_{Az,Sz}(t) * F_{Bz_1,Tz_1}(t) * F_{Az,Tz_1}(t)\right\}$$
$$\geq \left\{F_{z,z_1}(t) * F_{z,z}(t) * F_{z_1,z_1}(t) * F_{z,z_1}(t)\right\}$$
$$\geq F_{z,z_1}(t).$$

Using inequality (3.2), putting $x = z$ and $y = z_1$, we get

$$L_{Az,Bz_1}(kt) \leq \left\{L_{Sz,Tz_1}(t) \diamond L_{Az,Sz}(t) \diamond L_{Bz_1,Tz_1}(t) \diamond L_{Az,Tz_1}(t)\right\}$$
$$\leq \left\{L_{z,z_1}(t) \diamond L_{z,z}(t) \diamond L_{z_1,z_1}(t) \diamond L_{z,z_1}(t)\right\}$$
$$\leq L_{z,z_1}(t).$$

(By Lemma 2.2, $z = z_1$).

Thus z is the unique common fixed point of A, B, S and T.

If we increase the number of self maps from four to six, then we have the following.

Theorem 3.2. *Let $(X, F, L, *, \diamond)$ be a complete intuitionistic Menger space with $t * t \geq t$ and $(1-t)\diamond(1-t) \leq (1-t)$. Let A, B, S, T, I and J be selfmappings of X such that the following conditions are satisfied:*
(i) *$AB(X) \subseteq J(X)$ and $ST(X) \subseteq I(X)$,*
(ii) *I and J are continuous,*
(iii) *There exists $k \in (0,1)$ such that for every $x, y \in X$ and $t > 0$,*

$$F_{ABx,STy}(kt) \geq \Big\{F_{Ix,Jy}(t) * F_{ABx,Ix}(t) * F_{STy,Jy}(t) * F_{ABx,Jy}(t)\Big\}, \tag{3.8}$$

$$L_{ABx,STy}(kt) \leq \Big\{L_{Ix,Jy}(t)\diamond L_{ABx,Ix}(t)\diamond L_{STy,Jy}(t)\diamond L_{ABx,Jy}(t)\Big\}. \tag{3.9}$$

If the pair (AB, I) and (ST, J) are compatible mappings of type $(A\text{-}1)$, then AB, ST, I and J have a unique common fixed point in X. Furthermore, if the pairs $(A, B), (A, I), (B, I), (S, T), (S, J)$ and (T, J) are commuting mapping then A, B, S, T, I and J have a unique common fixed point.

Proof. From Theorem 3.1, z is the unique common fixed point of AB, ST, I and J.

Finally, we need to show that z is also a common fixed point of A, B, S, T, I, and J. For this, let z be the unique common fixed point of both the pairs (AB, I) and (ST, J). Then, by using commutativity of the pair $(A, B), (A, I)$, and (B, I), we obtain

$$\begin{aligned}
&Az = A(ABz) = A(BAz) = AB(Az),\\
&Az = A(Iz) = I(Az),\\
&Bz = B(ABz) = B(A(Bz)) = BA(Bz) = AB(Bz), \quad Bz = B(Iz) = I(Bz),
\end{aligned} \tag{3.10}$$

which show that Az and Bz are common fixed point of (AB, I), yielding thereby

$$Az = z = Bz = Iz = ABz, \tag{3.11}$$

in the view of uniqueness of the common fixed point of the pair (AB, I).

Similarly, using the commutativity of $(S, T), (S, J), (T, J)$ it can be shown that

$$Sz = Tz = Jz = STz = z. \tag{3.12}$$

Now, we need to show that $Az = Sz(Bz = Tz)$ also remains a common fixed point of both the pairs (AB, I) and (ST, J). For this, put $x = z$ and $y = z$ in (3.8) and using (3.11) and (3.12), we get

$$\begin{aligned}
F_{ABz,STz}(kt) &\geq \Big\{F_{Iz,Jz}(t) * F_{ABz,Iz}(t) * F_{STz,Jz}(t) * F_{ABz,Jz}(t)\Big\}\\
&\geq F_{Az,Sz}(t).
\end{aligned}$$

By (3.9), we get

$$\begin{aligned}
L_{ABz,STz}(kt) &\leq \Big\{L_{Iz,Jz}(t)\diamond L_{ABz,Iz}(t)\diamond L_{STz,Jz}(t)\diamond L_{ABz,Jz}(t)\Big\}\\
&\leq L_{Az,Sz}(t).
\end{aligned}$$

By Lemma 2.2, we get
$Az = Sz$. Similarly, it can be shown that $Bz = Tz$. Thus z is the unique common fixed point of A, B, S, T, I and J. □

Example 3.1. Let $X = [0, 1]$ with the metric d defined by $d(x, y) = |x - y|$ and for each $t \in [0, 1]$ define

$$F(x, y, t) = \frac{|x-y|}{t + |x-y|}, \qquad L(x, y, t) = \frac{t}{t + |x-y|},$$
$$F(x, y, 0) = 1, \qquad L(x, y, 0) = 0 \ \text{ for all } x, y \in X.$$

Clearly $(X, F, L, *, \diamond)$ is a complete intuitionistic Menger space where $*$ is defined by $a * b = \min\{a, b\}$ and $\diamond$ is defined by $a\diamond b = \max\{a, b\}$.

Define the self mappings A, B, S and T on X by

$$Ax = Bx = 0 \ \text{ for all } x \in X$$
$$Sx = \begin{cases} 0, & \text{if } 0 \leq x < 1\\ 1, & \text{if } x = 1,\end{cases}$$
$$Tx = x \ \text{ for all } \ x \in X.$$

If we define a sequence $\{x_n\}$ in X by $x_n = \left\{\frac{1}{n}\right\}$, then we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 0,$$

$$\lim_{n\to\infty} F(SAx_n, 0, t) \geq F(A0, 0, t) = 0,$$
$$\lim_{n\to\infty} L(SAx_n, 0, t) \leq L(A0, 0, t) = 1 \text{ and}$$
$$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = 0,$$
$$\lim_{n\to\infty} F(TBx_n, 0, t) \geq F(B0, 0, t) = 0,$$
$$\lim_{n\to\infty} L(TBx_n, 0, t) \leq L(B0, 0, t) = 1.$$

That is the pairs *(A,S), (B,T)* are compatible of type $(A-1)$ and S, T are continuous.

Therefore all the conditions of Theorem 3.1 are satisfied and so A, B, S and T have a unique common fixed point 0 in X.

Corollary 3.1. *Let $(X, F, L, *, \diamond)$ be a complete intuitionistic Menger space with $t * t \geq t$ and $(1-t)\diamond(1-t) \leq (1-t)$. Let S and T be self mappings of X and there exists $k \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$,*

$$F_{x,y}(kt) \geq \left\{F_{Sx,Ty}(t) * F_{x,Sx}(t) * F_{y,Ty}(t) * F_{x,Ty}(t)\right\} \tag{3.13}$$

$$\text{and} \quad L_{x,y}(kt) \leq \left\{L_{Sx,Ty}(t)\diamond L_{x,Sx}(t)\diamond L_{y,Ty}(t)\diamond L_{x,Ty}(t)\right\}. \tag{3.14}$$

Then S and T have a unique common fixed point in X.

Proof. If we set $A = B = I$ (the identity mapping) in Theorem 3.1, then it is easy to check that the pairs (I, S) and (I, T) are compatible of type $(A$-1) and the identity mapping I is continuous. Hence by Theorem 3.1, S and T have a unique common fixed point in X. $\square$

4. Conclusion

In this work, we proposed and proved some interesting common fixed point theorems in intuitionistic menger space for compatible mappings of type $(A$-1). Our results improve and develop many known common fixed point theorems available in the literature of intuitionistic menger fixed point theory.

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FOURTH HANKEL DETERMINANT FOR A NEW SUBCLASS OF BOUNDED TURNING FUNCTIONS

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Abstract

In this paper, we consider a new subfamily of holomorphic (analytic) functions with bounded turning in the open unit disk $\mathbb{U} = \{z; |z| < 1\}$. Here, in this paper, we focus the coefficient estimates and upper bounds of fourth Hankel determinant for this family. Moreover, the same bounds have been investigated for two-fold and three-fold symmetric functions.

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Keywords and Phrases: Analytic function, Bounded turning functions, Hankel determinant, m-fold symmetric function.

1. Introduction

Let A denote the class of all analytic functions which are normalized by $\psi(0) = 0$ and $\psi'(0) = 1$ in the open unit disc $\mathbb{U} = \{z; |z| < 1\}$. Geometrically the normalization condition satisfies as $\psi(0) = 0$ like the transformations of the image domain given by $\psi'(0) = 1$. The function ψ having the Taylor-Maclaurin series expansion given by

$$\psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n. \tag{1.1}$$

Also, let S be a subset of A which consists all the univalent functions. Without loss of generality, we can say an univalent functions can be written in the form (1.1). Next, let P denote the class of analytic functions with real part positive and has the form :

$$\rho(z) = 1 + \sum_{k=1}^{\infty} \rho_k z^k. \tag{1.2}$$

The Hankel determinant $H_{q,n}(\psi)$; $(q \in \mathbb{N}_0, n \in \mathbb{N})$ where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for a function $\psi \in S$ of the form (1.1) was defined by Pommerenke [18, 19] as

$$H_{q,n}(\psi) = \begin{vmatrix} \psi_n & \psi_{n+1} & \cdots & \psi_{n+q-1} \\ \psi_{n+1} & \psi_{n+2} & \cdots & \psi_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ \psi_{n+q-1} & \psi_{n+q} & \cdots & \psi_{n+2q-2} \end{vmatrix}. \tag{1.3}$$

The problem of figuring the upper bound of $H_{q,n}$ over various subfamilies of A is fascinating and extensively studied in the literature of Geometric Function Theory of Complex Analysis. For fixed q, n, the growth rate of $H_{q,n}$ as $n \to \infty$ has been studied by Noonan and Noor [15, 14] for different subfamilies of univalent function. Sharp upper bound of $H_{2,2}(\psi) = \psi_2\psi_4 - \psi_3^2$ of second Hankel determinant were obtained by various authors. It is worth citing a few of them, for example [6, 7, 8, 10]. Unfortunately, the sharp bound of $H_{2,2}(\psi)$ for the whole class S is still not known. In [17], Thomas conjectured that if $\psi \in S$, then $|H_{2,n}(\psi)| \leq 1$. As it was shown by Li and Srivastava in [13], this conjecture is not true for $n \geq 4$. Similarly, Răducanu and Zaprawa in [21] proved that it is also false for $n = 2$. In fact, they showed that $max\ \{|H_{2,2}(\psi)|\ ;\ \psi \in S\} \ \geq\ 1 : 175$.

One of the paper on $H_{3,1}(\psi)$ by Babalola [3] for the families of $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$ as 16, 0.714 and 0.742 respectively. Moreover, Babalola claimed that the extremal function for class of starlike function is the rotations of $\psi(z) = \frac{z}{(1-z)^2}$. The above estimates are true but not sharp. In fact, Zaprawa[23] meliorate [3] results by proving third Hankel determinant for $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$ as 1 , 0.090 , 0.683 respectively and alleged that improved bounds still not sharp. He considered functions with m-fold symmetry for the sharpness of the subfamilies of $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{R}$. Other related results can be found in [4, 5, 11, 22].

Here, we will study the class $\mathcal{R}_1 \subset A$ and satisfying the condition

$$Re(\psi'(z) + z\psi''(z)) > 0, (z \in \mathbb{U}), \tag{1.4}$$

and the class $\mathcal{R} \subset A$ satisfying $Re(\psi'(z)) > 0, z \in \mathbb{U}$.

The class $\mathcal{R}$ is said to be of bounded turning because $Re(\psi'(z)) > 0$ is equivalent to $|\arg \psi'(z)| < \frac{\pi}{2}$ and $\arg \psi(z)$ is the angle of rotation of the image of a line segment starting from z under the mapping ψ. They are of special interest since they are not part of a wide subclass of univalent functions known as starlike functions. See the counter example by Krzyz [12] showing that $\mathcal{S}^* \not\subseteq \mathcal{R}, \mathcal{R} \not\subseteq \mathcal{S}^*$. In addition, classes $\mathcal{R}$ and $\mathcal{R}_1$ are related in a same way as are the classes of starlike and convex functions, i.e., $\mathcal{R}_1 \subset \mathcal{R}$ [1] as $C \subset \mathcal{S}^*$, as

$$\psi \in \mathcal{R}_1 \iff z\psi'(z) \in \mathcal{R},$$

as

$$\psi \in C \iff z\psi'(z) \in \mathcal{S}^*.$$

Here we investigate the upper bounds of $H_{4,1}$ for the class $\mathcal{R}_1$ with the motivation of paper by Arif et al. [2] for the class of bounded turning functions $\mathcal{R}$ as if $\psi \in \mathcal{R}$ then $|H_{4,1}(\psi)| \le 0.78050$ and [9] for α bounded turning functions.

2. Preliminary Lemmas

To investigate fourth Hankel determinant for class $\mathcal{R}_1$ we use the following results as:

Lemma 2.1 ([20]). *If ρ belongs to P, then*

$$|\rho_k| \le 2; \quad k \in \mathbb{N}, \tag{2.1}$$

$$|\rho_{m+n} - \lambda\rho_m\rho_n| \le 2; \quad 0 \le \lambda \le 1, \tag{2.2}$$

$$|\rho_p\rho_q - \rho_r\rho_s| \le 4; \quad p+q = r+s, \tag{2.3}$$

Theorem 2.1 ([2]). *Let $k(z) = z + \sum_{r=1}^{\infty} k_r z^r \in \mathcal{S}^*$. Then*

$$|k_2^2(k_3 - \lambda k_2^2)| = \begin{cases} 4(3-4\lambda) & for \quad \lambda \le \frac{5}{8}; \\ \frac{1}{2(2\lambda-1)} & for \quad \lambda \in [\frac{5}{8}, \frac{3}{4}]; \\ \frac{1}{4(1-\lambda)} & for \quad \lambda \in [\frac{3}{4}, \frac{7}{8}]; \\ 4(4\lambda-3) & for \quad \lambda \ge \frac{7}{8}. \end{cases}$$

where λ is a real number.

Theorem 2.2 ([16]). *Let $\psi \in \mathcal{R}_1$, then*

$$|H_{3,1}(\psi)| \le 0.2121\cdots. \tag{2.4}$$

3. Bounds of fourth Hankel determinant

First of all, $H_{4,1}(\psi); \psi \in A$ of the form (1.1) is polynomial of six successive coefficients as:

$$H_{4,1}(\psi) = \psi_7 H_{3,1}(\psi) - \psi_6\Delta_1 + \psi_5\Delta_2 - \psi_4\Delta_3 \tag{3.1}$$

where

$$\Delta_1 = (\psi_3\psi_6 - \psi_4\psi_5) - \psi_2(\psi_2\psi_6 - \psi_3\psi_5) + \psi_4(\psi_2\psi_4 - \psi_3^2), \tag{3.2}$$

$$\Delta_2 = (\psi_4\psi_6 - \psi_5^2) - \psi_2(\psi_3\psi_6 - \psi_4\psi_5) + \psi_3(\psi_3\psi_5 - \psi_4^2), \tag{3.3}$$

$$\Delta_3 = \psi_2(\psi_4\psi_6 - \psi_5^2) - \psi_3(\psi_3\psi_6 - \psi_4\psi_5) + \psi_4(\psi_3\psi_5 - \psi_4^2). \tag{3.4}$$

Theorem 3.1. *Let $\psi \in \mathcal{R}_1$ then maximum of $|H_{4,1}(\psi)| \le 0.004495423$.*

Proof. Let $\psi \in \mathcal{R}_1$, then it follows that $\psi'(z) + z\psi''(z) = \rho(z)$ with $\rho(z) = 1 + \sum_{n=1}^{\infty}\rho_n z^n \in P$.

$$\left(1 + \sum_{n=2}^{\infty} n^2\psi_n z^{n-1}\right) = \left(1 + \sum_{n=1}^{\infty}\rho_n z^n\right).$$

Comparing the coefficients we yields that

$$\psi_n = \frac{\rho_{n-1}}{n^2}. \tag{3.5}$$

Utilizing (3.5) in (3.2), (3.3) and in (3.4), we obtain

$$\Delta_1 = \frac{\rho_2\rho_5}{324} - \frac{\rho_3\rho_4}{400} - \frac{\rho_1^2\rho_5}{576} + \frac{\rho_1\rho_2\rho_4}{900} + \frac{\rho_1\rho_3^2}{1024} - \frac{\rho_2^2\rho_3}{1296};$$

$$\Delta_2 = \frac{\rho_3\rho_5}{576} - \frac{\rho_4^2}{625} - \frac{\rho_1\rho_2\rho_5}{1296} + \frac{\rho_1\rho_3\rho_4}{1600} + \frac{\rho_2^2\rho_4}{2025} - \frac{\rho_2\rho_3^2}{2304};$$

$$\Delta_3 = \frac{\rho_1\rho_3\rho_5}{2304} - \frac{\rho_1\rho_4^2}{2500} - \frac{\rho_2^2\rho_5}{2916} + \frac{\rho_2\rho_3\rho_4}{1800} - \frac{\rho_3^3}{4096}.$$

Now rewrite the above equations as follow

$$\Delta_1 = \frac{\rho_5}{576}(\rho_2 - \rho_1^2) + \frac{\rho_3}{1296}(\rho_4 - \rho_2^2) - \frac{\rho_3}{1024}(\rho_4 - \rho_1\rho_3)$$

$$
\begin{aligned}
&- \frac{4759\rho_4}{2073600}(\rho_3 - \rho_1\rho_2) + \frac{2455\rho_2}{2073600}(\rho_5 - \rho_1\rho_4) + \frac{\rho_2\rho_5}{2073600},\\
\Delta_2 &= \frac{1}{1296}\rho_5(\rho_3 - \rho_1\rho_2) - \frac{1}{2025}\rho_4(\rho_4 - \rho_2^2) + \frac{1}{2304}\rho_3(\rho_5 - \rho_2\rho_3)\\
&- \frac{1559}{3240000}\rho_3(\rho_5 - \rho_1\rho_4) - \frac{56}{50625}\rho_4(\rho_4 - \rho_1\rho_3) + \frac{71}{1440000}\rho_3\rho_5,\\
\Delta_3 &= \frac{1}{2916}\rho_5(\rho_4 - \rho_2^2) - \frac{1}{2304}\rho_5(\rho_4 - \rho_1\rho_3) + \frac{1}{4096}\rho_3(\rho_6 - \rho_3^2) - \frac{1}{4096}\rho_3(\rho_6 - \rho_2\rho_4)\\
&+ \frac{1}{2500}\rho_4(\rho_5 - \rho_1\rho_4) - \frac{287}{921600}\rho_4(\rho_5 - \rho_2\rho_3) + \frac{4679}{1866240000}\rho_4\rho_5.
\end{aligned}
$$

By applying triangular inequality and the inequalities asserted by the Lemma in Section 2, we obtain

$$
|\Delta_1| \le \frac{1}{144} + \frac{1}{324} + \frac{1}{256} + \frac{4759}{518400} + \frac{2455}{518400} + \frac{345}{518400} = \frac{77}{2700}, \tag{3.6}
$$

$$
|\Delta_2| \le \frac{1}{324} + \frac{4}{2025} + \frac{1}{576} + \frac{224}{50625} + \frac{1559}{810000} + \frac{71}{1440000} = \frac{19003}{1440000}, \tag{3.7}
$$

$$
|\Delta_3| \le \frac{1}{729} + \frac{1}{576} + \frac{1}{512} + \frac{1}{625} + \frac{287}{230400} + \frac{4679}{466560000} = \frac{57}{7200}. \tag{3.8}
$$

Now, by substituting (3.6), (3.7), (3.8), (2.2) and (2.1) in (3.1), we find that

$$
\begin{aligned}
|H_{4,1}(\psi)| &\le \frac{312769}{361267200} + \frac{77}{48600} + \frac{342054}{324000000} + \frac{57}{57600}\\
&\le \frac{4027899}{896000000}.
\end{aligned}
$$

The proof of Theorem is now completed. □

4. Bounds for $H_{4,1}(\psi)$ for Two-fold and Three-fold Symmetric Functions

A function ψ is said to be m-fold symmetric if the following condition $\psi(\varepsilon z) = \varepsilon\psi(z)$ holds, where $\varepsilon = \exp(\frac{2\Pi\iota}{m})$ denotes the principal m-th root of 1. $S^{(m)}$ denotes the class of normalized univalent functions.

$$
S^{(m)} = \{\psi(z) \in S; \psi(z) = z + \textstyle\sum_{k=1}^{\infty}\psi_{mk+1}z^{mk+1}, \quad z \in \mathbb{U}. \tag{4.1}
$$

A function $\psi \in S^{(m)}$ in $\mathcal{R}_1^{(m)}$ if

$$
\psi'(z) + z\psi''(z) = \rho(z) \quad with \quad \rho \in P^{(m)},
$$

where

$$
P^{(m)} = \{\rho(z) : \rho(z) = 1 + \textstyle\sum_{k=1}^{\infty}\rho_{mk}z^{mk}\}. \tag{4.2}
$$

We see that if $\psi \in S^{(3)}$ then $\psi(z) = z + \psi_4 z^4 + \psi_7 z^7 + \cdots$ and consequently $H_{4,1}(\psi) = \psi_4^2(\psi_4^2 - \psi_7)$ and if $\psi \in S^{(2)}$ having odd functions of S and of the form $\psi(z) = z + \psi_3 z^3 + \psi_5 z^5 + \cdots$ so $H_{4,1}(\psi) = \psi_3\psi_5\psi_7 - \psi_3^3\psi_7 + \psi_3^2\psi_5^2 - \psi_5^3$.

Theorem 4.1 (Three-fold Symmetric Functions)**.** *Let $\psi \in \mathcal{R}_1^{(3)}$ then*

$$
|H_{4,1}(\psi)| \le \frac{79}{200704}.
$$

Proof. If $\psi \in \mathcal{R}_1^{(3)}$ then $\exists \tilde{h}(z) = z + h_4 z^4 + h_7 z^7 + \cdots \in \mathcal{S}^{*(3)}$ such that $\frac{z\tilde{h}'(z)}{\tilde{h}(z)}$ and $\psi \in \mathcal{R}_1^{(3)} \in S^{(m)}$ for $m = 3$, we have

$$
1 + 3h_4 z^3 + (6h_7 - 3h_4^2)z^6 + \cdots = 1 + 16\psi_4 z^3 + 49\psi_7 z^6 + \cdots
$$

.

Equating the coefficients, we get

$$
3h_4 = 16\psi_4, \qquad 6h_7 - 3h_4^2 = 49\psi_7. \tag{4.3}
$$

Since $\tilde{h} \in \mathcal{S}^{*(3)}$, $\exists k(z) = z + \sum_{r=1}^{\infty} k_r z^r \in \mathcal{S}^*$ such that $\tilde{h}(z) = \sqrt[3]{k(z^3)}$,

$$
\therefore z + h_4 z^4 + h_7 z^7 + \cdots = z + \frac{1}{3}k_2 z^4 + (\frac{1}{3}k_3 - \frac{1}{9}k_2^2)z^7 + \cdots
$$

Equating the coefficients

$$
h_4 = \frac{1}{3}k_2, \quad h_7 = (\frac{1}{3}k_3 - \frac{1}{9}k_2^2). \tag{4.4}
$$

Now rearranging (4.3) and (4.4), we find that

$$\psi_4 = \frac{1}{16}k_2, \quad \psi_7 = \frac{1}{49}(2k_3 - k_2^2). \tag{4.5}$$

We have already seen that $\psi_2 = \psi_3 = \psi_5 = \psi_6 = 0$. Hence $H_{4,1}(\psi) = \psi_4^2(\psi_4^2 - \psi_7)$. It follows that

$$|H_{4,1}(\psi)| = \frac{1}{6272}|k_2^2(k_3 - \frac{305}{512}k_2^2)|.$$

Thus, by utilizing (2.1) from Section 2, where $\lambda = \frac{305}{512} \leq \frac{5}{8}$, we get our desired result as asserted by Theorem 4.1. □

Theorem 4.2 (Two-fold Symmetric Functions)**.** *Let* $\psi \in \mathcal{R}_1^{(2)}$ *then*

$$|H_{4,1}(\psi)| \leq \frac{8}{11025}.$$

Proof. Here $\psi \in \mathcal{R}_1^{(2)}$ thereupon, $H_{4,1}(\psi) = \psi_3\psi_5\psi_7 - \psi_3^3\psi_7 + \psi_3^2\psi_5^2 - \psi_5^3$.

As long as $\psi \in \mathcal{S}^{*(2)}$ likewise $\rho \in P^{(2)}$. Hence $\psi'(z) + z\psi''(z) = \rho(z)$ with the help of expansions (4.1) and (4.2) for two fold, it follows that

$$1 + 9\psi_3 z^2 + 25\psi_5 z^4 + 49\psi_7 z^6 + ... = 1 + \rho_2 z^2 + \rho_4 z^4 + \rho_6 z^6 + \ldots.$$

$$\therefore \psi_3 = \frac{1}{9}\rho_2, \quad \psi_5 = \frac{1}{25}\rho_4, \quad \psi_7 = \frac{1}{49}\rho_6.$$

$$|H_{4,1}(\psi)| = \left|\frac{1}{11025}\psi_2\psi_4\psi_6 - \frac{1}{35721}\psi_2^3\psi_6 + \frac{1}{50625}\psi_2^2\psi_4^2 - \frac{1}{15625}\psi_4^3\right|.$$

$$\leq \frac{1}{11025}\left|\left(\rho_2\rho_6 - \frac{441}{625}\rho_4^2\right)\right|\left|\left(\rho_4 - \frac{25}{81}\rho_2^2\right)\right|.$$

Finally, applying the inequalities (2.2) and (2.3) of Lemma in Section 2, we get our Theorem proved. □

5. Conclusion

In this paper, we have presented a systematic study of class of analytic function associated with bounded turning function in the open disk $\mathbb{U}$. For function belonging to the subclass of analytic function class, which we have introduced and studied here, we have derived the estimates of fourth Hankel determinant. Furthermore, we have also investigated the same kind of bounds for some two-fold and three-fold functions.

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ANDREWS' TYPE WP-BAILEY LEMMA AND ITS APPLICATIONS

By

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Abstract

Over the years, the study of Bailey transform, Bailey lemma, Bailey pair, their variants and their applications are the major subjects of interest. Of course, it is due to the efficiency of the Bailey transform and lemma in producing many ordinary and q-hypergeometric identities, multiple series summation and transformation formulas, and the Rogers-Ramanujan type identities. Andrews investigated a WP-Bailey lemma and the pairs with the help of Bailey transform and used it to derive well-known summations and multiple series transformations. In this research paper, we investigate an Andrews' type WP-Bailey lemma and the pairs with the help of First Bailey Type Transform due to Joshi and Vyas. The investigated Andrews' type WP-Bailey lemma is then applied to obtain terminating multiple q-hypergeometric identities and construct the WP-Bailey type chains and a binary tree.

The paper is motivated by the observation that the basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) gamma and q-hypergeometric functions and basic (or q-) hypergeometric polynomials, are applicable particularly in several diverse areas including number theory, theory of partitions and combinatorial analysis as well as in the study of combinatorial generating functions.

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1. Introduction, Preliminaries and Motivation

The generalized basic (or q-) hypergeometric series ${}_r\Phi_s$ [10] (see also [9] and [36]) is defined by

$$ {}_r\Phi_s\left[\begin{matrix} a_1,\cdots,a_r\ ; \\ \\ b_1,\cdots,b_s\ ; \end{matrix}\ q,\ z\right] = \sum_{n=0}^{\infty}\frac{(a_1;q)_n\cdots(a_r;q)_n}{(b_1;q)_n\cdots(b_s;q)_n}\ \frac{z^n}{(q;q)_n}\left[(-1)^n\, q^{\binom{n}{2}}\right]^{1+s-r}, \tag{1.1}$$

where $q \neq 0$ when $r > s+1$. We also note that

$$ {}_{r+1}\Phi_r\left[\begin{matrix} a_1,\cdots,a_{r+1}\ ; \\ \\ b_1,\cdots,b_r\ ; \end{matrix}\ q,\ z\right] = \sum_{n=0}^{\infty}\frac{(a_1;q)_n\cdots(a_{r+1};q)_n}{(b_1;q)_n\cdots(b_r;q)_n}\ \frac{z^n}{(q;q)_n}, \tag{1.2}$$

where the q-shifted factorials are given by

$$(a;q)_n := (1-a)\cdots(1-aq^{n-1}), \qquad (n\in\mathbb{N}_0) \tag{1.3}$$

$$(a;q)_0 := 1. \tag{1.4}$$

In the definition (1.2), it is assumed that none of the denominator parameters are of the form q^{-k} and the series (1.2) will terminate if one of its numerator parameters is of the form q^{-k}, where k is a non-negative integer. A detailed discussion on convergence of the series in (1.2) can be found in [10, pp. 4–5].

Moreover, this series is very well-poised if $a_1q = a_2b_1 = \cdots = a_{r+1}b_r$ along with the condition $a_2 = q\sqrt{a_1}, a_3 = -q\sqrt{a_1}$.

The recent research papers [33, 34, 35, 42] and many others, cited therein, are examples of ongoing trend and interest in the field of q-analysis and q-calculus. Srivastava [33] used the concept of basic (or q-) calculus to introduce the families of q-extensions of starlike functions, which are associated with the Janowski functions (in the open unit disk U) of complex theory. He defined two general subclasses $\mathcal{S}_n^{\alpha}(\lim_{n\to\infty},\beta,b,q)$ and $\mathcal{G}_n^{\alpha}(\lim_{n\to\infty},\beta,b,q)$ of normalized analytic functions with complex order and negative coefficients and their associated coefficient estimates, radii of close-to-convexity, starlikeness and convexity, extreme points, and growth and distortion theorems. Moreover, Srivastava [34] presents an excellent set of discussion and comments on the study of post-quantum or (p,q)-version of the classical q-analysis. In a survey-cum-expository review article by Srivastava [35], the overview and recent developments in the theory of several extensively studied higher transcendental functions along with their applications in widely investigated areas of various sciences have been nicely presented. Further, several contiguous extensions of the q-analogues of the celebrated set of first, second and third summation theorems due to Kummer are investigated in a recent paper by Vyas et al. [42].

The present work is motivated essentially by the fact that the basic (or q-) series, basic (or q-) polynomials and basic (or q-) calculus, specifically the basic (or q-) hypergeometric functions and the basic (or q-) hypergeometric polynomials have demonstrated applications around number theory such as, for example, the theory of partitions and are also found to be useful in a wide range of fields including, for example, combinatorial analysis, finite vector spaces, lie theory, particle physics, quantum mechanics, mechanical engineering, theory of heat conduction, nonlinear electric circuit theory, cosmology and statistics (see, for details, [36, pp. 350–351], [33, p. 328], and [38, p. 1817]; see also the references cited therein). Further motivation for studying such quantum (or q-) hypergeometric functions in this paper can be found in the book entitled *quantum calculus* [14]. Here, in our present investigation, we are mainly concerned with the celebrated Bailey transform and lemma, their extensions and applications to obtain multiple q-hypergeometric identities.

1.1. Bailey Transform and Lemma

The well-known *Bailey transform* [31] is defined as:
If
$$\beta_n = \sum_{r=0}^{n}\alpha_r\, u_{n-r}\, v_{n+r} \tag{1.5}$$
and
$$\gamma_n = \sum_{r=n}^{\infty}\delta_r\, u_{r-n}\, v_{r+n}, \tag{1.6}$$
then, subject to convergence conditions,
$$\sum_{n=0}^{\infty}\alpha_n\gamma_n = \sum_{n=0}^{\infty}\beta_n\delta_n. \tag{1.7}$$
Bailey [6] studied the above transform, in the form of *Bailey lemma* as given below :
$$\beta_n = \sum_{r=0}^{n}\frac{\alpha_r}{(aq;q)_{n+r}\,(q;q)_{n-r}}, \tag{1.8}$$
and used it to obtain a number of identities of Rogers-Ramanujan type including the two well-known Rogers-Ramanujan identities. Following Bailey [6, Section 4], Slater [31] published a list of 130 identities of Rogers-Ramanujan type. Specifically, the mechanism of production of an infinite family of identities out of one identity is mentioned in [6, Section 4].
As explained in [6, Section 4], the Bailey transform (Equations (1.5) to (1.7)) and the following conjugate Bailey pair
$$\gamma_n = \sum_{r=n}^{\infty}\frac{\delta_r}{(aq;q)_{r+n}\,(q;q)_{r-n}} \tag{1.9}$$
with the choice $\delta_r = \dfrac{\left(b,c,q^{-N};q\right)_r q^r}{\left(\dfrac{bcq^{-N}}{a};q\right)_r}$, $u_r = \dfrac{1}{(q;q)_r}$ and $v_r = \dfrac{1}{(aq;q)_r}$ gives:
$$\sum_{n\geq 0}\left(\frac{(b,c;q)_n\left(\dfrac{aq}{bc}\right)^n\alpha_n}{\left(\dfrac{aq}{b},\dfrac{aq}{c};q\right)_n}\right)\frac{1}{(q;q)_{N-n}\,(aq;q)_{N+n}} = \sum_{n\geq 0}\frac{(b,c;q)_n\left(\dfrac{aq}{bc};q\right)_{N-n}\left(\dfrac{aq}{bc}\right)^n\beta_n}{(q;q)_{N-n}\left(\dfrac{aq}{b},\dfrac{aq}{c};q\right)_N}. \tag{1.10}$$
However, Bailey [6] ignored the above-mentioned conjugate pair due to the complexity involved in it but, it was Andrews [2] who observed the iteration of the Bailey lemma, that is, equation (1.10) is again an instance of equation (1.8) with
$$\alpha_r^{'} = \frac{(b,c;q)_r\left(\dfrac{aq}{bc}\right)^r\alpha_r}{\left(\dfrac{aq}{b},\dfrac{aq}{c};q\right)_r} \tag{1.11}$$

and

$$\beta_N^{'} = \sum_{n\geq 0} \frac{(b,c;q)_n \left(\frac{aq}{bc};q\right)_{N-n} \left(\frac{aq}{bc}\right)^n}{(q;q)_{N-n} \left(\frac{aq}{b},\frac{aq}{c};q\right)_N} \beta_n. \tag{1.12}$$

In this way, the Bailey lemma was iterated ad infinitum, which originated the concept of Bailey pairs and chains, see also [5].

1.2. WP-Bailey Lemma, Pairs and Chains

Andrews [3, p. 15, Def. 6.1] generalized the standard Bailey lemma as:

$$\beta_n(a,k) = \sum_{p=0}^{n} \frac{\left(\frac{k}{a};q\right)_{n-p} (k;q)_{n+p}}{(q;q)_{n-p} (aq;q)_{n+p}} \alpha_p(a,k), \tag{1.13}$$

where $\alpha_0(a,k) = 1$ and named it as WP-Bailey lemma. The sequences $\alpha_n(a,k)$ and $\beta_n(a,k)$ form a WP-Bailey pair. For $k = 0$, (1.13) converts back to the standard Bailey pair (1.8). He also explained the construction of two distinct WP-Bailey pairs as:

If the initial pair $(\alpha_n(a,k), \beta_n(a,k))$ satisfy (1.10), then the pairs $\left(\alpha_n^{'}(a,k), \beta_n^{'}(a,k)\right)$ and $\left(\widetilde{\alpha_n}(a,k), \widetilde{\beta_n}(a,k)\right)$ also, where

$$\alpha_n^{'}(a,k) = \frac{(b,c;q)_n}{\left(\frac{aq}{b},\frac{aq}{c};q\right)_n} \left(\frac{k}{\varrho}\right)^r \alpha_n(a,\varrho), \tag{1.14}$$

and

$$\beta_N^{'}(a,k) = \frac{\left(\frac{kb}{a},\frac{kc}{a};q\right)_n}{\left(\frac{aq}{b},\frac{aq}{c};q\right)_n} \sum_{p=0}^{n} \frac{\left(1-\varrho q^{2p}\right)(b,c;q)_p \left(\frac{k}{\varrho};q\right)_{n-p}}{(1-\varrho)\left(\frac{kb}{a},\frac{kc}{a};q\right)_p} \frac{(k;q)_{n+p}}{(q\varrho;q)_{n+p}} \beta_p(a,\varrho) \left(\frac{k}{\varrho}\right)^n, \tag{1.15}$$

with $\varrho = \frac{kbc}{aq}$, and

$$\widetilde{\alpha_n}(a,k) = \frac{\left(\frac{qa^2}{k};q\right)_{2n} \left(\frac{k^2}{qa^2}\right)^n}{(k;q)_{2n}} \alpha_n\left(a,\frac{qa^2}{k}\right), \tag{1.16}$$

$$\widetilde{\beta_n}(a,k) = \sum_{p=0}^{n} \frac{\left(\frac{k^2}{qa^2};q\right)_{n-p} \left(\frac{k^2}{qa^2}\right)^p}{(q;q)_{n-p}} \beta_p\left(a,\frac{qa^2}{k}\right). \tag{1.17}$$

Thus, the Bailey pair $(\alpha_n(a,k), \beta_n(a,k))$ generates two separate Bailey chains as:

$$(\alpha_n(a,k), \beta_n(a,k)) \to \left(\alpha_n^{'}(a,k), \beta_n^{'}(a,k)\right) \to \left(\alpha_n^{''}(a,k), \beta_n^{''}(a,k)\right) \to \cdots \tag{1.18}$$

and

$$(\alpha_n(a,k), \beta_n(a,k)) \to \left(\widetilde{\alpha_n}(a,k), \widetilde{\beta_n}(a,k)\right) \to \left(\widetilde{\alpha_n^{'}}(a,k), \widetilde{\beta_n^{'}}(a,k)\right) \to \cdots \tag{1.19}$$

The above two Bailey chains together form a binary tree originated from the initial pair $(\alpha_n(a,k), \beta_n(a,k))$. The repeated iteration of WP-Bailey lemma (1.13) with the different WP-Bailey pairs produced the well-known classical Jackson ${}_8\phi_7$ summation theorem [10, p. 356, Equation (II.22)], Bailey's ${}_{10}\phi_9$ identity [10, p. 363, Equation (III.28)] etc. (see [3]) which were not possible to derive with the help of standard Bailey lemma (1.8). Later on, many researchers, for example, [1, 4, 7, 8, 13, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 28, 30, 32, 39, 40, 41, 43, 44] and references therein have explored the concept of WP-Bailey lemma in different ways to produce new q-hypergeometric transformations and various types of combinatorial results containing mock-theta functions and elliptic functions. Specifically, Srivastava et al. [39] used the WP-Bailey pair and its conjugate pair to investigate many transformations and q-series identities. In a sequel, Srivastava et al. [40] investigated a derived WP-Bailey pair which was the limiting case of the WP-Bailey pair and utilized it to establish the families of derived WP-Bailey pairs and their applications in establishing many q-transformations. Recently, [41] studied different forms of Bailey transform and utilized them to establish some transformation formulas for a q-hypergeometric series as well as bi-basic hypergeometric series, together with several other related useful q-series identities.

1.3. First Bailey Type Transform (*FBTT*) by Joshi and Vyas [12]

Joshi and Vyas [11] gave new dimensions to Bailey transform (Equations (1.5) to (1.7)) by investigating its two different extensions with the help of series rearrangement technique [29, 37] and shown its efficiency in deriving several transformations and summations for ordinary and q-hypergeometric functions and the double series *Rogers-Ramanujan type identities*. Recently, Joshi and Vyas [12] developed two new series transforms of Bailey type, one of which will be utilized in this paper and is as follows:

First Bailey Type Transform (*FBTT*): If

$$\beta_{(n,l)} = \sum_{m=0}^{min(n,l)} \alpha_m u_{n-m} u'_{l-m} v_{n+m} v'_{l+m} t_{n-l} w_{l+n} \tag{1.20}$$

and

$$\gamma_m = \sum_{n\geq 0} \sum_{l\geq 0} \delta_{n+m} \delta'_{l+m} u_n u'_l v_{n+2m} v'_{l+2m} t_{n+m-l} w_{l+n+m}, \tag{1.21}$$

then, subject to convergence conditions

$$\sum_{m\geq 0} \alpha_m \gamma_m = \sum_{n,l\geq 0} \beta_{(n,l)} \delta_n \delta'_l. \tag{1.22}$$

Joshi and Vyas [12] utilized their Bailey type transforms in investigating some q-hypergeometric identities involving very well-poised ${}_{10}\phi_9$ or ${}_{12}\phi_{11}$ series. In addition, they obtained two Bailey type lemmas (that is, First Bailey Type Lemma (FBTL) and Second Bailey Type Lemma (SBTL)) and then applied the lemmas to investigate some q-hypergeometric identities involving very well-poised ${}_{10}\phi_9$ or ${}_{12}\phi_{11}$ as well as the *multiple series Rogers-Ramanujan type identities* [12, pp. 11–26]. They also explained the construction of two Bailey type chains. Zhang and Song [45] obtained two identities related to mock theta functions by employing the 2-fold Bailey lemma or FBTL. Further, in a series of papers, Patkowski [25, 26, 27, 28] utilized the new Bailey type pairs investigated by [12] to give applications in the field of number theory.

Although, Andrews [3] investigated the WP-Bailey lemma from the Bailey transform (Equations (1.5) to (1.7)), the Andrews' type WP-Bailey lemma from FBTT (Equations (1.20) to (1.22)), has not been investigated till now. It will be our endeavour in this research paper to obtain and study Andrews' type WP-Bailey lemma, WP-Bailey pairs and to point out their role in deriving a class of q-hypergeometric identities.

The proof of Andrews' type WP-Bailey lemma, new WP-Bailey type pair, and the generated chain is given in Section 2. The applications of Andrews' type WP-Bailey lemma in deriving the terminating multiple series hypergeometric identities are discussed in Section 3. In Section 4, we shall construct the alternative WP-Bailey type pair satisfying Andrews' type WP-Bailey lemma and the corresponding WP-Bailey type chain.

2. Andrews' Type WP-Bailey Lemma or *WP-BTL*

Theorem 2.1. *If $n, l \geq 0$, then sequences $\alpha_m(a,k,k')$ and $\beta_{(n,l)}(a,k,k')$ form an Andrews' Type WP-Bailey Pair or WP-BTP, provided they satisfy the following Andrews' Type WP-Bailey Lemma or WP-BTL:*

$$\beta_{(n,l)}(a,k,k') = \sum_{m=0}^{min(n,l)} \frac{\left(\frac{k}{a};q\right)_{n-m} \left(\frac{k'}{a};q\right)_{l-m} (k;q)_{n+m} (k';q)_{l+m}}{(q;q)_{n-m}(q;q)_{l-m}(aq;q)_{n+m}(aq;q)_{l+m}} \alpha_m(a,k,k'). \tag{2.1}$$

Also

$$\beta'_{(n,l)}(a,k,k') = \sum_{m=0}^{min(n,l)} \frac{\left(\frac{k}{a};q\right)_{n-m} \left(\frac{k'}{a};q\right)_{l-m} (k;q)_{n+m} (k';q)_{l+m}}{(q;q)_{n-m}(q;q)_{l-m}(aq;q)_{n+m}(aq;q)_{l+m}} \alpha'_m(a,k,k') \tag{2.2}$$

where

$$\alpha'_m(a,k,k') = \frac{(b,c,b',c';q)_m \left(\frac{a^2q^2}{bcb'c'}\right)^m}{\left(\frac{aq}{b},\frac{aq}{c},\frac{aq}{b'},\frac{aq}{c'};q\right)_m} \alpha_m\left(a,\frac{kbc}{aq},\frac{k'b'c'}{aq}\right), \tag{2.3}$$

and

$$\beta'_{(M,N)}(a,k,k') = \sum_{n,l\geq 0} \frac{\left(\frac{kb}{a},\frac{kc}{a};q\right)_M}{\left(\frac{aq}{b},\frac{aq}{c};q\right)_M} \frac{\left(\frac{k'b'}{a},\frac{k'c'}{a};q\right)_N}{\left(\frac{aq}{b'},\frac{aq}{c'};q\right)_N} \cdot \frac{\left(q\sqrt{\frac{kbc}{aq}},-q\sqrt{\frac{kbc}{aq}},b,c;q\right)_n}{\left(\sqrt{\frac{kbc}{aq}},-\sqrt{\frac{kbc}{aq}},\frac{kb}{a},\frac{kc}{a};q\right)_n} \frac{\left(\frac{aq}{bc};q\right)_{M-n}(k;q)_{M+n}\left(\frac{aq}{bc}\right)^n}{\left(\frac{kbc}{a};q\right)_{M+n}(q;q)_{M-n}}$$

$$\cdot \frac{\left(q\sqrt{q\frac{k'b'c'}{aq}},-q\sqrt{\frac{k'b'c'}{aq}},b',c';q\right)_l}{\left(\sqrt{\frac{k'b'c'}{aq}},-\sqrt{\frac{k'b'c'}{aq}},\frac{k'b'}{a},\frac{k'c'}{a};q\right)_l} \frac{\left(\frac{aq}{b'c'};q\right)_{N-l}(k';q)_{N+l}\left(\frac{aq}{b'c'}\right)^l}{\left(\frac{k'b'c'}{a};q\right)_{N+l}(q;q)_{N-l}} \beta_{(n,l)}\left(a,\frac{kbc}{aq},\frac{k'b'c'}{aq}\right). \tag{2.4}$$

Proof. For the choices

$$\delta_r = \frac{\left(q\sqrt{\frac{kbc}{aq}},-q\sqrt{\frac{kbc}{aq}},b,c,kq^M,q^{-M};q\right)_r q^r}{\left(q\sqrt{\frac{kbc}{aq}},-q\sqrt{\frac{kbc}{aq}},\frac{kb}{a},\frac{kc}{a},\frac{kbcq^M}{a},\frac{bcq^{-M}}{a};q\right)_r},$$

$$\delta'_r = \frac{\left(q\sqrt{\frac{k'b'c'}{aq}},-q\sqrt{\frac{k'b'c'}{aq}},b',c',k'q^N,q^{-N};q\right)_r q^r}{\left(\sqrt{\frac{k'b'c'}{aq}},-\sqrt{\frac{k'b'c'}{aq}},\frac{k'b'}{a},\frac{k'c'}{a},\frac{k'b'c'q^N}{a},\frac{b'c'q^{-N}}{a};q\right)_r},$$

$$u_r = \frac{\left(\frac{kbc}{a^2q};q\right)_r}{(q;q)_r}, \quad u'_r = \frac{\left(\frac{k'b'c'}{a^2q};q\right)_r}{(q;q)_r},$$

$$v_r = \frac{\left(\frac{kbc}{aq};q\right)_r}{(aq;q)_r}, \quad v'_r = \frac{\left(\frac{k'b'c'}{aq};q\right)_r}{(aq;q)_r},$$

and $t_r = w_r = 1$ in (1.21) and then, the use of Jackson summation theorem [10, p. 238, Equation (II. 22)], leads to

$$\gamma_m = \frac{\left(\frac{kbc}{a},\frac{aq}{b},\frac{aq}{c};q\right)_M}{\left(aq,\frac{kb}{a},\frac{kc}{a},\frac{aq}{bc};q\right)_M} \frac{\left(\frac{k'b'c'}{a},\frac{aq}{b'},\frac{aq}{c'};q\right)_N}{\left(aq,\frac{k'b'}{a},\frac{k'c'}{a},\frac{aq}{b'c'};q\right)_N}$$
$$\cdot \frac{\left(b,c,kq^M,q^{-M},b',c',k'q^N,q^{-N};q\right)_m}{\left(\frac{aq}{b},\frac{aq}{c},aq^{1+M},\frac{aq}{b'},\frac{aq}{c'},aq^{1+N};q\right)_m} \left(\frac{k}{a};q\right)_{M-m}\left(\frac{k'}{a};q\right)_{N-m} q^{-m^2+m}\left(\frac{a^2q^{M+N+2}}{bcb'c'}\right)^m. \tag{2.5}$$

Now, we shall discuss the proof of (2.2).
Consider the right side of (2.2) as:

$$\Omega = \sum_{m=0}^{min(M,N)} \frac{\left(\frac{k}{a};q\right)_{M-m}\left(\frac{k'}{a};q\right)_{N-m}(k;q)_{M+m}(k';q)_{N+m}}{(q;q)_{M-m}(q;q)_{N-m}(aq;q)_{M+m}(aq;q)_{N+m}} \alpha_m{}'(a,k,k'). \tag{2.6}$$

Substituting the value of $\alpha'_m(a,k,k')$ from (2.3), we get

$$\Omega = \sum_{m=0}^{min(M,N)} \frac{\left(\frac{k}{a};q\right)_{M-m}\left(\frac{k'}{a};q\right)_{N-m}(k;q)_{M+m}(k';q)_{N+m}}{(q;q)_{M-m}(q;q)_{N-m}(aq;q)_{M+m}(aq;q)_{N+m}} \frac{(b,c,b',c';q)_m\left(\frac{a^2q^2}{bcb'c'}\right)^m}{\left(\frac{aq}{b},\frac{aq}{c},\frac{aq}{b'},\frac{aq}{c'};q\right)_m} \alpha_m\left(a,\frac{kbc}{aq},\frac{k'b'c'}{aq}\right) \tag{2.7}$$

$$= \frac{(k;q)_M\,(k';q)_N}{(aq,q;q)_M\,(aq,q;q)_N} \sum_{m=0}^{min(M,N)} \frac{\left(\frac{k}{a};q\right)_{M-m}\left(\frac{k'}{a};q\right)_{N-m}}{(aq^{1+M},aq^{1+N};q)_m} \frac{\left(b,c,kq^M,q^{-M},b',c',k'q^N,q^{-N};q\right)_m}{\left(\frac{aq}{b},\frac{aq}{c},\frac{aq}{b'},\frac{aq}{c'};q\right)_m} \tag{2.8}$$

$$\cdot\, q^{-m^2+m}\left(\frac{a^2q^{M+N+2}}{bcb'c'}\right)^m \alpha_m\left(a,\frac{kbc}{aq},\frac{k'b'c'}{aq}\right)$$

$$= \frac{\left(k,\frac{kb}{a},\frac{kc}{a},\frac{aq}{bc};q\right)_M}{\left(\frac{kbc}{a},\frac{aq}{b},\frac{aq}{c},q;q\right)_M} \frac{\left(k',\frac{k'b'}{a},\frac{k'c'}{a},\frac{aq}{b'c'};q\right)_N}{\left(\frac{k'b'c'}{a},\frac{aq}{b'},\frac{aq}{c'},q;q\right)_N} \sum_{m=0}^{min(M,N)} \alpha_m\left(a,\frac{kbc}{aq},\frac{k'b'c'}{aq}\right)\gamma_m \tag{2.9}$$

$$= \frac{\left(k,\frac{kb}{a},\frac{kc}{a},\frac{aq}{bc};q\right)_M}{\left(\frac{kbc}{a},\frac{aq}{b},\frac{aq}{c},q;q\right)_M} \frac{\left(k',\frac{k'b'}{a},\frac{k'c'}{a},\frac{aq}{b'c'};q\right)_N}{\left(\frac{k'b'c'}{a},\frac{aq}{b'},\frac{aq}{c'},q;q\right)_N} \sum_{n=0}^{M}\sum_{l=0}^{N}\beta_{(n,l)}\left(a,\frac{kbc}{aq},\frac{k'b'c'}{aq}\right)\delta_n\delta'_l \tag{2.10}$$

$$= \frac{\left(k,\frac{kb}{a},\frac{kc}{a},\frac{aq}{bc};q\right)_M}{\left(\frac{kbc}{a},\frac{aq}{b},\frac{aq}{c},q;q\right)_M} \frac{\left(k',\frac{k'b'}{a},\frac{k'c'}{a},\frac{aq}{b'c'};q\right)_N}{\left(\frac{k'b'c'}{a},\frac{aq}{b'},\frac{aq}{c'},q;q\right)_N} \sum_{n=0}^{M}\sum_{l=0}^{N}\beta_{(n,l)}\left(a,\frac{kbc}{aq},\frac{k'b'c'}{aq}\right)$$

$$\cdot \frac{\left(q\sqrt{\frac{kbc}{aq}},-q\sqrt{\frac{kbc}{aq}},b,c,kq^M,q^{-M};q\right)_n q^n}{\left(\sqrt{\frac{kbc}{aq}},-\sqrt{\frac{kbc}{aq}},\frac{kb}{a},\frac{kc}{a},\frac{kbcq^M}{a},\frac{bcq^{-M}}{a};q\right)_n} \tag{2.11}$$

$$\cdot \frac{\left(q\sqrt{\frac{k'b'c'}{aq}},-q\sqrt{\frac{k'b'c'}{aq}},b',c',k'q^M,q^{-N};q\right)_l q^l}{\left(\sqrt{\frac{k'b'c'}{aq}},-\sqrt{\frac{k'b'c'}{aq}},\frac{k'b'}{a},\frac{k'c'}{a},\frac{k'b'c'q^N}{a},\frac{b'c'q^{-N}}{a};q\right)_l}$$

$$= \frac{\left(\frac{kb}{a},\frac{kc}{a};q\right)_M}{\left(\frac{aq}{b},\frac{aq}{c};q\right)_M} \frac{\left(\frac{k'b'}{a},\frac{k'c'}{a};q\right)_N}{\left(\frac{aq}{b'},\frac{aq}{c'};q\right)_N} \sum_{n=0}^{M}\sum_{l=0}^{N}\beta_{(n,l)}\left(a,\frac{kbc}{aq},\frac{k'b'c'}{aq}\right)$$

$$\cdot \frac{\left(q\sqrt{\frac{kbc}{aq}},-q\sqrt{\frac{kbc}{aq}},b,c;q\right)_n}{\left(\sqrt{\frac{kbc}{aq}},-\sqrt{\frac{kbc}{aq}},\frac{kb}{a},\frac{kc}{a};q\right)_n}$$

$$\cdot \frac{\left(q\sqrt{\frac{k'b'c'}{aq}},-q\sqrt{\frac{k'b'c'}{aq}},b',c';q\right)_l}{\left(\sqrt{\frac{k'b'c'}{aq}},-\sqrt{\frac{k'b'c'}{aq}},\frac{k'b'}{a},\frac{k'c'}{a};q\right)_l} \frac{(k;q)_{M+m}\left(\frac{aq}{bc};q\right)_{M-n}\left(\frac{aq}{bc}\right)^n}{\left(\frac{kbc}{a};q\right)_{M+n}(q;q)_{M-m}} \frac{(k';q)_{N+l}\left(\frac{aq}{bc};q\right)_{N-l}\left(\frac{aq}{bc}\right)^l}{\left(\frac{k'b'c'}{a};q\right)_{N+l}(q;q)_{N-l}} = \beta'_{(M,N)}(a,k,k') \tag{2.12}$$

□

Remark 2.1. If $\alpha_m(a,k,k')$ and $\beta_{(n,l)}(a,k,k')$ form a WP-BTP, then so do $\alpha'_m(a,k,k')$ and $\beta'_{(n,l)}(a,k,k')$, which we refer to as new WP-BTP and are given by (2.3) and (2.4).

Remark 2.2. Taking $k,\ k'=0$ in Theorem 2.1 leads to the First Bailey type lemma due to [12].

Remark 2.3. The notion of repeated application of WP-BTL gives rise to the concept of WP-Bailey type chain which takes initial WP-BTP, that is,

$(\alpha_m(a,k,k'),\beta_{(n,l)}(a,k,k'))$ where

$$\alpha_m(a,k,k')=\frac{\left(a,q\sqrt{a},-q\sqrt{a},\frac{a^2q}{kk'};q\right)_m\left(\frac{kk'}{a^2q}\right)^m}{\left(a,\sqrt{a},-\sqrt{a},\frac{kk'}{a};q\right)_m} \tag{2.13}$$

and

$$\beta_{(n,l)}(a,k,k')=\frac{\left(k,\frac{aq}{k'};q\right)_n\left(k',\frac{aq}{k};q\right)_l}{\left(q,\frac{kk'}{a};q\right)_n\left(q,\frac{kk'}{a};q\right)_l}\frac{\left(\frac{kk'}{a};q\right)_{n+l}\left(\frac{k}{a};q\right)_{n-l}\left(\frac{kk'}{a^2q}\right)^l}{(aq;q)_{n+l}\left(\frac{aq}{k'};q\right)_{n-l}} \tag{2.14}$$

as input and generates new WP-BTP $\left(\alpha'_m(a,k,k'),\beta'_{(n,l)}(a,k,k')\right)$. Now, by taking $\left(\alpha'_m(a,k,k'),\beta'_{(n,l)}(a,k,k')\right)$ as initial WP-BTP in WP-BTL, we can obtain next pair as $\left(\alpha''_m(a,k,k'),\beta''_{(n,l)}(a,k,k')\right)$. Continuing this process, we obtain a sequence of WP-BTP's called WP-Bailey type chain as:

$$\begin{aligned}&(\alpha_m(a,k,k'),\beta_{(n,l)}(a,k,k'))\to\left(\alpha'_m(a,k,k'),\beta'_{(n,l)}(a,k,k')\right)\to\\&\left(\alpha''_m(a,k,k'),\beta''_{(n,l)}(a,k,k')\right)\to\cdots.\end{aligned} \tag{2.15}$$

3. Certain Terminating q-hypergeometric Identities

In this section, we provide certain terminating multiple series identities involving ${}_{12}\phi_{11}$ and ${}_{16}\phi_{15}$, as applications of WP-BTL.

Theorem 3.1. *The following assertion holds true:*

$$\frac{\left(\frac{k}{a},\frac{kbc}{a},\frac{aq}{b},\frac{aq}{c};q\right)_M}{\left(aq,\frac{kc}{a},\frac{kb}{a},\frac{aq}{bc};q\right)_M}\frac{\left(\frac{k'}{a},\frac{k'b'c'}{a},\frac{aq}{b'},\frac{aq}{c'};q\right)_N}{\left(aq,\frac{k'c'}{a},\frac{k'b'}{a},\frac{aq}{b'c'};q\right)_N}$$

$$\cdot\ {}_{12}\Phi_{11}\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},b,c,b',c',\mu,kq^M,k'q^N,q^{-M},q^{-N}\ ;\\ \sqrt{a},-\sqrt{a},\frac{aq}{b},\frac{aq}{c},\frac{aq}{b'},\frac{aq}{c'},\frac{aq}{\mu},\frac{aq^{1-M}}{k},\frac{aq^{1-N}}{k'},aq^{1+M},aq^{1+N}\ ;\end{matrix}\ q,\ q\right]$$

$$=\sum_{n,l\geq 0}\frac{\left(\frac{aq}{\mu};q\right)_{n+l}\left(\frac{kbc}{a^2q};q\right)_{n-l}}{(aq;q)_{n+l}\left(\frac{a^2q^2}{k'b'c'};q\right)_{n-l}}$$

$$\cdot\frac{\left(q\sqrt{\frac{kbc}{aq}},-q\sqrt{\frac{kbc}{aq}},b,c,\frac{kbc}{aq},\frac{a^2q^2}{k'b'c'},kq^M,q^{-M};q\right)_n q^n}{\left(q,\sqrt{\frac{kbc}{aq}},-\sqrt{\frac{kbc}{aq}},\frac{kb}{a},\frac{kc}{a},\frac{kbcq^M}{a},\frac{bcq^{-M}}{a},\frac{aq}{\mu};q\right)_n}$$

$$\cdot \frac{\left(q\sqrt{\frac{k'b'c'}{aq}}, -q\sqrt{\frac{k'b'c'}{aq}}, b', c', \frac{k'b'c'}{aq}, \frac{a^2q^2}{kbc}, k'q^N, q^{-N}; q\right)_l \left(\frac{aq}{\mu}\right)^l}{\left(\sqrt{\frac{k'b'c'}{aq}}, -\sqrt{\frac{k'b'c'}{aq}}, \frac{k'b'}{a}, \frac{k'c'}{a}, \frac{k'b'c'q^N}{a}, \frac{b'c'q^{-N}}{a}, \frac{aq}{\mu}; q\right)_l} \tag{3.1}$$

where $\mu = \dfrac{a^4q^3}{kk'bcb'c'}$

Proof. The q-hypergeometric summation in Theorem 3.1 is the outcome of the assertion that the *WP-BTP* $\left(\alpha'_m(a,k,k'), \beta'_{(n,l)}(a,k,k')\right)$ satisfies the *WP-BTL* (Theorem 2.1) where the initial pair $(\alpha_m(a,k,k'), \beta_{(n,l)}(a,k,k'))$ is given by (2.13) and (2.14). □

Remark 3.1. The case $k = \dfrac{Aaq}{bc}$ and $k' = \dfrac{Baq}{b'c'}$ of Theorem 3.1 leads to a hypergeometric identity due to [10, Equation (3.2)].

Theorem 3.2. *The following assertion holds true:*

$$\begin{aligned}
&{}_{16}\Phi_{15}\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, b', c', d, e, d', e', \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{b'}, \frac{aq}{c'}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{d'}, \frac{aq}{e'}, \end{matrix}\right. \\
&\qquad\qquad\left.\begin{matrix} \nu, kq^M, k'q^N, q^{-M}, q^{-N}; \\ \frac{aq}{\nu}, \frac{aq^{1-M}}{k}, \frac{aq^{1-N}}{k'}, aq^{1+M}, aq^{1+N}; \end{matrix} q\right] \\
&= \frac{\left(aq, \frac{kd}{a}, \frac{ke}{a}, \frac{aq}{de}; q\right)_M}{\left(\frac{aq}{d}, \frac{aq}{e}, \frac{kde}{a}, \frac{k}{a}; q\right)_M} \frac{\left(aq, \frac{k'd'}{a}, \frac{k'e'}{a}, \frac{aq}{d'e'}; q\right)_N}{\left(\frac{aq}{d'}, \frac{aq}{e'}, \frac{k'd'e'}{a}, \frac{k'}{a}; q\right)_N} \\
&\quad \cdot \sum_{i,j\geq 0} \frac{\left(q\sqrt{\frac{kde}{aq}}, -q\sqrt{\frac{kde}{aq}}, d, e, kq^M, q^{-M}; q\right)_i q^i}{\left(\sqrt{\frac{kde}{aq}}, -\sqrt{\frac{kde}{aq}}, \frac{kd}{a}, \frac{ke}{a}, \frac{kdeq^M}{a}, \frac{deq^{-M}}{a}; q\right)_i} \\
&\quad \cdot \frac{\left(q\sqrt{\frac{k'd'e'}{aq}}, -q\sqrt{\frac{k'd'e'}{aq}}, d', e', k'q^N, q^{-N}; q\right)_j q^j}{\left(\sqrt{\frac{k'd'e'}{aq}}, -\sqrt{\frac{k'd'e'}{aq}}, \frac{k'd'}{a}, \frac{k'e'}{a}, \frac{k'd'e'q^N}{a}, \frac{d'e'q^{-N}}{a}; q\right)_j}
\end{aligned} \tag{3.2}$$

$$\cdot \sum_{n,l\geq 0} \frac{\left(\frac{kcde}{a^2q}, \frac{kbde}{a^2q}, \frac{kde}{aq}, \frac{aq}{bc}; q\right)_i}{\left(q, \frac{aq}{b}, \frac{aq}{c}, \frac{kbcde}{a^2q}; q\right)_i} \frac{\left(\frac{k'c'd'e'}{a^2q}, \frac{k'b'd'e'}{a^2q}, \frac{k'd'e'}{aq}, \frac{aq}{b'c'}; q\right)_j}{\left(q, \frac{aq}{b'}, \frac{aq}{c'}, \frac{k'b'c'd'e'}{a^2q}; q\right)_j}$$

$$\cdot \frac{\left(q\sqrt{\frac{kbcde}{a^2q^2}}, -q\sqrt{\frac{kbcde}{a^2q^2}}, b, c, \frac{kbcde}{a^2q^2}, \frac{a^3q^3}{k'b'c'd'e'}, \frac{kde}{aq}q^i, q^{-i}; q\right)_n q^n}{\left(q, \sqrt{\frac{kbcde}{a^2q^2}}, -\sqrt{\frac{kbcde}{a^2q^2}}, \frac{kcde}{a^2q}, \frac{kbde}{a^2q}, \frac{aq}{\nu}, \frac{kbcde}{a^2q}q^i, \frac{bcq^{-i}}{a}; q\right)_n}$$

$$\cdot \frac{\left(q\sqrt{\frac{k'b'c'd'e'}{a^2q^2}}, -q\sqrt{\frac{k'b'c'd'e'}{a^2q^2}}, b', c', \frac{k'b'c'd'e'}{a^2q^2}; q\right)_l}{\left(q, \sqrt{\frac{k'b'c'd'e'}{a^2q^2}}, -\sqrt{\frac{k'b'c'd'e'}{a^2q^2}}, \frac{k'c'd'e'}{a^2q}, \frac{k'b'd'e'}{a^2q}, \frac{aq}{\nu}; q\right)_l}$$

$$\cdot \frac{\left(\frac{a^3q^3}{kbcde}, \frac{k'd'e'}{aq}q^j, q^{-j}; q\right)_l q^l \left(\frac{1}{\nu}\right)^l}{\left(\frac{k'b'c'd'e'}{a^2q}q^i, \frac{b'c'q^{-j}}{a}; q\right)_l} \frac{\left(\frac{aq}{\nu}; q\right)_{n+l} \left(\frac{kbcde}{a^3q^2}; q\right)_{n-l}}{(aq;q)_{n+l} \left(\frac{a^3q^3}{k'b'c'd'e'}; q\right)_{n-l}}$$

where $\nu = \dfrac{a^6q^5}{kk'bcb'c'ded'e'}$

Proof. First, we obtain the *WP-BTP* $\left(\alpha''_m(a,k,k'), \beta''_{(n,l)}(a,k,k')\right)$ using equations (2.3) and (2.4) with new parameters d, e, d' and e' and then apply it to *WP-BTL* (Theorem 2.1). Further simplification using the pair $\left(\alpha'_m(a,k,k'), \beta'_{(n,l)}(a,k,k')\right)$ defined in the previous theorem completes the proof of the above theorem. □

4. Alternative WP-Bailey Type Pairs

In the present section, we obtain alternative WP-Bailey type pairs or alternative WP-BTP, which cannot be reduced to (2.3) and (2.4) by any means.

Theorem 4.1. *If* $n, l \geq 0$ *and the two sequences* $(\alpha_m(a,k,k'))$ *and* $\beta_{(n,l)}(a,k,k')$ *form a WP-BTP, then so do* $\widetilde{\alpha_m}(a,k,k')$ *and* $\widetilde{\beta_{(n,l)}}(a,k,k')$*, where*

$$\widetilde{\alpha_m}(a,k,k') = \frac{\left(\frac{a^2q}{k}, \frac{a^2q}{k'}; q\right)_{2m} \left(\frac{k^2k'^2}{a^4q^2}\right)^m}{(k,k';q)_{2m}} \alpha_m\left(a, \frac{a^2q}{k}, \frac{a^2q}{k'}\right) \tag{4.1}$$

and

$$\widetilde{\beta_{(M,N)}}(a,k,k') = \sum_{n,l\geq 0} \frac{\left(\frac{k^2}{a^2q}; q\right)_{M-n}, \left(\frac{k'^2}{a^2q}; q\right)_{N-l} \left(\frac{k^2}{a^2q}\right)^n \left(\frac{k'^2}{a^2q}\right)^l}{(q;q)_{M-n}(q;q)_{N-l}} \cdot \beta_{(n,l)}\left(a, \frac{a^2q}{k}, \frac{a^2q}{k'}\right). \tag{4.2}$$

Proof. Following the method similar to the proof of Theorem 2.1 and applying the q-Saalschütz summation theorem twice, the assertion in Theorem 4.1 can be deduced easily. □

Remark 4.1. The first iteration of the investigated lemma mentioned in Theorem 2.1 with the pair $\left(\widetilde{\alpha_m}(a,k,k'), \widetilde{\beta_{(n,l)}}(a,k,k')\right)$ gives the following identity:

$$\frac{\left(k,\frac{k}{a};q\right)_M\left(k',\frac{k'}{a};q\right)_N}{\left(aq,\frac{k^2}{a^2q};q\right)_M\left(aq,\frac{k'^2}{a^2q};q\right)_N}\ {}_{16}\Phi_{15}\left[\begin{matrix} a,q\sqrt{a},-q\sqrt{a},\sqrt{\frac{a^2q}{k}},-\sqrt{\frac{a^2q}{k}},\sqrt{\frac{a^2q^2}{k}},\\ \sqrt{a},-\sqrt{a},\sqrt{kq},-\sqrt{kq},\sqrt{k},-\sqrt{k},\end{matrix}\right.$$

$$\left.\begin{matrix} -\sqrt{\frac{a^2q^2}{k}},\sqrt{\frac{a^2q}{k'}},-\sqrt{\frac{a^2q}{k'}},\sqrt{\frac{a^2q^2}{k'}},-\sqrt{\frac{a^2q^2}{k'}},\frac{kk'}{a^2q},kq^M,k'q^N,q^{-M},q^{-N};\\ \sqrt{k'},-\sqrt{k'},\sqrt{k'q},-\sqrt{k'q},\frac{a^3q^2}{kk'},\frac{aq^{1-M}}{k},\frac{aq^{1-N}}{k'},aq^{1+M},aq^{1+N};\end{matrix}\ q\right]$$

$$=\sum_{n,l\geq 0}\frac{\left(\frac{a^2q}{k},\frac{k'}{a},q^{-M};q\right)_n\left(\frac{a^2q}{k'},\frac{k}{a},q^{-N};q\right)_l}{\left(q,\frac{a^3q^2}{kk'},\frac{a^2q^{2-M}}{k^2};q\right)_n\left(q,\frac{a^3q^2}{kk'},\frac{a^2q^{2-N}}{k'^2};q\right)_l}$$

$$\cdot\frac{\left(\frac{a^3q^2}{kk'};q\right)_{n+l}\left(\frac{aq}{k};q\right)_{n-l}}{(aq;q)_{n+l}\left(\frac{k'}{a};q\right)_{n-l}}q^n\left(\frac{a^2q^2}{kk'}\right)^l \tag{4.3}$$

Further iteration of the lemma mentioned in Theorem 2.1 produces many additional identities similar to (4.1).

Remark 4.2. Now, the alternative WP-Bailey type chain generated from the initial WP-Bailey pair $(\alpha_m(a,k,k'),\beta_{(n,l)}(a,k,k'))$ is as follows:

$$\begin{aligned}&(\alpha_m(a,k,k'),\beta_{(n,l)}(a,k,k'))\to\left(\widetilde{\alpha_m}(a,k,k'),\widetilde{\beta_{(n,l)}}(a,k,k')\right)\to\\&\left(\widetilde{\alpha'_m}(a,k,k'),\widetilde{\beta'_{(n,l)}}(a,k,k')\right)\to\cdots\end{aligned} \tag{4.4}$$

The two types of Bailey chains (Equations (2.15) and (4.4)) are generated by a single WP-BTP (initial pair) $(\alpha_m(a,k,k'),\beta_{(n,l)}(a,k,k'))$ and thus forms a binary tree.

5. Conclusion

The *FBTT* (Equations (1.20) to (1.22)) developed by Joshi and Vyas [12] is explored in the study of *WP-BTL*, pairs, chains and a binary tree. The *WP-BTL* is then utilized to obtain the terminating very well-poised q-multiple hypergeometric identities of order higher than the identities due to Joshi and Vyas [12]. Furthermore, by the repeated application of investigated *WP-BTL*, one can develop the $2(p-2)$ fold identities along the line of Andrews [3].

Basic (or q-) series and basic (or q-) polynomials, especially the basic (or q-) gamma and q-hypergeometric functions and basic (or q-) hypergeometric polynomials, are applicable particularly in several diverse areas (see, for example, [36, pp. 350–351] and [34, p. 328]). Moreover, in this recently-published survey-cum-expository review article by Srivastava [34], the so-called (p,q)-calculus was exposed to be a rather trivial and inconsequential variation of the classical q-calculus, the additional parameter p being redundant (see, for details, [33, p. 340] and [35, pp. 1511–1512]). This observation by Srivastava [33, 35] will indeed apply also to any future attempt to produce the rather straightforward (p,q)-variants of the results which we have presented in this paper.

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SOME RESULTS ON VALUE DISTRIBUTION THEORY FOR *E*-VALUED MEROMORPHIC FUNCTION

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Abstract

In this paper, we investigate analogous of Milloux inequality and Hayman's alternative for *E*-valued meromorphic functions from the complex plane $\mathbb{C}$ to an infinite dimensional complex Banach space *E* with a Schauder basis. As an application of our results, we deduce some interesting analogous results for *E*-valued meromorphic functions from the complex plane $\mathbb{C}$ to an infinite dimensional complex Banach space *E* with a Schauder basis. And also we have given the applications of homogeneous differential polynomials to the Nevanlinna's theory of *E*-valued meromorphic functions from the complex plane $\mathbb{C}$ to an infinite dimensional complex Banach space E with a Schauder basis and given some generalizations by these polynomials.

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1. Introduction

In the 1970s, the value distribution theory of meromorphic function is expanded to the vector-valued meromorphic function from the complex plane $\mathbb{C}$ to a finite dimensional space $\mathbb{C}^n$ (see Ziegler [25]). After that, some works related to vector-valued meromorphic function in finite dimensional spaces were done by several authors ([4]-[6],[7]-[19]). C. G. Hu and Q. J. Hu [2] investigated Nevanlinnas first and second fundamental theorems for an *E*-valued meromorphic function from the disk to infinite-dimensional Banach spaces *E* with a Schauder basis. Bhoosnurmath and Pujari [1] established some interesting results for the *E*-valued Borel exceptional values of meromorphic functions, Wu and Xuan [21, 22] proved remarkable results on the characteristic functions, exceptional values, and deficiency of *E*-valued meromorphic function, and Hu [3] proved the advancements of the Nevanlinna theory of E-valued meromorphic functions and studied its related Paley problems.

2. The Value Distribution Theory on Banach Spaces

Nevanlinna theory for *E*-valued meromorphic function will play a key role in the proof of theorems. We shall use standard notations of value distribution theory for *E*-valued meromorphic function, $V(a,\infty,f)$, $V(a,f)$ $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$,$T(r,f)$,...([1]-[6], [20]-[25]).

Theorem 2.1 ([2]). *(the E-valued Nevanlinna's first fundamental theorem). Let $f(z)$ be a non-constant E-valued meromorphic function in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$. then for $0 < r < R$, $a \in E$ and $f(z) \not\equiv a$,*

$$T(r,a) = V(r,a) + N(r,a) + m(r,a) + \log^+ \|c_q(a)\| + \varepsilon(r,a)$$

Here $\varepsilon(r,a)$ is a function such that

$$|\varepsilon(r,a)| \leq \log^+ \|a\| + \log 2, \quad \varepsilon(r,0) = 0$$

and $c_q(a) \in E$ is the coefficient of the first term in the Laurent series at the point a.

Theorem 2.2 ([2]). *(the E-valued Nevanlinnas second fundamental theorem). Let $f(z)$ be a nonconstant E-valued meromorphic function of compact projection in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$. then for $0 < r < R$ and $a_k \in E$ $(k = 1,2,...,q)$ be $q \geq 3$ distinct points. Then for $0 < r < R$,*

$$(q-2)T(r,f) + G(r,f) \leq \sum_{k=1}^{q}[V(r,a_k) + \overline{N}(r,a_k)] + S(r,f)$$

and

$$(q-1)T(r,f) + G(r,f) \leq \sum_{k=1}^{q}[V(r,a_k) + \overline{N}(r,a_k)] + \overline{N}(r,\infty) + S(r,f),$$

where

$$G(r,f) = V(r,0,f') = \int_0^r \frac{1}{2\pi t}dt\int_{C_r} \Delta \log \|f(\xi)\| dx \bigwedge dy, \quad \xi = x + iy.$$

or

Let $f(z)$ be a nonconstant E-valued meromorphic function of compact projection in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$. then for $0 < r < R$ and $a_k \in E$ $(k = 1,2,...,q)$ be $q \geq 3$ distinct points. Then for $0 < r < R$,

$$\sum_{k=1}^{q} m(r,a_k,f) \leq 2T(r,f) - N^1(r,f) + V(r,f) + S(r,f),$$

where $N^1(r,f) = 2N(r,f) - N(r.f') + N(r,1/f')$.

We use $\overline{N}^{k)}(r, 1/f - a_j)$ to denote the zeros of $f(z) - a$ whose multiplicities are no greater than k and are counted only once. Likewise, we use $\overline{C}^{(k+1}_{\alpha,\beta}(r, 1/f - a_j)$ to denote the zeros of $f(z) - a$ in $|z| < r$ whose multiplicities are greater than k and are counted only once.

3. Main Results

In the value distribution theory, it is very important to introduce and study the derivative of a given function. It is natural to ask whether can we establish the analogous of Milloux inequality and Hayman's alternative for E-valued meromorphic function of compact projection in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$.

By adopting the notations of Nevanlinna functions in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$, we proved the following theorems and establish an interesting and remarkable result of the Milloux inequality and Heyman's alternative in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$.

Theorem 3.1. *Let $f(z)$ be a transcendental E-valued meromorphic function of compact projection in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$. Let*

$$\Theta(z) = \textstyle\sum_{l=0}^{k} a_l f^{(l)}(z), \tag{3.1}$$

for any positive integer k, where $a_0, a_1, a_2, a_3, ..., a_k$ are small functions of $f(z)$. Then

$$m\left(r, \frac{\Theta}{f}\right) = S(r, f) \tag{3.2}$$

and

$$T(r, \Theta) \leq (k+1)T(r, f) + G(r, f) + S(r, f). \tag{3.3}$$

Theorem 3.2. *Let $f(z)$ be a transcendental E-valued meromorphic function of compact projection in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$ and $\Theta(z)$ be the function defined by (3.1). If $\Theta(z)$ is not a constant, then*

$$\begin{aligned} T(r, f) \quad < \quad & \left[\overline{N}(r, f) + V(r, f)\right] + [N\left(r, \frac{1}{f}\right)V(r, \frac{1}{f})] + [\overline{N}\left(r, \frac{1}{\Theta - a}\right) + V\left(r, \frac{1}{\Theta - a}\right)\Big] \\ & -\left[N^{(0)}\left(r, \frac{1}{\Theta'}\right) + V\left(r, \frac{1}{\Theta'}\right)\right] + S(r, f), \end{aligned} \tag{3.4}$$

where $(a \neq 0, \infty)$ and $N^{(0)}\left(r, \frac{1}{\Theta'}\right)$ counts only zeros of Θ' but not the repeated roots of $\Theta = a$ in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$.

Theorem 3.3. *Let $f(z)$ be a transcendental E-valued meromorphic function of compact projection in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$, $\Theta = f^{(k)}$ and $N^{(0)}\left(r, \frac{1}{\Theta'}\right)$ be defined as in Theorem 3.2. Then*

$$\begin{aligned} k[N^1(r, f) + V(r, f)] \quad \leq \quad & \left[\overline{N}^{(2}(r, f) + V(r, f)\right] + [\overline{N}\left(r, \frac{1}{\Theta - a}\right) + V\left(r, \frac{1}{\Theta - a}\right)\Big] \\ & + [N^{(0)}\left(r, \frac{1}{\Theta'}\right) + \left(r, \frac{1}{\Theta'}\right)] + S(r, f), \end{aligned} \tag{3.5}$$

where $N^1(r, f)$ counts the simple poles of $f(z)$ and $\overline{N}^{(2}(r, f)$ counts the multiple poles of $f(z)$, not including multiplicity in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$.

Theorem 3.4. *Let $f(z)$ be a transcendental E-valued meromorphic function of compact projection in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$. Then*

$$\begin{aligned} T(r, f) \quad \leq \quad & \left(2 + \frac{1}{k}\right)[N\left(r, \frac{1}{f}\right) + V\left(r, \frac{1}{f}\right)] \\ & + \left(2 + \frac{2}{k}\right)[\overline{N}\left(r, \frac{1}{\Theta - a}\right) + V\left(r, \frac{1}{\Theta - a}\right)] + S(r, f). \end{aligned} \tag{3.6}$$

By replacing $\Theta = f^{(k)}(z)$ in the Theorem 3.2, we get the following result.

Corollary 3.1. *Let $f(z)$ be a transcendental E-valued meromorphic function of compact projection in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$ and k is any positive integer. Then*

$$T(r, f) \quad \leq \quad [\overline{N}(r, f) + V(r, f)] + [N\left(r, \frac{1}{f}\right) + V\left(r, \frac{1}{f}\right)]$$

$$+[\overline{N}\left(r, \frac{1}{f^{(k)}-a}\right) + V\left(r, \frac{1}{f^{(k)}-a}\right)]$$
$$-[N^{(0)}\left(r, \frac{1}{f^{(k+1)}}\right) + V\left(r, \frac{1}{f^{(k+1)}}\right)] + S(r, f). \tag{3.7}$$

By Theorem 3.2*, we get the following Corollary.*

Corollary 3.2. *Let $f(z)$ be a transcendental E-valued meromorphic function of compact projection in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \le \infty$ with only a finite number of zeros and poles. Then every function Θ as defined in* (3.1) *assumes every finite complex value, except possibly zero, infinitely often or else is identically constant in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \le \infty$.*

By replacing $\Theta = f^{(k)}(z)$ in the Theorem 3.4*, we get the following result.*

Corollary 3.3. *Let $f(z)$ be a transcendental E-valued meromorphic function of compact projection in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \le \infty$. Then*

$$\begin{aligned} T(r, f) \quad \le \quad & \left(2 + \frac{1}{k}\right)[N\left(r, \frac{1}{f}\right) + V\left(r, \frac{1}{f}\right)] \\ & + \left(2 + \frac{2}{k}\right)[\overline{N}\left(r, \frac{1}{f^{(k)}-a}\right) + V\left(r, \frac{1}{f^{(k)}-a}\right)] + S(r, f). \end{aligned}$$

By replacing the value of $F = \frac{f-\omega_1}{\omega_2}$, where ω_1 and ω_2 be complex numbers $\omega_2 \neq 0$ and $T(r, F) = T(r, f) + O(1)$ in Theorem 3.4*. Then we get the following result.*

Corollary 3.4. *(Hayman's Alternative in annuli). Let $f(z)$ be a transcendental E-valued meromorphic function of compact projection in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \le \infty$. Then either f assumes every finite value infinitely often or $f^{(k)}$ assumes every finite value except possibly zero infinitely often in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \le \infty$.*

4. Proof of the main results

Proof of the Theorem 3.1.

First of all, we prove the Theorem 3.1 for the case $\Theta(z) = f^{(k)}$ using induction on the number k and then deduce the conclusion of the Theorem 3.1 for the general case.

By Theorem 2.1, we have

$$\begin{aligned} T(r, f') &= T\left(r, f\frac{f'}{f}\right) \le T(r, f) + T\left(r, \frac{f'}{f}\right) + O(1) \\ &= T(r, f) + m\left(r, \frac{f'}{f}\right) + N\left(r, \frac{f'}{f}\right) + V\left(r, \frac{f'}{f}\right) + O(1) \\ &\le T(r, f) + \overline{N}(r, f) + G(r, f) + S(r, f) \\ &\le 2T(r, f) + G(r, f) + S(r, f). \end{aligned}$$

Hence the result is true for $k = 1$.

Suppose that the theorem is true for $k = n$. Then by assumption, we have

$$m\left(r, \frac{f^{(n)}}{f}\right) = S(r, f) \tag{4.1}$$

and

$$T(r, f^{(n)}) \le (n+1)T(r, f) + G(r, f) + S(r, f). \tag{4.2}$$

Also we have,

$$m\left(r, f^{(n+1)}\right) = m\left(r, f^{(n)}\right) + m\left(r, \frac{f^{(n+1)}}{f^{(n)}}\right) \tag{4.3}$$

and

$$\begin{aligned} N(r, f^{(n+1)}) + V(r, f^{(n+1)}) &= N(r, f^{(n)}) + V(r, f^{(n)}) + \overline{N}(r, f^{(n)}) + V(r, f^{(n)}) \\ &= N(r, f^{(n)}) + V(r, f^{(n)}) + \overline{N}(r, f) + V(r, f) \\ &\le N(r, f^{(n)}) + N(r, f) + G(r, f). \end{aligned} \tag{4.4}$$

By Theorem 2.1, we have

$$\begin{aligned} m\left(r, \frac{f^{(n+1)}}{f}\right) &\leq m\left(r, \frac{f^{(n+1)}}{f^{(n)}}\right) + m\left(r, \frac{f^{(n)}}{f}\right) \\ &\leq S(r, f^{(n)}) + S(r, f) \\ &\leq S(r, f) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} T(r, f^{(n+1)}) &= m\left(r, f^{(n+1)}\right) + N(r, f^{(n+1)}) + V(r, f^{(n+1)}) \\ &\leq m(r, f^{(n)}) + m\left(r, \frac{f^{(n+1)}}{f^{(n)}}\right) \\ &\quad + N(r, f^{(n)}) + V(r, f^{(n)}) + N(r, f) + V(r, f) + O(1) \\ &\leq T(r, f^{(n)}) + N(r, f) + G(r, f) + S(r, f) \\ &\leq (n+1)T(r, f) + T(r, f) + G(r, f) + S(r, f) \\ &\leq (n+2)T(r, f) + G(r, f) + S(r, f). \end{aligned} \tag{4.6}$$

Hence the result is true for all positive integer k.
Now, we consider the general case.
By above case, it is obvious that

$$\begin{aligned} m\left(r, \frac{\Theta}{f}\right) &\leq \sum_{l=0}^{k} m\left(r, \frac{a_l f^{(l)}}{f}\right) + \log(k+1) \\ &\leq \sum_{l=0}^{k} m(r, a_l) + m\left(r, \frac{f^{(l)}}{f}\right) + \log(k+1) \leq S(r, f). \end{aligned} \tag{4.7}$$

Thus, we have

$$\begin{aligned} m(r, \Theta) &\leq m\left(r, \frac{\Theta}{f}\right) + m(r, f) \\ &\leq m(r, f) + S(r, f). \end{aligned} \tag{4.8}$$

On the other hand, we derive

$$\begin{aligned} N(r, \Theta) + V(r, \Theta) &\leq N\left(r, f^{(k)}\right) + V\left(r, f^{(k)}\right) \\ &\leq N(r, f) + V(r, f) + k\overline{N}(r, f) + V(r, f). \end{aligned} \tag{4.9}$$

Therefore, by (4.8) and (4.9), we have

$$\begin{aligned} T(r, \Theta) &= m(r, \Theta) + N(r, \Theta) + V(r, \Theta) \\ &\leq m(r, f) + N(r, f) + V(r, f) + k\overline{N}(r, f) + V(r, f) + S(r, f) \\ &\leq T(r, f) + k\overline{N}(r, f) + V(r, f) + S(R, f) \\ &\leq (k+1)T(r, f) + G(r, f) + S(r, f) \\ T(r, \Theta) &\leq (k+1)T(r, f) + G(r.f) + G(r, f) \end{aligned}$$

which completes the proof of Theorem 3.1.
Proof of the Theorem 3.2. By Theorem 2.2, we have

$$\begin{aligned} & m(r, \Theta) + m\left(r, \frac{1}{\Theta}\right) + m\left(R, \frac{1}{\Theta - a}\right) \\ & \leq 2T(r, \Theta) - N^{(1)}(r, f) + V(r, f) + S(r, \Theta). \end{aligned} \tag{4.10}$$

By Theorem 2.1, we have

$$\begin{aligned} & 2T(r, \Theta) - N^{(1)}(r, f) \\ = \; & m(r, \Theta) + m(r, a, \Theta) + N(r, \Theta) + V(r, \Theta) + N(r, a, \Theta) + V(r, a, \Theta) \\ & - \left[2[N(r, \Theta) + V(r, \Theta)] - N(r, \Theta') + V(r, \Theta') + N\left(r, \frac{1}{\Theta'}\right) + V\left(r, \frac{1}{\Theta'}\right)\right] \\ = \; & m(r, \Theta) + m(r, a, \Theta) + N(r, a, \Theta) + V(r, a, \Theta) \end{aligned}$$

$$-N\left(r,\frac{1}{\Theta'}\right)+V\left(r,\frac{1}{\Theta'}\right)+N\left(r,\Theta'\right)+V\left(r,\Theta'\right)-N\left(r,\Theta\right)+V\left(r,\Theta\right). \tag{4.11}$$

It is obvious that

$$N\left(r,\Theta'\right)+V\left(r,\Theta'\right)-\left[N\left(r,\Theta\right)+V\left(r,\Theta\right)\right] \quad \leq \quad \overline{N}\left(r,f\right)+V(r,f) \tag{4.12}$$

and

$$\begin{aligned} & N\left(r,\frac{1}{\Theta-a}\right)+V\left(r,\frac{1}{\Theta-a}\right)-[N\left(r,\frac{1}{\Theta'}\right)+V\left(r,\frac{1}{\Theta'}\right)] \\ &= \overline{N}\left(r,\frac{1}{\Theta-a}\right)+\left(r,\frac{1}{\Theta-a}\right)-[N^{(0)}\left(r,\frac{1}{\Theta'}\right)+V\left(r,\frac{1}{\Theta'}\right)]. \end{aligned} \tag{4.13}$$

Hence it follows from (4.10), (4.11), (4.12) and (4.13) that

$$\begin{aligned} m\left(r,\frac{1}{\Theta}\right) \quad \leq \quad & \overline{N}\left(r,f\right)+V(r,f)+\overline{N}\left(r,\frac{1}{\Theta-a}\right)+\left(r,\frac{1}{\Theta-a}\right) \\ & -[N^{0}\left(r,\frac{1}{\Theta'}\right)+V\left(r,\frac{1}{\Theta'}\right)]+S\left(R,\Theta\right). \end{aligned} \tag{4.14}$$

From (3.3), we have

$$S\left(r,\Theta\right)=S\left(r,f\right).$$

By Theorem 2.1, we obtain

$$\begin{aligned} T\left(r,f\right) \quad &= \quad m\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f}\right)+V\left(r,\frac{1}{f}\right)+\ O(1) \\ &\leq \quad m\left(r,\frac{1}{\Theta}\right)+m\left(r,\frac{\Theta}{f}\right) \\ &\quad\quad +N\left(r,\frac{1}{f}\right)+V\left(r,\frac{1}{f}\right)+\ O(1) \\ &\leq \quad m\left(r,\frac{1}{\Theta}\right)+N\left(r,\frac{1}{f}\right)+V\left(r,\frac{1}{f}\right)+S\left(r,f\right). \end{aligned} \tag{4.15}$$

From (4.14) and (4.15), we have

$$\begin{aligned} T\left(r,f\right) \quad \leq \quad & \overline{N}\left(r,f\right)+V(r,f)+N\left(r,\frac{1}{f}\right)+V\left(r,\frac{1}{f}\right) \\ & +\overline{N}\left(r,\frac{1}{\Theta-a}\right)+V\left(r,\frac{1}{\Theta-a}\right) \\ & -[N^{(0)}\left(r,\frac{1}{\Theta'}\right)+V\left(r,\frac{1}{\Theta'}\right)]+S\left(R,f\right) \end{aligned}$$

which completes the proof of Theorem 3.2.

Proof of the Theorem 3.3.

We first define the function

$$g=\frac{\left(f^{(k+1)}\right)^{k+1}}{\left(a-f^{(k)}\right)^{k+2}}=\frac{(\Theta')^{k+1}}{(a-\Theta)^{k+2}}. \tag{4.16}$$

Suppose f has a simple pole at z_0, in $\mathbb{C}_{\mathbb{R}}=\{|z|<R\}$, $0<R\leq\infty$. i,e $f(z)=b(z-z_0)^{-1}+O(1)$ for some $b\neq 0$. Then differentiating k times,

$$f^{(k)}(z)=\frac{(-1)^k a\,k!}{(z-z_0)^{k+1}}\ \left(1+O((z-z_0)^{k+1})\right).$$

Differentiating again and then substituting it into g, we find that

$$g=\frac{(-1)^k(k+1)^{k+1}}{a\,k!}\ \left(1+O((z-z_0)^{k+1})\right).$$

Thus, at a simple pole of f, $g\neq 0,\infty$, in $\mathbb{C}_{\mathbb{R}}=\{|z|<R\}$, $0<R\leq\infty$ but g' has a zero of order at least k in $\mathbb{C}_{\mathbb{R}}=\{|z|<R\}$, $0<R\leq\infty$. Now we apply first Theorem 3.1 to $\frac{g'}{g}$, assuming g to be non constant, giving

$$m\left(r,\frac{g'}{g}\right)-m\left(r,\frac{g}{g'}\right)+\ O(1)$$

$$
\begin{aligned}
&= N\left(r, \frac{g}{g'}\right) + V\left(r, \frac{g}{g'}\right) - [N\left(r, \frac{g'}{g}\right) + V\left(r, \frac{g'}{g}\right)] \\
&= N(r, g) + V(r, g) + N\left(r, \frac{1}{g'}\right) + \left(r, \frac{1}{g'}\right) - [N(r, g') + V(r, g')] \\
&\quad - [N\left(r, \frac{1}{g}\right) + V\left(r, \frac{1}{g}\right)] \\
&= N\left(r, \frac{1}{g'}\right) + V\left(r, \frac{1}{g'}\right) - [N\left(r, \frac{1}{g}\right) + V\left(r, \frac{1}{g}\right)] - [\overline{N}(r, g) + V(r, g)] \\
&= N^{(0)}\left(r, \frac{1}{g'}\right) + V\left(r, \frac{1}{g'}\right) - [\overline{N}\left(r, \frac{1}{g}\right) + V\left(r, \frac{1}{g}\right)] - [\overline{N}(r, g) + V(r, g)].
\end{aligned}
\tag{4.17}
$$

Thus using (4.17), Theorem 2.1 and the property that $m\left(r, \frac{g}{g'}\right)$ is non negative, we establish

$$
\begin{aligned}
k[N^1(r, f) + V(r, f)] &\leq N^{(0)}\left(r, \frac{1}{g'}\right) + V\left(r, \frac{1}{g'}\right) \leq [\overline{N}\left(r, \frac{1}{g}\right) + V\left(r, \frac{1}{g}\right)] \\
&\quad + \overline{N}(r, g) + V(r, g) + m\left(r, \frac{g'}{g}\right) + O(1) \\
&\leq \overline{N}\left(r, \frac{1}{g}\right) + V\left(r, \frac{1}{g}\right) + \overline{N}(r, g) + V(r, g) + S(r, g).
\end{aligned}
\tag{4.18}
$$

By (4.18) and zeros and poles of g can only occur at multiple poles of f, a-points of Θ or zeros of Θ' which are not a-points of Θ in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$ and so

$$
\begin{aligned}
&\overline{N}\left(r, \frac{1}{g}\right) + V\left(r, \frac{1}{g}\right) + \overline{N}(r, g) + V(r, g) \\
\leq\ & \overline{N}\left(r, \frac{1}{\Theta - a}\right) + V\left(r, \frac{1}{\Theta - a}\right) \\
&+ \overline{N}^{(2}(r, f) + V(r, f) + N^{(0)}\left(r, \frac{1}{\Theta'}\right) + V\left(r, \frac{1}{\Theta'}\right).
\end{aligned}
$$

Hence by (4.15), we derive

$$
\begin{aligned}
k[N^1(r, f) + V(r, f)] &\leq \overline{N}^{(2}(r, f) + V(r, f) + \overline{N}\left(r, \frac{1}{\Theta - a}\right) \\
&\quad + V\left(r, \frac{1}{\Theta - a}\right) + N^{(0)}\left(r, \frac{1}{\Theta'}\right) + V\left(r, \frac{1}{\Theta'}\right) + S(r, f).
\end{aligned}
$$

Proof of the Theorem 3.4.

We start by noting that in $N(r, f)$, multiple poles are counted at least twice in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$ and then apply (3.4)

$$
\begin{aligned}
&N^1(r, f) + V(r, f) + 2[\overline{N}^{(2}(r, f) + V(r, f)] \leq T(r, f) \\
&\qquad \leq \overline{N}(r, f) + V(r, f) + N\left(r, \frac{1}{f}\right) + V\left(r, \frac{1}{f}\right) \\
&\qquad + \overline{N}\left(r, \frac{1}{\Theta - a}\right) + V\left(r, \frac{1}{\Theta - a}\right) - [N^{(0)}\left(r, \frac{1}{\Theta'}\right) + V\left(r, \frac{1}{\Theta'}\right)] + S(r, f).
\end{aligned}
$$

Since $\overline{N}(r, f) + V(r, f) = N^1(r, f) + V(r, f) + \overline{N}^{(2}(r, f) + V(r, f)$, hence by (4.19), we get

$$
\begin{aligned}
&\overline{N}^{(2}(r, f) + V(r, f) \leq N\left(r, \frac{1}{f}\right) + V\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{\Theta - a}\right) + \left(r, \frac{1}{\Theta - a}\right) \\
&- [N^{(0)}\left(r, \frac{1}{\Theta'}\right) + V\left(r, \frac{1}{\Theta'}\right)] + S(r, f).
\end{aligned}
\tag{4.19}
$$

By (4.19) and (3.5), we obtain

$$\begin{aligned}
& k\,[N^1(r,f)+V(r,f)] \\
\leq\ & N\left(r,\frac{1}{f}\right)+V\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{\Theta-a}\right)+V\left(r,\frac{1}{\Theta-a}\right) \\
& -[N^{(0)}\left(r,\frac{1}{\Theta'}\right)+V\left(r,\frac{1}{\Theta'}\right)] \\
& +\overline{N}\left(r,\frac{1}{\Theta-a}\right)+V\left(r,\frac{1}{\Theta-a}\right)+N^{(0)}\left(r,\frac{1}{\Theta'}\right)+V\left(r,\frac{1}{\Theta'}\right)+S(r,f)
\end{aligned}$$

$$\begin{aligned}
k[N^1(r,f)+V(r,f)] \quad \leq \quad & N\left(r,\frac{1}{f}\right)+V\left(r,\frac{1}{f}\right) \\
& +2\,[\overline{N}\left(r,\frac{1}{\Theta-a}\right)+V\left(r,\frac{1}{\Theta-a}\right)]+S(r,f).
\end{aligned} \tag{4.20}$$

Now making an appeal to (4.19) and (4.20), we can write

$$\begin{aligned}
& \overline{N}(r,f)+V(r,f) \\
=\ & N^1(r,f)+V(r,f)+\overline{N}^{(2}(r,f)+V(r,f) \\
\leq\ & \frac{1}{k}[N\left(r,\frac{1}{f}\right)+V\left(r,\frac{1}{f}\right)]+\frac{2}{k}\,[\overline{N}\left(r,\frac{1}{\Theta-a}\right)+V\left(r,\frac{1}{\Theta-a}\right)] \\
& +N\left(r,\frac{1}{f}\right)+V\left(r,\frac{1}{f}\right) \\
& +\overline{N}\left(r,\frac{1}{\Theta-a}\right)+V\left(r,\frac{1}{\Theta-a}\right)-[N^{(0)}\left(r,\frac{1}{\Theta'}\right)+V\left(r,\frac{1}{\Theta'}\right)]+S(r,f)
\end{aligned}$$

$$\begin{aligned}
& \overline{N}(r,f)+V(r,f) \\
\leq\ & \left(1+\frac{1}{k}\right)[N\left(r,\frac{1}{f}\right)+V\left(r,\frac{1}{f}\right)]+\left(1+\frac{2}{k}\right)\overline{N}\left(r,\frac{1}{\Theta-a}\right)+V\left(r,\frac{1}{\Theta-a}\right)] \\
& -[N^{(0)}\left(r,\frac{1}{\Theta'}\right)+V\left(r,\frac{1}{\Theta'}\right)]+S(r,f).
\end{aligned} \tag{4.21}$$

Since $N^{(0)}\left(r,\frac{1}{\Theta'}\right)\geq 0$, therefore applying (4.21) into (3.4), we obtain

$$\begin{aligned}
T(r,f) \quad \leq \quad & \left(2+\frac{1}{k}\right)[N\left(r,\frac{1}{f}\right)+V\left(r,\frac{1}{f}\right)] \\
& +\left(2+\frac{2}{k}\right)[\overline{N}\left(r,\frac{1}{\Theta-a}\right)+V\left(r,\frac{1}{\Theta-a}\right)]+S(r,f).
\end{aligned}$$

5. On the deficiencies of differential polynomials for E-valued meromorphic functions

We shall concerned with E-valued meromorphic functions P which are polynomials in the E-valued meromorphic function $f(z)$ and derivatives of $f(z)$ with coefficients of the form $a(z)$ in $\mathbb{C}_{\mathbb{R}}=\{|z|<R\}$, $0<R\leq\infty$.

Let

$$F_k=a\,(f)^{t_0}[f^1]^{t_1}[f^2]^{t_2}...[f^m]^{t_m}$$

and

$$P=\sum_{k=1}^{N}F_k,$$

where $f^{(1)},f^{(2)},...,f^{(m)}$ are the successive derivatives of f in $\mathbb{C}_{\mathbb{R}}=\{|z|<R\}$, $0<R\leq\infty$ and $t_0,t_1,...,t_m$ are non negative integers.

Definition 5.1. *If $t_0+t_1+...+t_m$ for a fixed positive integer in every term of P, then P is called a E-valued homogeneous differential polynomial in $f(z)$ of degree n in $\mathbb{C}_{\mathbb{R}}=\{|z|<R\}$, $0<R\leq\infty$.*

Lemma 5.1. *Let $f(z)$ be a transcendental E-valued meromorphic function of compact projection in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$ and $a_j \in \overline{\mathbb{C}}$ $(j = 1, 2, ..., q)$ be q distinct complex numbers. Then we have*

$$\sum_{j=1}^{q} m\left(r, \frac{1}{f - a_j}\right) = m\left(r, \sum_{j=1}^{q} \frac{1}{f - a_j}\right) + O(1).$$

Proof. The proof of the Lemma 5.1 follows on similar lines as in the [25]. Therefore we omitted the proof of Lemma 5.1. □

We introduce some Lemmas which are important and useful in further investigation.

Lemma 5.2. *If P is a E-valued homogeneous differential polynomial in f in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$ of degree $n \geq 1$, then*

$$m\left(r, \frac{P}{f^n}\right) = S(r, f) \tag{5.1}$$

Proof. We know that

$$m\left(r, \frac{f^{(i)}}{f}\right) = S(r, f),$$

for $i = 1, 2, 3, ...$

By definition, P is the sum of finite number of terms of the type

$$F_k = a\,(f)^{t_0}[f^1]^{t_1}[ff^2]^{t_2}...[f^m]^{t_m},$$

where $t_0 + t_1 + ... + t_m$ are non-negative integers satisfying

$$\sum_{i=0}^{m} t_i = n.$$

Then

$$\frac{F_k}{f^n} = a\left(\frac{f^{(1)}}{f}\right)^{t_1}\left(\frac{f^{(2)}}{f}\right)^{t_2}...\left(\frac{f^{(1)}}{f}\right)^{t_m}.$$

So,

$$\begin{aligned} m\left(r, \frac{F^{(k)}}{f^n}\right) &\leq m(r, a) + \sum_{i=0}^{m} t_i m\left(r, \frac{f^{(i)}}{f}\right) \\ &\leq S(r, f). \end{aligned}$$

Thus,

$$\begin{aligned} m\left(r, \frac{P}{f^n}\right) &= m\left(r, \sum_k \frac{F^{(k)}}{f^n}\right) \\ &\leq \sum_k m\left(r, \frac{F^{(k)}}{f^n}\right) + O(1) \\ &\leq S(r, f), \end{aligned}$$

which proves the Lemma 5.2. □

Lemma 5.3. *Let P be a E-valued homogeneous differential polynomial in f of degree n in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$ and suppose that P is a E-valued homogeneous polynomial of degree n in $ff^{(1)}, f^{(2)}, ..., f^{(m)}$ with coefficients of the form $a(z)$ in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$. If P is not a constant and $a_1, a_2, ..., a_q$ are distinct elements of $\mathbb{C}$ where q is any positive integer, then*

$$n\sum_{i=1}^{q} m\left(r, \frac{1}{f - a}\right) \leq T(r, P) - [N\left(r, \frac{1}{P}\right) + V\left(r, \frac{1}{P}\right)] + S(r, f) \tag{5.2}$$

or

$$n\,q\,T(r, f) \leq T(r, P) + n\sum_{i=1}^{q}[N\left(r, \frac{1}{f - a}\right) + V\left(r, \frac{1}{f - a}\right)] - [N\left(r, \frac{1}{P}\right) + V\left(r, \frac{1}{P}\right)] + S(r, f) \tag{5.3}$$

Proof. We may assume that $q \geq 2$.

Let

$$F(z) = \sum_{i=1}^{q} \frac{1}{(f(z)-a)^n}.$$

By Lemma 5.1, we have

$$\begin{aligned} m(r,P)+O(1) &\geq \sum_{i=1}^{q} m\left(r, \frac{1}{f-a_i}\right)^n \\ &= n\sum_{i=1}^{q} m\left(r, \frac{1}{f-a_i}\right). \end{aligned} \tag{5.4}$$

Thus,

$$\begin{aligned} n\sum_{i=1}^{q} m\left(r, \frac{1}{f-a_i}\right) &\leq m(r,F)+O(1) \\ &\leq m(r,PF)+m\left(r,\frac{1}{P}\right)+O(1) \\ &\leq m(r,\frac{1}{P})+\sum_{i=1}^{q} m\left(r, \frac{P}{f-a_i}\right)^n. \end{aligned} \tag{5.5}$$

Now for $1 \leq i \leq q$, P is a E-valued homogeneous differential polynomial of degree n in $f-a_i$ in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$, since the successive derivative of $f-a_i$ are precisely those of f and so by Lemma 5.1, we have

$$m\left(r, \frac{P}{f-a_i}\right)^n = S(r,f),$$

for $i = 1, 2, 3, ..., q$.

Hence from (5.5), we have

$$n\sum_{i=1}^{q} m\left(r, \frac{1}{f-a_i}\right) \leq m\left(r,\frac{1}{P}\right)+S(r,f).$$

So,

$$\begin{aligned} & n\sum_{i=1}^{q} m\left(r, \frac{1}{f-a_i}\right)+N\left(r,\frac{1}{P}\right)+V\left(r,\frac{1}{P}\right) \\ \leq\ & m\left(r,\frac{1}{P}\right)+N\left(r,\frac{1}{P}\right)+V\left(r,\frac{1}{P}\right)+S(r,f) \end{aligned}$$

$$\begin{aligned} & n\sum_{i=1}^{q} m\left(r, \frac{1}{f-a_i}\right) \\ \leq\ & T\left(r,\frac{1}{P}\right)-[N\left(r,\frac{1}{P}\right)+V\left(r,\frac{1}{P}\right)]+S(r,f) \\ \leq\ & T(r,P)-[N\left(r,\frac{1}{P}\right)+V\left(r,\frac{1}{P}\right)]+S(r,f), \end{aligned} \tag{5.6}$$

which proves (5.2).

Next by (5.6), we prove (5.3)

$$\begin{aligned} & n\sum_{i=1}^{q} m\left(r, \frac{1}{f-a_i}\right)+n\sum_{i=1}^{q}\left[N\left(r,\frac{1}{f-a_i}\right)+V\left(r,\frac{1}{f-a_i}\right)\right] \\ \leq\ & T(r,P)+n\sum_{i=1}^{q}\left[N\left(r,\frac{1}{f-a_i}\right)+V\left(r,\frac{1}{f-a_i}\right)\right]-\left[N\left(r,\frac{1}{P}\right)+V\left(r,\frac{1}{P}\right)\right]+S(r,f) \end{aligned}$$

$$n\,q\,T\left(r,\frac{1}{f-a_i}\right) \leq T\left(r,\frac{1}{P}\right)+n\sum_{i=1}^{q}\left[N\left(r,\frac{1}{f-a_i}\right)+V\left(r,\frac{1}{f-a_i}\right)\right]-\left[N\left(r,\frac{1}{P}\right)+V\left(r,\frac{1}{P}\right)\right]+S(r,f)$$

$$n\,q\,T(r,f) \leq T\left(r,\frac{1}{P}\right)+n\sum_{i=1}^{q}\left[N\left(r,\frac{1}{f-a_i}\right)+V\left(r,\frac{1}{f-a_i}\right)\right]-\left[N\left(r,\frac{1}{P}\right)+V\left(r,\frac{1}{P}\right)\right]+S(r,f),$$

which proves (5.3). □

Theorem 5.1. *Let P be a E-valued homogeneous differential polynomial in f of degree n in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$ and $a \neq b$. If f is a non constant E-valued meromorphic function in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$, then*

$$\begin{aligned} T(r,f) \quad \leq \quad & N(r,P) + V(r,P) + N\left(r,\frac{1}{f-a}\right) + V\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{f-b}\right) + V\left(r,\frac{1}{f-b}\right) \\ & -[N(r,f) + V(r,f)] - \left[N\left(r,\frac{1}{P}\right) + V\left(r,\frac{1}{P}\right)\right] + S(r,f). \end{aligned} \tag{5.7}$$

Proof. Since $a \neq b$, we have

$$\frac{1}{f-b} = \left(\frac{P}{f-b} - \frac{P}{f-a}\right)\left(\frac{f-a}{P}\right)\frac{1}{b-a}.$$

Therefore by Lemma 5.1, we obtain

$$\begin{aligned} & m\left(r,\frac{1}{f-b}\right) \\ \leq \quad & m\left(r,\frac{P}{f-b}\right) + m\left(r,\frac{P}{f-a}\right) + m\left(r,\frac{f-a}{P}\right) + O(1) \\ \leq \quad & m\left(r,\frac{P}{f-b}\right) + m\left(r,\frac{P}{f-a}\right) + m\left(r,\frac{P}{f-a}\right) \\ & +N\left(r,\frac{P}{f-a}\right) + V\left(r,\frac{P}{f-a}\right) - \left[N\left(r,\frac{f-a}{P}\right) + V\left(r,\frac{f-a}{P}\right)\right] + O(1) \\ \leq \quad & N(r,P) + V(r,P) + N\left(r,\frac{1}{f-a}\right) + V\left(r,\frac{1}{f-a}\right) - \left[N\left(r,\frac{1}{P}\right) + V\left(r,\frac{1}{P}\right)\right] \\ & -[N(r,f) + V(r,f)] + S(r,f), \end{aligned} \tag{5.8}$$

where

$$m\left(r,\frac{P}{f-b}\right) \leq S(r,f)$$

and

$$\begin{aligned} & N\left(r,\frac{P}{f-a}\right) - \left[N\left(r,\frac{f-a}{P}\right) + V\left(r,\frac{f-a}{P}\right)\right] \\ = \quad & N(r,P) + V(r,P) + N\left(r,\frac{1}{f-a}\right) + V\left(r,\frac{1}{f-a}\right) \\ & - \left[N\left(r,\frac{1}{P}\right) + V\left(r,\frac{1}{P}\right)\right] - [N(r,f) + V(r,f)]. \end{aligned}$$

If we add the term $N\left(r,\frac{1}{f-b}\right)$ on both sides of the inequality (5.8), we get

$$\begin{aligned} T(r,f) \quad \leq \quad & N(r,P) + V(r,P) + N\left(r,\frac{1}{f-a}\right) + V\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{f-b}\right) + V\left(r,\frac{1}{f-b}\right) \\ & - \left[N\left(r,\frac{1}{P}\right) + V\left(r,\frac{1}{P}\right)\right] - [N(r,f) + V(r,f)] + S(r,f). \end{aligned} \tag{5.9}$$

If we restrict $P = f'(z)$, the inequality (5.9) becomes

$$\begin{aligned} T(r,f) \quad \leq \quad & \overline{N}(r,f) + V(r,f) + N\left(r,\frac{1}{f-a}\right) + V\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{f-b}\right) + V\left(r,\frac{1}{f-b}\right) \\ & - \left[N\left(r,\frac{1}{f'}\right) + V\left(r,\frac{1}{f'}\right)\right] - [N(r,f) + V(r,f)] + S(r,f). \end{aligned}$$

□

Theorem 5.2. *Let P be a E-valued homogeneous differential polynomial in f of degree n and $b \neq 0$ in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$. If f is a non constant E-valued meromorphic function in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$, then*

$$\begin{aligned} T(r,f) \quad \leq \quad & \overline{N}(r,f) + V(r,f) + N\left(r,\frac{1}{f-a}\right) + V\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{P-b}\right) + V\left(r,\frac{1}{P-b}\right) \\ & - \left[N^{(0)}\left(r,\frac{1}{P'}\right) + V\left(r,\frac{1}{P'}\right)\right] + S(r,f). \end{aligned} \tag{5.10}$$

Proof. Since $b \neq 0$, we have

$$\frac{1}{f-a} = \left(\frac{P}{f-a} - \frac{P'}{f-a}\,\frac{P-b}{P'}\right)\frac{1}{b}.$$

By Lemma 5.1, we derive

$$\begin{aligned}
m\left(r, \frac{1}{f-a}\right) &\leq m\left(r, \frac{P}{f-a}\right) + m\left(r, \frac{P'}{f-a}\right)\\
&\quad + m\left(r, \frac{P-b}{P'}\right) + O(1)\\
&\leq N\left(r, \frac{P'}{P-b}\right) + V\left(r, \frac{P'}{P-b}\right) + N\left(r, \frac{P-b}{P'}\right)\\
&\quad + V\left(r, \frac{P-b}{P'}\right) + S(r, f),
\end{aligned}$$

where

$$m\left(r, \frac{P}{f-a}\right) + m\left(r, \frac{P'}{f-a}\right) m\left(r, \frac{P-b}{P'}\right) \leq S(r, f).$$

Therefore,

$$\begin{aligned}
m\left(r, \frac{1}{f-a}\right) &\leq N(r, P') + V(r, P') + N\left(r, \frac{1}{P-b}\right) + V\left(r, \frac{1}{P-b}\right)\\
&\quad - \left[N\left(r, \frac{1}{P'}\right) + V\left(r, \frac{1}{P'}\right)\right] - [N(r, P) + V(r, P)] + S(r, f)\\
&\leq \overline{N}(r, f) + V(r, f) + N\left(r, \frac{1}{P-b}\right) + V\left(r, \frac{1}{P-b}\right)\\
&\quad - \left[N\left(r, \frac{1}{P'}\right) + V\left(r, \frac{1}{P'}\right)\right] + S(r, f)\\
&\leq \overline{N}(r, f) + V(r, f) + \overline{N}\left(r, \frac{1}{P-b}\right) + V\left(r, \frac{1}{P-b}\right)\\
&\quad - \left[N\left(r, \frac{1}{P'}\right) + V\left(r, \frac{1}{P'}\right)\right] + S(r, f).
\end{aligned} \tag{5.11}$$

Thus,

$$\begin{aligned}
&N\left(r, \frac{1}{P-b}\right) + V\left(r, \frac{1}{P-b}\right) - \left[N\left(r, \frac{1}{P'}\right) + V\left(r, \frac{1}{P'}\right)\right]\\
&= \overline{N}\left(r, \frac{1}{P-b}\right) + V\left(r, \frac{1}{P-b}\right) - \left[\overline{N}\left(r, \frac{1}{P-b}\right) + V\left(r, \frac{1}{P-b}\right)\right]\\
&+ N\left(r, \frac{1}{P'}\right) + V\left(r, \frac{1}{P'}\right) - \left[N\left(r, \frac{1}{P-b}\right) + N\left(r, \frac{1}{P-b}\right)\right]\\
&= \overline{N}\left(r, \frac{1}{P-b}\right) + V\left(r, \frac{1}{P-b}\right) - N^{(0)}\left(r, \frac{1}{P'}\right) + V\left(r, \frac{1}{P'}\right),
\end{aligned}$$

where

$$N^{(0)}\left(r, \frac{1}{P'}\right) + V\left(r, \frac{1}{P'}\right) = \overline{N}\left(r, \frac{1}{P-b}\right) + V\left(r, \frac{1}{P-b}\right) + N\left(r, \frac{1}{P'}\right) + V\left(r, \frac{1}{P'}\right) - \left[N\left(r, \frac{1}{P-b}\right) + V\left(r, \frac{1}{P-b}\right)\right].$$

If we add the term $N\left(r, \frac{1}{f-b}\right)$ on both sides of the inequality (5.11) we get

$$\begin{aligned}
T(r, f) &\leq \overline{N}(r, f) + V(r, f) + N\left(r, \frac{1}{f-a}\right) + V\left(r, \frac{1}{f-a}\right)\\
&\quad + N\left(r, \frac{1}{P-b}\right) + V\left(r, \frac{1}{P-b}\right) - \left[N^{(0)}\left(r, \frac{1}{P'}\right) + V\left(r, \frac{1}{P'}\right)\right] + S(r, f).
\end{aligned}$$

If we restrict $P = W^{(k)}(z)$, the inequality (5.10) becomes

$$T(r, f) \leq \overline{N}(r, f) + V(r, f) + N\left(r, \frac{1}{f-a}\right) + V\left(r, \frac{1}{f-a}\right)$$

$$+N\left(r,\frac{1}{f^{(k)}-b}\right)+V\left(r,\frac{1}{f^{(k)}-b}\right)-\left[N^{(0)}+V\left(r,\frac{1}{f^{(k+1)}}\right)\right]+S(r,f),$$

which is one of the Milloux result. □

Theorem 5.3. *Let P be a homogeneous differential polynomial in f of degree n in $\mathbb{C}_{\mathbb{R}}=\{|z|<R\}$, $0<R\leq\infty$. If f is a non constant E-valued meromorphic function in $\mathbb{C}_{\mathbb{R}}=\{|z|<R\}$, $0<R\leq\infty$, then*

$$\begin{aligned} T(r,f) \quad \leq \quad & N\left(r,\frac{1}{f-a}\right)+V\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{P-b}\right)+V\left(r,\frac{1}{P-b}\right) \\ & +N\left(r,\frac{1}{P-c}\right)+V\left(r,\frac{1}{P-c}\right)-\left[N^{1}(r,P)+V(r,P)\right]+S(r,f), \end{aligned} \tag{5.12}$$

where

$$N^{1}(r,P)+V(r,P)=2[N(r,P)+V(r,P)]-[N(r,P')+V(r,P')]+N\left(r,\frac{1}{P'}\right)+V\left(r,\frac{1}{P'}\right).$$

are non negative.

Proof. It is easy to write

$$\frac{1}{f-a}=\frac{1}{P}\,\frac{P}{f-a}.$$

By Lemma 5.1, we have

$$\begin{aligned} m\left(r,\frac{1}{f-a}\right) \quad \leq \quad & m\left(r,\frac{1}{P}\right)+m\left(r,\frac{P}{f-a}\right) \\ \leq \quad & m\left(r,\frac{1}{P}\right)+S(r,f) \\ \leq \quad & T(r,P)-\left[N\left(r,\frac{1}{P}\right)+V\left(r,\frac{1}{P}\right)\right]+S(r,f). \end{aligned} \tag{5.13}$$

By Lemma 5.2, we have

$$\begin{aligned} T(r,P) \quad \leq \quad & N\left(r,\frac{1}{P}\right)+V\left(r,\frac{1}{P}\right)+N\left(r,\frac{1}{P-b}\right)+V\left(r,\frac{1}{P-b}\right) \\ & +N\left(r,\frac{1}{P-c}\right)+V\left(r,\frac{1}{P-c}\right)-[N^{1}(r,P)+V(r,P)]+S(r,f). \end{aligned}$$

If we use Lemma 5.1 in the equality (5.13), we have

$$\begin{aligned} T(r,f) \quad \leq \quad & N\left(r,\frac{1}{f-a}\right)+V\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{P}\right)+V\left(r,\frac{1}{P}\right) \\ & +N\left(r,\frac{1}{P-b}\right)+V\left(r,\frac{1}{P-b}\right)+N\left(r,\frac{1}{P-c}\right)+V\left(r,\frac{1}{P-c}\right) \\ & -[N^{1}(r,P)+V(r,P)]-\left[N\left(r,\frac{1}{P}\right)+V\left(r,\frac{1}{P}\right)\right]+S(r,f) \end{aligned}$$

or

$$T(r,f) \quad \leq \quad N\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{P-b}\right)+N\left(r,\frac{1}{P-c}\right)-N^{1}(r,P)+S(r,f)$$

If we restrict $P=f^{(k)}(z)$, the inequality (5.12) becomes

$$\begin{aligned} T(r,f) \quad \leq \quad & N\left(r,\frac{1}{f-a}\right)+V\left(r,\frac{1}{f-a}\right)+N\left(r,\frac{1}{f^{(k)}-b}\right)+V\left(r,\frac{1}{f^{(k)}-b}\right) \\ & +N\left(r,\frac{1}{f^{(k)}-c}\right)+V\left(r,\frac{1}{f^{(k)}-c}\right) \\ & -\left[N^{(1)}\left(r,f^{(k)}\right)+V\left(r,f^{(k)}\right)\right]+S(r,f). \end{aligned}$$

□

Theorem 5.4. *Let P be a homogeneous differential polynomial in f of degree n in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$. If f is a non constant E-valued meromorphic function in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$, then*

$$\begin{aligned} nq\,T(r,f) \quad \leq \quad & \overline{N}(r,f) + V(r,f) + \overline{N}\left(r, \frac{1}{P-b}\right) + V\left(r, \frac{1}{P-b}\right) \\ & + n\textstyle\sum_{i=1}^{q}\left[N\left(r, \frac{1}{f-a_i}\right) + V\left(r, \frac{1}{f-a_i}\right)\right] \\ & - \left[N\left(r, \frac{1}{P'}\right) + V\left(r, \frac{1}{P'}\right)\right] + S(r,f). \end{aligned} \tag{5.14}$$

Proof. By Lemma 5.2, we have

$$\begin{aligned} T(r,P) \quad \leq \quad & N\left(r, \frac{1}{P}\right) + V\left(r, \frac{1}{P}\right) + N(r,P) + V(r,P) \\ & + N\left(r, \frac{1}{P-b}\right) + V\left(r, \frac{1}{P-b}\right) - [N^1(r,P) + V(r,P)] + S(r,P), \end{aligned}$$

where

$$N^1(r,P) = 2[N(r,P) + V(r,P)] - [N(r,P') + V(r,P')] + N\left(r, \frac{1}{P'}\right) + V\left(r, \frac{1}{P'}\right).$$

$$\begin{aligned} T(r,P) \quad \leq \quad & N\left(r, \frac{1}{P}\right) + V\left(r, \frac{1}{P}\right) + N(r,P) + V(r,P) \\ & + N\left(r, \frac{1}{P-b}\right) + V\left(r, \frac{1}{P-b}\right) - \left[N\left(r, \frac{1}{P'}\right) + V\left(r, \frac{1}{P'}\right)\right] \\ & - 2[N(r,P) + V(r,P)] + N(r,P') + V(r,P') + S(r,P) \\ \leq \quad & \overline{N}(r,P) + V(r,P) + N\left(r, \frac{1}{P}\right) + V\left(r, \frac{1}{P}\right) + \overline{N}\left(r, \frac{1}{P-b}\right) + V\left(r, \frac{1}{P-b}\right) \\ & - \left[N\left(r, \frac{1}{P'}\right) + V\left(r, \frac{1}{P'}\right)\right] + S(r,P). \end{aligned}$$

Therefore,

$$\begin{aligned} T(r,P) \quad \leq \quad & \overline{N}(r,P) + V(r,P) + N\left(r, \frac{1}{P}\right) + V\left(r, \frac{1}{P}\right) \\ & + \overline{N}\left(r, \frac{1}{P-b}\right) + V\left(r, \frac{1}{P-b}\right) - \left[N\left(r, \frac{1}{P'}\right) + V\left(r, \frac{1}{P'}\right)\right] + S(r,P). \end{aligned}$$

On the other hand, it is easy to write $\overline{N}(r,P) + V(r,P) \leq \overline{N}(r,f) + V(r,f) + S(r,f)$. If we use the inequality (5.7), we can write

$$\begin{aligned} n\,q\,T(r,f) \quad \leq \quad & \overline{N}(r,P) + V(r,P) + N\left(r, \frac{1}{P}\right) + V\left(r, \frac{1}{P}\right) \\ & + \overline{N}\left(r, \frac{1}{P-b}\right) + V\left(r, \frac{1}{P-b}\right) - \left[N\left(r, \frac{1}{P'}\right) + V\left(r, \frac{1}{P'}\right)\right] \\ & + n\textstyle\sum_{i=1}^{q}\left[N\left(r, \frac{1}{f-a_i}\right) + V\left(r, \frac{1}{f-a_i}\right)\right] \\ & - \left[N\left(r, \frac{1}{P'}\right) + V\left(r, \frac{1}{P'}\right)\right] + S(r,f) \end{aligned}$$

or

$$\begin{aligned} n\,q\,T(r,f) \quad \leq \quad & \overline{N}(r,f) + V(r,f) + \overline{N}\left(r, \frac{1}{P-b}\right) + V\left(r, \frac{1}{P-b}\right) \\ & + n\textstyle\sum_{i=1}^{q}\left[N\left(r, \frac{1}{f-a_i}\right) + V\left(r, \frac{1}{f-a_i}\right)\right] - \left[N\left(r, \frac{1}{P'}\right) + V\left(r, \frac{1}{P'}\right)\right] + S(r,f). \end{aligned}$$

If $n = 1$ and $q = 1$ the inequality (5.14) gives the inequality (5.10). That is, the inequality (5.14) is the generalization of the inequality (5.10). □

Theorem 5.5. *Let P be a E-valued homogeneous differential polynomial in f of degree n in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$ and $s = 1, 2, 3, ...,$ if f is a non constant E-valued meromorphic function in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$.*

Then

$$\begin{aligned}(s-1)n\,q\,T(r,f) \quad \leq \quad & \overline{N}(r,f) + V(r,f) + (s-1)\,n\sum_{i=1}^{q}\left[N\left(r,\frac{1}{f-a_i}\right) + V\left(r,\frac{1}{f-a_i}\right)\right] \\ & + \sum_{j=1}^{s}\left[\overline{N}\left(r,\frac{1}{P-b_j}\right) + V\left(r,\frac{1}{P-b_j}\right)\right] \\ & - \left[N\left(r,\frac{1}{P'}\right) + V\left(r,\frac{1}{P'}\right)\right] + S(r,f)\end{aligned} \tag{5.15}$$

If $s = 3, 4, 5, ...,$ then

$$\begin{aligned}(s-2)n\,q\,T(r,f) \quad \leq \quad & (s-2)\,n\sum_{i=1}^{q}\left[N\left(r,\frac{1}{f-a_i}\right) + V\left(r,\frac{1}{f-a_i}\right)\right] \\ & + \sum_{j=1}^{s}\left[\overline{N}\left(r,\frac{1}{P-b_j}\right) + V\left(r,\frac{1}{P-b_j}\right)\right] \\ & - \left[N\left(r,\frac{1}{P'}\right) + V\left(r,\frac{1}{P'}\right)\right] + S(r,f).\end{aligned} \tag{5.16}$$

Proof. By Lemma 5.2 in terms of the E-valued homogeneous differential polynomial P in $\mathbb{C}_{\mathbb{R}} = \{|z| < R\}$, $0 < R \leq \infty$, then we have

$$\begin{aligned}(s-1)T(r,f) \quad \leq \quad & \overline{N}(r,f) + V(r,f) + \sum_{j=1}^{s}\left[\overline{N}\left(r,\frac{1}{P-b_j}\right) + V\left(r,\frac{1}{P-b_j}\right)\right] \\ & - \left[N\left(r,\frac{1}{P'}\right) + V\left(r,\frac{1}{P'}\right)\right] + S(r,f)\end{aligned} \tag{5.17}$$

and

$$\begin{aligned}(s-2)T(r,f) \quad \leq \quad & \sum_{j=1}^{s}\left[\overline{N}\left(r,\frac{1}{P-b_j}\right) + V\left(r,\frac{1}{P-b_j}\right)\right] \\ & - \left[N^{(1)}(r,P) + V(r,P)\right] + S(r,f),\end{aligned} \tag{5.18}$$

where

$$N(1)(r,P) + V(r,P) = 2[N(r,P) + V(r,P)] - [N(r,P') + V(r,P')] + N\left(r,\frac{1}{P'}\right) + V\left(r,\frac{1}{P'}\right)$$

non negative.

If we use the inequality (5.17) and (5.18) in the equality (5.7), we get

$$\begin{aligned}(s-1)n\,q\,T(r,f) \quad \leq \quad & \overline{N}(r,f) + V(r,f) + (s-1)\,n\sum_{i=1}^{q}\left[N\left(r,\frac{1}{f-a_i}\right) + V\left(r,\frac{1}{f-a_i}\right)\right] \\ & + \sum_{j=1}^{s}\left[\overline{N}\left(r,\frac{1}{P-b_j}\right) + V\left(r,\frac{1}{P-b_j}\right)\right] \\ & - \left[N\left(r,\frac{1}{P'}\right) + V\left(r,\frac{1}{P'}\right)\right] + S(r,f)\end{aligned}$$

and

$$\begin{aligned}(s-2)n\,q\,T(r,f) \quad \leq \quad & (s-2)\,n\sum_{i=1}^{q}\left[N\left(r,\frac{1}{f-a_i}\right) + V\left(r,\frac{1}{f-a_i}\right)\right] \\ & + \sum_{j=1}^{s}\left[\overline{N}\left(r,\frac{1}{P-b_j}\right) + V\left(r,\frac{1}{P-b_j}\right)\right] \\ & - \left[N\left(r,\frac{1}{P'} + V\left(r,\frac{1}{P'}\right)\right) + \right] + S(r,f).\end{aligned}$$

□

6. Conclusion

In this article, we obtain analogous of Milloux inequality and Hayman's alternative for *E*-valued meromorphic functions from the complex plane $\mathbb{C}$ to an infinite dimensional complex Banach space E with a Schauder basis. As an application of our theorems, we deduce some interesting analogous results for *E*-valued meromorphic functions from the complex plane $\mathbb{C}$ to an infinite dimensional complex Banach space *E* with a Schauder basis. And also we have given the applications of homogeneous differential polynomials to the Nevanlinna's theory of *E*-valued meromorphic functions from the complex plane $\mathbb{C}$ to an infinite dimensional complex Banach space *E* with a Schauder basis and given some generalizations by these polynomials.

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