***Jñānābha*, Vol. 50(2) (2020), 59-62**
(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

ON INTEGRALS INVOLVING A PRODUCT OF EXTENDED BESSEL MAITLAND FUNCTION AND I^*-FUNCTION

By

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(*Received : May* 01, 2020 *; Revised: June* 03, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50207

Abstract

The object of this paper is to establish some interesting integrals involving the product of extended Bessel-Maitland function and I^*- function and then, express them in terms of extended I^*- function. Further some special cases of our results are also deduced.
2010 Mathematics Subject Classifications: 33C60, 33C45.
Keywords and phrases: I^*-function, Wright's generalized Bessel-Maitland function, I-function, definite integrals.

1 Introduction

In the last decade, many authors (for example, see [1], [4] - [9]) have developed numerous integral formulas involving a variety of special functions. Such integrals play a very important role in many diverse fields of engineering and sciences ([5], [11], [14]).

Recently, Ghayasuddin and Khan [4], Khan et al. ([6] - [9]) and Ali et al. [1] obtained certain interesting new class of integrals involving the generalized Bessel-Maitland function and expressed in terms of generalized (Wright) hypergeometric function.

In order to derive our main results, we are required to express following definitions and formulae of some well known special functions:

The I- function [13] defined in terms of following Mellin - Barnes type integral is given by

$$(1.1)\quad I^{m,n}_{p_i,q_i;r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n}; & (a_{ji},\alpha_{ji})_{n+1,p_i}\\ (b_j,\beta_j)_{1,m}; & (b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right] = \frac{1}{2\pi\omega}\int_L \varphi(\xi) z^{\xi} d\xi, \omega = \sqrt{(-1)},$$

where

$$\varphi(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji})(+\beta_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji})(-\alpha_{ji}\xi)\right]}$$

$p_i, q_i (i = 1, 2, \ldots, r), m$ and n are integers satisfying $0 \le n \le p_i, 0 \le m \le q_i (i = 1, 2, \ldots, r)$, r is finite,$\alpha_j, \beta_j, \alpha_{ji}, \beta_{ji}$ are real and positive and a_j, b_j, a_{ji}, b_{ji} are numbers such that $\alpha_k(b_h + v) \neq \beta_h(a_k - 1 - k)$ for $k, v = 0, 1, 2\ldots; h = 1, 2, \ldots, m; i = 1, 2, .., r$.

L is contour running form $\sigma - i\infty$ to $\sigma + i\infty$, where σ is real in the complex ξ-plane such that the poles

$$\xi = \frac{(a_j - 1 - v)}{\alpha_j}, j = 1, 2, \ldots n; v = 0, 1, 2, \cdots$$

$$\xi = \frac{(b_j + v)}{\beta_j}, j = 1, 2 \cdots, m; v = 0, 1, 2, \cdots$$

to the left and right hand sides of L, respectively.

The I^*-function [5], related to the I-function [13], is introduced as a contour integral in complex ξ-plane given by

$$(1.2)\quad I^{m,n}_{p_i,q_i;r}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n} : (a_{ji},\alpha_{ji})_{1,pi}\\ (b_j,\beta_j)_{1,m} : (a_{ji},\beta_{ji})_{1,qi}\end{matrix}\right.\right] = \frac{1}{2\pi\omega}\int_L \varphi(\xi) z^{\xi} d\xi,$$

in which

$$\varphi(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left[\prod_{j=1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right]}$$

Here, for finite value of r, all $p_i, q_i (i = 1, 2, \ldots r), m$ and n are positive integers, satisfying the inequalities; $0 \leq n \leq p, p_i \geq n \geq 1 (i = 1, 2, \ldots ..r), 1 \leq m \leq q, q_i \geq m \geq 1$ $(i = 1, 2, \ldots ..r), \alpha_j (j = 1, \ldots ., n), \beta_j (j = 1, \ldots, m)$, $\alpha_{ji} (1 \leq j \leq p_i, i = 1, 2, \ldots r)$, $\beta_{ji} (1 \leq j \leq q_i, i = 1, 2, \ldots r)$ are real and positive and $a_j (j = 1, \ldots ., n), b_j (j = 1, \ldots, m), a_{ji} (1 \leq j \leq p_i, i = 1, 2, \ldots r), b_{ji} (1 \leq j \leq q_i, i = 1, 2, \ldots r)$ are complex numbers such that $\alpha_k (b_h + \nu) \neq \beta_h (a_k - 1 - l) for l, \nu = 0, 1, 2, \ldots; h = 1, 2, \ldots, m$.

L is contour running from $\sigma - i\infty$ to $\sigma + i\infty$, where σ is real in the complex ξ- plane such that the poles $\xi = \frac{(a_j - 1 - l)}{\alpha_j}$, j=1,2,...,n; l=0,1,2,...,$\xi = \frac{(b_j + \nu)}{\beta_j}, j = 1, 2, \ldots, m; \nu = 0, 1, 2, \ldots$ lie to the left and right hand sides of the contour L respectively, the empty product is represented as 1.

The I^*- function converges absolutely in the ξ- plane if $|\arg z| < \frac{\pi}{2} A$.

(1.3) where, $A = \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{m} \beta_j - \max_{1 \leq i \leq r} \left[\sum_{j=1}^{p_i} \alpha_{ji} + \sum_{j=1}^{q_i} \beta_{ji} \right] > 0$

***Property* 1.1** *The I^*- function* (1.2) - (1.3) *is most probably identical to I-function given in* (1.1).

***Property* 1.2** *For $r = 1$, the I^*-function defined in* (1.2) - (1.3) *has a relation with Fox's H- function* [3] *as*

$$I^{m,n}_{p_1, q_1; 1} \left[z \middle| \begin{matrix} (a_j, \alpha_j)_{1,n} : (a_{j1}, \alpha_{j1})_{1,p_1} \\ (b_j, \beta_j)_{1,m} : (a_{j1}, \beta_{j1})_{1,q_1} \end{matrix} \right] = H^{m,n}_{p_1+n, q_1+m} \left[z \middle| \begin{matrix} (a_j, \alpha_j)_{1,n} : (a_{j1}, \alpha_{j1})_{n+1,p_1}, (a_{j1}, \alpha_{j1})_{1,n} \\ (b_j, \beta_j)_{1,m} : (b_{j1}, \beta_{j1})_{m+1,q_1}, (b_{j1}, \beta_{j1})_{1,m} \end{matrix} \right]$$

where, $|\arg z| < \frac{\pi}{2} A$ *and* $A = \sum_{j=1}^{n} \alpha_j + \sum_{j=1}^{m} \beta_j - \left[\sum_{j=1}^{p_1} \alpha_{j1} + \sum_{j=1}^{q_1} \beta_{j1} \right] > 0$.

A new extension of Bessel-Maitland function is introduced by Khan et al. [9] *as*

(1.4) $J^{\beta,\delta,\sigma,t}_{\alpha,\gamma,\mu,\rho,\nu,s}(z) = \sum_{h=0}^{\infty} \frac{(\beta)_{\delta h} (\sigma)_{th} (-z)^h}{\Gamma(\gamma h + \alpha + 1)(\nu)_{sh} (\mu)_{\rho h}}$,

where $\alpha, \gamma, \mu, \rho, \nu, \beta, \delta, \sigma \in C; (\beta) > 0, (\gamma) > 0, (\delta) > 0, (\mu) > 0, (\rho) > 0, (\sigma) > 0, (\nu) > 0, (\alpha) \geq -1; s, t > 0$ *and* $t < (\gamma) + s$ *and* $(\beta)_0 = 1, (\beta)_{\delta h} = \frac{\Gamma(\beta + \delta h)}{\Gamma(\beta)}$ *denotes the generalized Pochhammer symbol.*

Some formulae of definite integrals are found by the authors:

By Edward [2] *as*

(1.5) $\int_0^1 \int_0^1 y^k (1-x)^{k-1} (1-y)^{l-1} (1-xy)^{1-k-l} dx dy = \frac{\Gamma(k)\Gamma(l)}{\Gamma(k+l)}$,

provided $0 < (l) < (k)$.

By Oberhettinger [12] *given as*

(1.6) $\int_0^{\infty} x^{\alpha-1} (x + c + \sqrt{x^2 + 2cx})^{-\beta} dx = 2\beta c^{-\beta} (\frac{c}{2})^{\alpha} \frac{\Gamma(2\alpha)\Gamma(\beta-\alpha)}{\Gamma(1+\alpha+\beta)}$,

provided $0 < (\alpha) < (\beta)$.

MacRobert [10] *presented as*

(1.7) $\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} [cx + d(1-x)]^{-\alpha-\beta} dx = \frac{1}{c^{\alpha} d^{\beta}} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$,

provided $(\alpha) > 0, (\beta) > 0$ *and c, d are non zero constants and the expression* $cx + d(1-x)$, *where* $0 \leq x \leq 1$ *is non zero.*

2 Main Results

In this section we derive following theorems on extended Bessel-Maitland function and I^*- function.

***Theorem* 2.1** *If* $\alpha, \gamma, \mu, \rho, \nu, \beta, \delta, \sigma \in C; (\beta) > 0, (\gamma) > 0, (\delta) > 0, (\mu) > 0, (\rho) > 0, (\sigma) > 0, (\nu) > 0, (\alpha) \geq -1; s, t > 0$ *and* $t < (\gamma) + s$ *and* $0 < (l) < (k)$. *Then following integral exists*

(2.1) $$\int_0^1 \int_0^1 y^k (1-x)^{k-1} (1-y)^{l-1} (1-xy)^{1-k-l} J^{\beta,\delta,\sigma,t}_{\alpha,\gamma,\mu,\rho,\nu,s} \left[\frac{y(1-x)(1-y)}{(1-xy)^2} \right] I^{m,n}_{p_i, q_i; r} \left[\frac{zy(1-x)(1-y)}{(1-xy)^2} \middle| \begin{matrix} (a_j, A_j)_{1,n} : (a_{ji}, A_{ji})_{1,pi} \\ (b_j, B_j)_{1,m} : (a_{ji}, B_{ji})_{1,qi} \end{matrix} \right] dx dy$$

$$= \sum_{h=0}^{\infty} \frac{(-1)^h (\beta)_{\delta h} (\sigma)_{th}}{\Gamma(\gamma h + \alpha + 1)(\nu)_{sh} (\mu)_{\rho h}} I^{m,n+2}_{p_i, q_i+1; r} \left[z \middle| \begin{matrix} (a_j, A_j)_{1,n}, (1-k-h, 1), (1-l-h, 1) : (a_{ji}, A_{ji})_{1,pi} \\ (b_j, B_j)_{1,m} : (a_{ji}, B_{ji})_{1,qi}, (1-k-l-2h, 2) \end{matrix} \right],$$

provided that $|\arg z| < \frac{\pi}{2} A', A' = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \max_{1 \leq i \leq r} [\sum_{j=1}^{p_i} A_{ji} + \sum_{j=1}^{q_i} B_{ji}] > 0$.

Proof. In left hand side of (2.1), expand $J^{\beta,\delta,\sigma,t}_{\alpha,\gamma,\mu,\rho,\nu,s}(.)$, in the series by (1.4) and the function $I^{m,n}_{p_i, q_i; r}[.]$, in the Mellin integral (1.2) and then, interchange the order of summation and integration, we get

(2.2) $\sum_{h=0}^{\infty} \frac{(-1)^h (\beta)_{\delta h} (\sigma)_{th}}{\Gamma(\gamma h + \alpha + 1)(\nu)_{sh} (\mu)_{\rho h}} \frac{1}{2\pi\omega} \int_L \varphi(\xi) z^{\xi} \int_0^1 \int_0^1 y^{k+h+\xi} (1-x)^{k+h+\xi-1} (1-y)^{l+h+\xi-1} (1-xy)^{1-k-l-2h-2\xi} dx dy d\xi$.

Now, in (2.2) apply the formula (1.5) to achieve

(2.3) $\sum_{h=0}^{\infty} \frac{(-1)^h (\beta)_{\delta h} (\sigma)_{th}}{\Gamma(\gamma h + \alpha + 1)(\nu)_{sh} (\mu)_{\rho h}} \frac{1}{2\pi\omega} \int_L \varphi(\xi) \frac{\Gamma(k+h+\xi)\Gamma(l+h+\xi)}{\Gamma(k+l+2h+2\xi)} z^{\xi} d\xi$,

By definition of (1.2), the expression (2.3) immediately gives the result (2.1).

***Theorem* 2.2** *If* $\alpha,\gamma,\mu,\rho,v,\beta,\delta,\sigma \in C; (\beta) > 0, (\gamma) > 0, (\delta) > 0, (\mu) > 0, (\rho) > 0, (\sigma) > 0, (v) > 0, (\alpha) \geq -1; s, t > 0, t < (\gamma) + s, 0 < (\tau) + 1 < (), |\arg z| < \frac{\pi}{2}A'$ *and* $A' = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \max_{1\leq i\leq r}[\sum_{j=1}^{p_i} A_{ji} + \sum_{j=1}^{q_i} B_{ji}] > 0,$ *then following integral exists*

$$(2.4)\quad \int_0^\infty x^\tau (x + c + \sqrt{x^2 + 2cx})^{-} J^{\beta,\delta,\sigma,t}_{\alpha,\gamma,\mu,\rho,v,s}\left(\frac{x}{x+c+\sqrt{x^2+2cx}}\right) I^{m,n}_{p_i,q_i;r}\left[\frac{zx}{(x+c+\sqrt{x^2+2cx})}\middle|\begin{matrix}(a_j, A_j)_{1,n} : (a_{ji}, A_{ji})_{1,pi}\\(b_j, B_j)_{1,m} : (a_{ji}, B_{ji})_{1,qi}\end{matrix}\right]dx$$

$$= (c)^{1+\tau-\lambda}(\frac{1}{2})^{1+\tau}(-\tau)\sum_{h=0}^\infty \frac{(-\frac{1}{2})^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(v)_{sh}(\mu)_{\rho h}} I^{m,n+1}_{p_i,q_i+1;r}\left[\frac{z}{2}\middle|\begin{matrix}(-1-2\tau-2h,2),(a_j, A_j)_{1,n} : (a_{ji}, A_{ji})_{1,pi}\\(b_j, B_j)_{1,m} : (a_{ji}, B_{ji})_{1,qi},(-1--\tau-2h,2)\end{matrix}\right]$$

$$+(c)^{1+\tau-\lambda}(\frac{1}{2})^{1+\tau}(-\tau-1)\sum_{h=0}^\infty \frac{(-\frac{1}{2})^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(v)_{sh}(\mu)_{\rho h}} I^{m,n+1}_{p_i,q_i+1;r}\left[\frac{z}{2}\middle|\begin{matrix}(-1-2\tau-2h,2),(a_j, A_j)_{1,n} : (a_{ji}, A_{ji})_{1,pi}\\(b_j, B_j)_{1,m} : (a_{ji}, B_{ji})_{1,qi},(--\tau-2h,2)\end{matrix}\right],$$

Proof. In the integrand of (2.4) define by the definitions given in (1.2) and (1.4) and then interchanging the order of integration and summation to get that

$$(2.5)\quad \sum_{h=0}^\infty \frac{(-1)^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(v)_{sh}(\mu)_{\rho h}} \frac{1}{2\pi\omega}\int_L \varphi(\xi)z^\xi \int_0^\infty x^{\tau+h+\xi}(x + c + \sqrt{x^2 + 2cx})^{--h-\xi}dxd\xi.$$

In the inner integral of (2.5), use the result (1.6) to find that

$$(2.6)\quad (c)^{1+\tau-\lambda}(\frac{1}{2})^{1+\tau}(-\tau)\sum_{h=0}^\infty \frac{(-\frac{1}{2})^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(v)_{sh}(\mu)_{\rho h}}\frac{1}{2\pi\omega}\int_L \varphi(\xi)\frac{(2+2\tau+2h+2\xi)}{(2+\lambda+\tau+2h+2\xi)}(\frac{z}{2})^\xi d\xi$$

$$+(c)^{1+\tau-\lambda}(\frac{1}{2})^{1+\tau}(-\tau-1)\sum_{h=0}^\infty \frac{(-\frac{1}{2})^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(v)_{sh}(\mu)_{\rho h}}\frac{1}{2\pi\omega}\int_L \varphi(\xi)\frac{(2+2\tau+2h+2\xi)}{(1+\lambda+\tau+2h+2\xi)}(\frac{z}{2})^\xi d\xi.$$

Finally, use the definition (1.2) in the contour integrals of (2.6), we obtain the right hand side of (2.4) as

$$(2.7)\quad (c)^{1+\tau-\lambda}(\frac{1}{2})^{1+\tau}(-\tau)\sum_{h=0}^\infty \frac{(-\frac{1}{2})^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(v)_{sh}(\mu)_{\rho h}} I^{m,n+1}_{p_i,q_i+1;r}\left[\frac{z}{2}\middle|\begin{matrix}(-1-2\tau-2h,2),(a_j, A_j)_{1,n} : (a_{ji}, A_{ji})_{1,pi}\\(b_j, B_j)_{1,m} : (a_{ji}, B_{ji})_{1,qi},(-1--\tau-2h,2)\end{matrix}\right]$$

$$+(c)^{1+\tau-\lambda}(\frac{1}{2})^{1+\tau}(-\tau-1)\sum_{h=0}^\infty \frac{(-\frac{1}{2})^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(v)_{sh}(\mu)_{\rho h}} I^{m,n+1}_{p_i,q_i+1;r}\left[\frac{z}{2}\middle|\begin{matrix}(-1-2\tau-2h,2),(a_j, A_j)_{1,n} : (a_{ji}, A_{ji})_{1,pi}\\(b_j, B_j)_{1,m} : (a_{ji}, B_{ji})_{1,qi},(--\tau-2h,2)\end{matrix}\right].$$

In the result (2.7), apply the conditions of the results (1.3), (1.4) and (1.6), we get the conditions given in the ***Theorem* 2.2**. Hence, the ***Theorem* 2.2** is followed.

***Theorem* 2.3** *If* $\alpha,\gamma,\mu,\rho,v,\beta,\delta,\sigma \in C; (\beta) > 0, (\gamma) > 0, (\delta) > 0, (\mu) > 0, (\rho) > 0, (\sigma) > 0, (v) > 0, (\alpha) \geq -1; s, t > 0, t < (\gamma) + s, (k) > 0, (l) > 0, |\arg z| < \frac{\pi}{2}A'$ *and* $A' = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \max_{1\leq i\leq r}[\sum_{j=1}^{p_i} A_{ji} + \sum_{j=1}^{q_i} B_{ji}] > 0,$ *then following integral holds good*

$$(2.8)\quad \int_0^1 x^{k-1}(1-x)^{l-1}[cx + d(1-x)]^{-k-l} J^{\beta,\delta,\sigma,t}_{\alpha,\gamma,\mu,\rho,v,s}\left[\frac{cdx(1-x)}{[cx+d(1-x)]^2}\right] I^{m,n}_{p_i,q_i;r}[\frac{zx(1-x)}{[cx+d(1-x)]^2}\left|\begin{matrix}(a_j, A_j)_{1,n} : (a_{ji}, A_{ji})_{1,pi}\\(b_j, B_j)_{1,m} : (a_{ji}, B_{ji})_{1,qi}\end{matrix}\right.]dx$$

$$= \frac{1}{c^k d^l}\sum_{h=0}^\infty \frac{(-1)^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(v)_{sh}(\mu)_{\rho h}} I^{m,n+2}_{p_i,q_i+1;r}\left[\frac{z}{cd}\middle|\begin{matrix}(1-k-h,1),(1-l-h,1),(a_j, A_j)_{1,n} : (a_{ji}, A_{ji})_{1,pi}\\(b_j, B_j)_{1,m} : (a_{ji}, B_{ji})_{1,qi},(1-k-l-2h,2)\end{matrix}\right],$$

Proof. In the similar manner of the ***Theorems* 2.1** and **2.2**, and using the result of Eqn. (1.7), we obtain the result (2.7) of the ***Theorem* 2.3**.

3 Special cases

Here, among numerous special cases of the results in ***Section* 2**, only three of which are presented.

1. On replacing α by $\alpha - 1$ in the ***Theorem* 2.1**,we get

$$(3.1)\quad \int_0^1\int_0^1 y^k(1-x)^{k-1}(1-y)^{l-1}(1-xy)^{1-k-l} E^{\beta,\delta,\sigma,t}_{\alpha,\gamma,\mu,\rho,v,s}\left[\frac{y(1-x)(1-y)}{(1-xy)^2}\right] I^{m,n}_{p_i,q_i;r}\left[\frac{zy(1-x)(1-y)}{(1-xy)^2}\middle|\begin{matrix}(a_j, A_j)_{1,n} : (a_{ji}, A_{ji})_{1,pi}\\(b_j, B_j)_{1,m} : (a_{ji}, B_{ji})_{1,qi}\end{matrix}\right]dxdy$$

$$= \sum_{h=0}^\infty \frac{(-1)^h(\beta)_{\delta h}(\sigma)_{th}}{\Gamma(\gamma h+\alpha)(v)_{sh}(\mu)_{\rho h}} I^{m,n+2}_{p_i,q_i+1;r}\left[z\middle|\begin{matrix}(a_j, A_j)_{1,n},(1-k-h,1),(1-l-h,1) : (a_{ji}, A_{ji})_{1,pi}\\(b_j, B_j)_{1,m} : (a_{ji}, B_{ji})_{1,qi},(1-k-l-2h,2)\end{matrix}\right],$$

provided that $|\arg z| < \frac{\pi}{2}A', A' = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \max_{1\leq i\leq r}[\sum_{j=1}^{p_i} A_{ji} + \sum_{j=1}^{q_i} B_{ji}] > 0,$ and $\gamma,\mu,\rho,v,\beta,\delta,\sigma \in C; (\beta) > 0, (\gamma) > 0, (\delta) > 0, (\mu) > 0, (\rho) > 0, (\sigma) > 0, (v) > 0, (\alpha) > 0; s, t > 0 and t < (\gamma) + s, 0 < (l) < (k).$

Here, $E^{\beta,\delta,\sigma,t}_{\alpha,\gamma,\mu,\rho,v,s}(z)$, a generalized Mittag - Leffler function, is defined by Khan and Ahmed [6].

2. On setting $r = 1$ in ***Theorem* 2.1**, we obtain

$$(3.2)\quad \int_0^1\int_0^1 y^k(1-x)^{k-1}(1-y)^{l-1}(1-xy)^{1-k-l} J^{\beta,\delta,\sigma,t}_{\alpha,\gamma,\mu,\rho,v,s}\left[\frac{y(1-x)(1-y)}{(1-xy)^2}\right] I^{m,n}_{p_1,q_1;1}\left[\frac{zy(1-x)(1-y)}{(1-xy)^2}\middle|\begin{matrix}(a_j, A_j)_{1,n} : (a_{j1}, A_{j1})_{1,p_1}\\(b_j, B_j)_{1,m} : (a_{j1}, B_{j1})_{1,q_1}\end{matrix}\right]dxdy$$

$$= \sum_{h=0}^{\infty} \frac{(-1)^h (\beta)_{\delta h} (\sigma)_{th}}{\Gamma(\gamma h+\alpha+1)(\nu)_{sh}(\mu)_{\rho h}} H^{m,n+2}_{p_1+n+2,q_1+m+1}\left[z \middle| \begin{matrix} (a_j,\alpha_j)_{1,n}, (1-k-h,1), (1-l-h,1) : (a_{j1},\alpha_{j1})_{n+1,p_1}, (a_{j1},\alpha_{j1})_{1,n} \\ (b_j,\beta_j)_{1,m} : (b_{j1},\beta_{j1})_{m+1,q_1}, (b_{j1},\beta_{j1})_{1,m}, (1-k-l-2h,2) \end{matrix}\right],$$

provided that $\alpha, \gamma, \mu, \rho, \nu, \beta, \delta, \sigma \in C; (\beta) > 0, (\gamma) > 0, (\delta) > 0, (\mu) > 0, (\rho) > 0, (\sigma) > 0, (\nu) > 0, (\alpha) \geq -1; s, t > 0, t < (\gamma) + s, 0 < (l) < (k), |\arg z| < \frac{\pi}{2} A'$ and $A' = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - [\sum_{j=1}^{p} A_j + \sum_{j=1}^{q} B_j] > 0$.

3. On setting $\rho = s = t = \delta = 0$ in ***Theorem* 2.1**, we obtain

$$(3.3)\quad \int_0^1 \int_0^1 y^k (1-x)^{k-1} (1-y)^{l-1} (1-xy)^{1-k-l} J^{\gamma}_{\alpha}\left[\frac{y(1-x)(1-y)}{(1-xy)^2}\right] I^{m,n}_{p_i,q_i;r}\left[\frac{zy(1-x)(1-y)}{(1-xy)^2} \middle| \begin{matrix} (a_j,A_j)_{1,n} : (a_{ji},A_{ji})_{1,pi} \\ (b_j,B_j)_{1,m} : (a_{ji},B_{ji})_{1,qi} \end{matrix}\right] dxdy$$

$$= \sum_{h=0}^{\infty} \frac{(-1)^h}{h!\Gamma(\gamma h+\alpha+1)} I^{m,n+2}_{p_i,q_i+1;r}\left[z \middle| \begin{matrix} (a_j,A_j)_{1,n}, (1-k-h,1), (1-l-h,1) : (a_{ji},A_{ji})_{1,pi} \\ (b_j,B_j)_{1,m} : (a_{ji},B_{ji})_{1,qi}, (1-k-l-2h,2) \end{matrix}\right],$$

provided that $z \in C, (\beta) > 0, (\alpha) > -1, 0 < (l) < (k), |\arg z| < \frac{\pi}{2} A$ and $A = \sum_{j=1}^{n} A_j + \sum_{j=1}^{m} B_j - \max_{1 \leq i \leq r}[\sum_{j=1}^{p_i} A_{ji} + \sum_{j=1}^{q_i} B_{ji}] > 0$, and $J^{\gamma}_{\alpha}(z)$ is Bessel-Maitland function [9, Eq.(8.3)].

Acknowledgement. We wish to express our sincere thanks to the Editor and Reviewer for their valuable suggestions to improve the paper in its present form.

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