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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

FUZZY PREOPEN SETS AND FUZZY PRE-CONTINUITY

By

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Abstract

In the present paper, we introduce fuzzy preopen (closed) sets and fuzzy pre-continuity in Sostak fuzzy topological space. Also we investigate their significant characteristic properties.
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1 Introduction

The concept of fuzzy sets was introduced by Zadeh [9] and later Chang [2] defined fuzzy topological spaces. Sostak [8] introduced a new fuzzy topological space exploiting the idea of partial openness of fuzzy sets. This generalized fuzzy topological space was later rephrased by Chattopadhyay et.al. [3]. Several mathematicians have worked on this space (see [4], [5]).

The concepts of fuzzy preopen sets, fuzzy strong preopen sets and strong pre continuity (see [6], [7]) have been introduced in case of classical fuzzy topological spaces introduced by Chang [2]. In the present paper, we introduce fuzzy preopen (closed) sets and fuzzy pre continuity in the Sostak fuzzy topological space redefined by Chattopdhyay [3]. Further we establish their significant properties.

2 Preliminaries

Let X be a non-empty set and $I \equiv [0,\ 1]$ be the unit closed interval of real line. Let I^X denote the set of all fuzzy sets on X. A fuzzy set A on X is a mapping $A : X \to I$, where for any $x \in X$, $A(x)$ denotes the degree of membership of element x in fuzzy set A. The null fuzzy set 0 and whole fuzzy set 1 are the constant mappings from X to $\{0\}$ and $\{1\}$ respectively.

A family τ of fuzzy sets on X is called a fuzzy topology (see [2]) on X if (i) 0 and 1 belong to τ, (ii) Any union of members of τ is in τ, (iii) a finite intersection of members of τ is in τ. The system consisting of X equipped with fuzzy topology τ defined on it, is called a fuzzy topological space and is denoted as (X, τ). Now we define the So-fuzzy topological space (see [3], [8]).

A So-fuzzy topology on a non-empty set X is a family τ of fuzzy sets on X satisfying the following axioms with respect to a mapping $\tau : I^X \to I$ such that

1. $\tau(0) = \tau(1) = 1$;
2. $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$; for any $A,\ B \in I^X$;
3. $\tau(\cup_{i\in J} A_i) \geq \wedge_{i\in J} \tau(A_i)$, for any arbitrary family $\{A_i\ : i \in J\} \subseteq I^X$.

The system (X, τ) is called So-fuzzy topological space and the real number $\tau(A)$ is called the degree (or grade) of openness of fuzzy set A. We note that

***Proposition* 2.1** *Let X be a non-empty set. Then the map $\tau : I^X \to I$ given by $\tau(0) = 1$ and $\tau(A) = \inf\{A(x) : x \in suppA\}$, if $A \neq 0$, satisfies the axioms of gradation of openness.*

If (X, τ) is a So-fuzzy topological space, then we observe that (see [2]) for any $\rho \in [0, 1]$, the family $\tau_\rho \equiv \{ A \in I^X : \tau(A) \geq \rho\}$ is actually a fuzzy topology in sense of Chang [2] and it is called ρ-level fuzzy topology on X with respect to the gradation of openness τ. All fuzzy sets belonging to τ_ρ are called fuzzy-ρ-open sets and their complements are called fuzzy-ρ-closed sets.

For any fuzzy set A, the interior and closure of A with respect to τ_ρ are defined as follows:

$$Int_\rho(A) = \cup\{G \in I^X :\ G \subseteq A \text{ and } G \in \tau_\rho\}$$

$$Cl_\rho(A) = \cap\{K \in I^X :\ A \subseteq\ K \text{ and } K^c \in \tau_\rho\}$$

3 Fuzzy-ρ-Pre Open (Closed) Sets

In this section, we define fuzzy-ρ-pre open sets and fuzzy-ρ-pre closed sets in So-fuzzy topological space and investigate their properties.

***Definition* 3.1** *Let $(X,\ \tau)$ be a So-fuzzy topological space and $A \in I^X$ be a fuzzy set. Then for any $\rho \in I$, a fuzzy set A is said to be a*

(i) *Fuzzy-ρ-pre open set in X iff $A \subseteq Int_\rho(Cl_\rho(A))$,*

(ii) *Fuzzy-ρ-pre closed set in X iff $A \supseteq Cl_\rho(Int_\rho(A))$.*

Clearly fuzzy sets 0 *and* 1 *are both trivially fuzzy ρ-pre open as well as fuzzy ρ-pre closed sets in X.*

***Remark* 3.1** *It is clear that every fuzzy-ρ-open (closed) set is a fuzzy-ρ-pre open (closed) set, but converse of these may not be true in general.*

***Example* 3.1** *Let $X = \{a,\ b\}$ and $A,\ B,\ C \in I^X$ be fuzzy sets defined as follows:*

$A = \{(a,\ 0.6),\ (b,\ 0.3)\}$; $\quad B = \{(a,\ 0.4),\ (b,\ 0.2)\}$; $\quad C = \{(a,\ 0.8),\ (b,\ 0.5)\}$.

Define a map $\tau : I^X \to I$ as follows:

$$\tau(F) = \begin{cases} 1, & if\ F = 0,\ 1 \\ 0.3, & if\ F = A \\ 0.2, & if\ F = B \\ 0, & otherwise. \end{cases}$$

Suppose $\rho = 0.1$. *We see that fuzzy set C is a fuzzy-ρ-pre open set because $Int_\rho(Cl_\rho(C)) = 1 \supseteq C$. But it is not a fuzzy-$\rho$-open set (because $\tau(C) = 0 \not\geq 0.1$).*

***Theorem* 3.1** *Let (X, τ) be a So-fuzzy topological space. Then for any $\rho \in I$,*

(a) *Any union of fuzzy-ρ-pre open sets is a fuzzy-ρ-pre open set;*

(b) *Any intersection of fuzzy-ρ-pre closed sets is a fuzzy-ρ-pre closed set.*

Proof. **(a)** Let $\{A_i\ :\ i \in J\}$ be an arbitrary collection of fuzzy-ρ-pre open sets in So-fuzzy topological space $(X,\ \tau)$. Then for each $i \in J$, we have $A_i \subseteq Int_\rho(Cl_\rho(A_i))$. Hence

$$\cup_{i\in J} A_i \subseteq\ \cup_{i\in J} Int_\rho(Cl_\rho(A_i)) \subseteq Int_\rho(\cup_{i\in J} Cl_\rho(A_i)) \subseteq Int_\rho(Cl_\rho(\cup_{i\in J} A_i)).$$

Thus $\cup_{i\in J}A_i$ is a fuzzy-ρ-pre open set. We can prove (b) similarly.

***Definition* 3.2** *Let $(X,\ \tau)$ be a So-fuzzy topological space and $A \in I^X$ be a fuzzy set. Then for each $\rho \in I$, fuzzy-ρ-pre interior and fuzzy-ρ-pre closure of fuzzy set A denoted as P-$int_\rho(A)$ and P-$cl_\rho(A)$ are defined as follows:*

$$P - int_\rho(A) = \cup\{G \in I^X : G \subseteq A \text{ and } G \text{ is a } fuzzy - \rho - pre\ open\ set\ in\ X\},$$

$$P - cl_\rho(A) \ \ = \cap\{K \in I^X : K \supseteq A \text{ and } K \text{ is a } fuzzy - \rho - pre\ closed\ set\ in\ X\}.$$

***Theorem* 3.2** *Let $(X,\ \tau)$ be a So-fuzzy topological space and $A \in I^X$ be a fuzzy set. Then for any $\rho \in I$,*

(i) *P-$cl_\rho(1 - A) = 1 -$ P-$int_\rho(A)$,*

(ii) *P-$int_\rho(1 - A) = 1 -$ P-$cl_\rho(A)$.*

Proof. **(i)** Suppose $\{G_i\}_{i\in J}$ is the family of all fuzzy-ρ-preopen sets in X contained in A. Then

$$P - int_\rho(A) =\ \cup_{i\in J}G_i = 1 - \cap_{i\in J}\ G_i^c.$$

Since $G_i \subseteq A$, we have $G_i^c \supseteq A^c,\ \forall\ i \in J$. Thus $\{G_i^c\}_{i\in J}$ is the collection of all fuzzy-ρ-preclosed sets containing A^c. Hence $\cap_{i\in J}G_i^c =$ P-$cl_\rho(A^c) =$ P-$cl_\rho(1 - A)$. Thus P-$int_\rho(A) = 1 -$ P-$cl_\rho(1 - A)$. Hence P-$cl_\rho(1 - A) = 1 -$ P-$int_\rho(A)$.

Proof. **(ii)** It can be proved in a similar manner.

***Theorem* 3.3** *Let $(X,\ \tau)$ be a So-fuzzy topological space. Then for any $\rho \in I$, a fuzzy set $A \in I^X$ is a*

(a) *Fuzzy-ρ-pre open set iff P-$int_\rho(A) = A$;*

(b) *Fuzzy-ρ-pre closed set iff P-$cl_\rho(A) = A$.*

Proof. **(a)** Let A be fuzzy-ρ-pre open set in X. Let $\{G_i\}_{i\in J}$ be the family of all fuzzy ρ-pre open sets contained in A. Since each $G_i \subseteq A,\ \ i \in J$, we have $\cup_{i\in J} G_i \subseteq A$. Therefore

(3.1) $P - int_\rho = \cup_{i\in J}\{G_i \in I^X : G_i \subseteq A$ and G_i is a fuzzy $- \rho - preopen\ set\} \subseteq A$.

Since $A \subseteq A$ and A is a fuzzy-ρ-preopen set in X, hence $A \in \{G_i\}_{i\in J}$. Therefore

(3.2) $A \subseteq\ \cup_{i\in J} G_i \equiv P - int_\rho(A)$.

From equations (3.3.1) and (3.3.2), $A = P\text{-}int_\rho(A)$.

Conversely; suppose A is a fuzzy set in So-fuzzy topological space $(X,\ \tau)$ such that $A = P\text{-}int_\rho(A)$. Then

(3.3) $A = P - int_\rho(A) = \cup\{G_i \in I^X\ : G_i \subseteq A$ and $G_i$is a fuzzy $- \rho -$ pre open set$\}$.

Since any union of fuzzy-ρ-preopen sets is a fuzzy-ρ-preopen set, in view of (3.3.3), set A is a fuzzy-ρ-pre open set in X.

Proof. **(b)** This can be proved in a similar manner.

***Theorem* 3.4** *Let $(X,\ \tau)$ be a So-fuzzy topological space. Then for any $\rho \in I$, the following properties hold for fuzzy-ρ-pre closure:*

(i) $P\text{-}cl_\rho(0) = 0$;
(ii) $P\text{-}cl_\rho(A)$ is a fuzzy-ρ-pre closed set in X;
(iii) $P\text{-}cl_\rho(A) \subseteq P\text{-}cl_\rho(B)$, if $A \subseteq B$;
(iv) $P\text{-}cl_\rho(P\text{-}cl_\rho(A)) = P\text{-}cl_\rho(A)$;
(v) $P\text{-}cl_\rho(A \cup B) \supseteq P\text{-}cl_\rho(A) \cup P\text{-}cl_\rho(B)$;
(vi) $P\text{-}cl_\rho(A \cap B) \subseteq P\text{-}cl_\rho(A) \cap P\text{-}cl_\rho(B)$.

Proof. It is easy to prove.

***Theorem* 3.5** *Let $(X,\ \tau)$ be a So-fuzzy topological space and $A,\ B \in I^X$ be fuzzy sets. Then for any $\rho \in I$,*

(i) $P\text{-}int_\rho(1) = 1$;
(ii) $P\text{-}int_\rho(A)$ is a fuzzy-ρ-pre open set in X;
(iii) $P\text{-}int_\rho(A) \subseteq P\text{-}int_\rho(B)$, if $A \subseteq B$;
(iv) $P\text{-}int_\rho(P\text{-}int_\rho(A)) = P\text{-}int_\rho(A)$;
(v) $P\text{-}int_\rho(A \cup B) \supseteq P\text{-}int_\rho(A) \cup P\text{-}int_\rho(B)$;
(vi) $P\text{-}int_\rho(A \cap B) \subseteq P\text{-}int_\rho(A) \cap P\text{-}int_\rho(B)$.

4 Fuzzy-ρ-Pre Continuous Map

In this section, we define a fuzzy-ρ-pre continuous map from one So-fuzzy topological space to another and investigate its characteristic properties. We know fuzzy-ρ-continuous map is defined (see [3]) as follows:

***Definition* 4.1** *Let $(X,\ \tau)$ and $(Y,\ \sigma)$ be two So-fuzzy topological spaces. A map $f : X \to Y$ is said to be a fuzzy-ρ-continuous map if $\tau(f^{-1}(B)) \geq \sigma(B)$, for each fuzzy set $B \in I^Y$ such that $\sigma(B) \geq \rho$.*

Now we define fuzzy-ρ-pre continuous map as follow:

***Definition* 4.2** *Let $(X,\ \tau)$ and $(Y,\ \sigma)$ be two So-fuzzy topological spaces. A map f from X to Y is called a fuzzy-ρ-pre continuous map iff $f^{-1}(B)$ is a fuzzy-ρ-pre open set for any fuzzy set $B \in I^Y$ such that $\sigma(B) \geq \rho$.*

***Remark* 4.1** *It is obvious that every fuzzy-ρ-continuous map is a fuzzy-ρ-pre continuous map, but converse may not be true.*

***Example* 4.1** *Let $X = \{a,\ b\}$, $Y = \{u,\ v\}$ and $A,\ B \in I^X,\ C \in I^Y$ be fuzzy sets defined as follows:*

$A = \{(a,\ 0.7),\ (b,\ 0.2)\}$; $B = \{(a,\ 0.5),\ (b,\ 0.6)\}$; $C = \{(a,\ 0.7),\ (b,\ 0.6)\}$;
$D = \{(a,\ 0.5),\ (b,\ 0.2)\}$; $E = \{(u,\ 0.8),\ (v,\ 0.7)\}$.

We define fuzzy topologies $\tau : I^X \to I$ and $\sigma : I^Y \to I$ as follows:

$$\tau(F) = \begin{cases} 1, & \text{if } F = 0,\ 1 \\ 0.2, & \text{if } F = A,\ D \\ 0.5, & \text{if } F = B \\ 0.6 & \text{if } F = C \\ 0, & \text{otherwise,} \end{cases}$$

$$\sigma(F) = \begin{cases} 1, & \text{if } F = 0,\ 1 \\ 0.7, & \text{if } F = E \\ 0, & \text{otherwise.} \end{cases}$$

Consider a map $f : (X,\ \tau) \to (Y,\ \sigma)$ defined as $f(A) = u,\ f(b) = v$. Suppose $\rho = 0.1$. We see that $f^{-1}(E) \subseteq Int_\rho(Cl_\rho(f^{-1}(E)))$. Hence $f^{-1}(E)$ is a fuzzy-ρ-pre open set. Similarly $f^{-1}(0) \equiv 0$ and $f^{-1}(1) \equiv 1$ are also fuzzy-ρ-pre open sets. Thus f is a fuzzy-ρ-pre continuous map. But f is not a fuzzy-ρ-continuous map because $f^{-1}(E)$ is not a fuzzy-ρ-open set.

***Theorem* 4.1** *Let $f : (X.\tau) \to (Y,\ \sigma)$ be a map from one So-fuzzy topological space to another such that $\tau^*(f^{-1}(B)) \geq \rho$, for each $B \in I^Y$ with $\sigma^*(B) \geq \rho$, then f is a fuzzy-ρ-pre continuous map.*

Proof. Let $f : (X,\ \tau) \to (Y,\ \sigma)$ be a map such that $\tau^*(f^{-1}(B)) \geq \rho$, for each $B \in I^Y$ for which $\sigma^*(B) \geq \rho$. Since $f^{-1}(B) \in I^X$ and $\tau^*(f^{-1}(B)) = \tau((f^{-1}(B))^c) = \tau(f^{-1}(B^c)) \geq \rho$, we conclude that $f^{-1}(B^c)$ is a fuzzy-ρ-open set in X. Since every fuzzy ρ-open set is a fuzzy ρ-pre open set, $f^{-1}(B^c)$ is a fuzzy-ρ-pre open set in X. Further $\sigma(B^c) = \sigma^*(B) \geq \rho$. Thus $f^{-1}(B^c)$ is a fuzzy-ρ-pre open set in X for each $B^c \in I^Y$ such that $\sigma(B^c) \geq \rho$. Therefore f is a fuzzy-ρ-pre continuous map.

***Theorem* 4.2** *Let $f : (X.\ \tau) \to (Y,\ \sigma)$ be a map from one So-fuzzy topological space to another. Then for any $\rho \in I$, following statements are equivalent:*

(a) f is a fuzzy-ρ-pre continuous map;
(b) $f^{-1}(B)$ is a fuzzy-ρ-pre closed set for each fuzzy-ρ-closed set B in Y;
(c) $Cl_\rho(Int_\rho(f^{-1}(B))) \subseteq f^{-1}(Cl_\rho(B))$, for each fuzzy set B in Y;
(d) $f(Cl_\rho(Int_\rho(A))) \subseteq Cl_\rho(f(A))$, for each fuzzy set A in X.

Proof. Let $(X,\ \tau)$ and $(Y,\ \sigma)$ be two So-fuzzy topological spaces. We will prove this theorem in following steps:

(i) $(a)arrow(b)$: Let $f : X \to Y$ be a fuzzy-ρ-pre continuous map for any $\rho \in I$. Let B be a fuzzy-ρ-closed set in Y. Then B^c is a fuzzy-ρ-open set in Y so that $\sigma(B^c) \geq \rho$. Since f is a fuzzy ρ-continuous map, we find that $f^{-1}(B^c)$ is a fuzzy-ρ-pre open set in X. Therefore $(f^{-1}(B^c))^c = f^{-1}(B)$ is a fuzzy-ρ-pre closed set in X. Similarly we can prove (b)$arrow$(a).

(ii) $(b)arrow(c)$: Let B be a fuzzy set in Y, then $Cl_\rho(B)$ is a fuzzy-ρ-closed set in Y and hence by (b), $f^{-1}(Cl_\rho(B))$ is a fuzzy-ρ-pre closed set in X. Therefore by definition, $f^{-1}(Cl_\rho(B)) \supseteq Cl_\rho(Int_\rho(f^{-1}(Cl_\rho(B)))) \supseteq Cl_\rho(Int_\rho(f^{-1}(B)))$. Thus $Cl_\rho(Int_\rho(f^{-1}(B))) \subseteq f^{-1}(Cl_\rho(B))$.

(iii) $(c)arrow(d)$: Let $A \in I^X$ be any fuzzy set, then $f(A) \in I^Y$. Now by (c), $Cl_\rho(Int_\rho(f^{-1}(f(A)))) \subseteq f^{-1}(Cl_\rho(f(A)))$. It implies that $Cl_\rho(Int_\rho(A)) \subseteq f^{-1}(Cl_\rho(f(A)))$. Hence $f(Cl_\rho(Int_\rho(A))) \subseteq f(f^{-1}(Cl_\rho(f(A)))) \subseteq Cl_\rho(f(A))$.

(iv) $(d)arrow(b)$: can be proved easily.

***Theorem* 4.3** *Let $(X,\ \tau)$, $(Y,\ \sigma)$ and $(Z,\ \delta)$ be three So-fuzzy topological spaces and let $\rho \in I$ be any real number. If $f : X \to Y$ is a fuzzy-ρ-pre continuous map and $g : Y \to Z$ is a fuzzy-ρ-continuous map, then $g \circ f : X \to Z$ is a fuzzy-ρ-pre continuous map.*

Proof. Let C be a fuzzy-ρ-open set in Z so that $\delta(C) \geq \rho$, then $\sigma(g^{-1}(C)) \geq \delta(C) \geq \rho$. Thus by hypothesis $g^{-1}(C)$ is a fuzzy-ρ-open set in Y. Since f is a fuzzy-ρ-pre continuous map, we get that $f^{-1}(g^{-1}(C))$ is a fuzzy-ρ-pre open set in X. Now $f^{-1}(g^{-1}(C)) = (g \circ f)^{-1}(C)$. Hence $(g \circ f)^{-1}(C)$ is a fuzzy-ρ-pre open set in X. Now $g \circ f : (X,\ \tau) \to (Z,\ \delta)$ is a map and we have derived that for any fuzzy-ρ-open set C in Z, fuzzy set $(g \circ f)^{-1}(C)$ is a fuzzy-ρ-pre open set in X. Hence $(g \circ f)$ is a fuzzy-ρ-pre continuous map.

5 Conclusion

In the present paper, we have defined fuzzy pre open (closed) sets and fuzzy pre-continuity in Sostak fuzzy topological space. The concept is introduced as an extension of concepts of fuzzy preopen sets introduced in [6]. Several significant results have been obtained.

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