FUZZY PREOPEN SETS AND FUZZY PRE-CONTINUITY

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Abstract
In the present paper, we introduce fuzzy preopen (closed) sets and fuzzy pre-continuity in Sostak fuzzy topological space. Also we investigate their significant characteristic properties.

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1 Introduction
The concept of fuzzy sets was introduced by Zadeh [9] and later Chang [2] defined fuzzy topological spaces. Sostak [8] introduced a new fuzzy topological space exploiting the idea of partial openness of fuzzy sets. This generalized fuzzy topological space was later rephrased by Chattopadhyay et.al. [3]. Several mathematicians have worked on this space (see [4], [5]).

The concepts of fuzzy preopen sets, fuzzy strong preopen sets and strong pre continuity (see [6], [7]) have been introduced in case of classical fuzzy topological spaces introduced by Chang [2]. In the present paper, we introduce fuzzy preopen (closed) sets and fuzzy precontinuity in the Sostak fuzzy topological space redefined by Chattopadhyay [3]. Further we establish their significant properties.

2 Preliminaries
Let $X$ be a non-empty set and $I \equiv [0, 1]$ be the unit closed interval of real line. Let $I^X$ denote the set of all fuzzy sets on $X$. A fuzzy set $A$ on $X$ is a mapping $A : X \to I$, where for any $x \in X$, $A(x)$ denotes the degree of membership of element $x$ in fuzzy set $A$. The null fuzzy set 0 and whole fuzzy set 1 are the constant mappings from $X$ to $[0]$ and $[1]$ respectively.

A family $\tau$ of fuzzy sets on $X$ is called a fuzzy topology (see [2]) on $X$ if (i) 0 and 1 belong to $\tau$, (ii) Any union of members of $\tau$ is in $\tau$, (iii) a finite intersection of members of $\tau$ is in $\tau$. The system consisting of $X$ equipped with fuzzy topology $\tau$ defined on it, is called a fuzzy topological space and is denoted as $(X, \tau)$. Now we define the So-fuzzy topological space (see [3], [8]).

A So-fuzzy topology on a non-empty set $X$ is a family $\tau$ of fuzzy sets on $X$ satisfying the following axioms with respect to a mapping $\tau : I^X \to I$ such that

1. $\tau(0) = \tau(1) = 1$;
2. $\tau(A \cap B) \geq \tau(A) \wedge \tau(B)$; for any $A, B \in I^X$;
3. $\tau(\bigcup_{i \in J} A_i) \geq \bigwedge_{i \in J} \tau(A_i)$, for any arbitrary family $\{A_i : i \in J\} \subseteq I^X$.

The system $(X, \tau)$ is called So-fuzzy topological space and the real number $\tau(A)$ is called the degree (or grade) of openness of fuzzy set $A$. We note that

**Proposition 2.1** Let $X$ be a non-empty set. Then the map $\tau : I^X \to I$ given by $\tau(0) = 1$ and $\tau(A) = \inf\{A(x) : x \in \text{supp} A\}$, if $A \neq 0$, satisfies the axioms of gradation of openness.

If $(X, \tau)$ is a So-fuzzy topological space, then we observe that (see [2]) for any $\rho \in [0, 1]$, the family $\tau_\rho = \{ A \in I^X : \tau(A) \geq \rho\}$ is actually a fuzzy topology in sense of Chang [2] and it is called $\rho$-level fuzzy topology on $X$ with respect to the gradation of openness $\tau$. All fuzzy sets belonging to $\tau_\rho$ are called fuzzy-$\rho$-open sets and their complements are called fuzzy-$\rho$-closed sets.

For any fuzzy set $A$, the interior and closure of $A$ with respect to $\tau_\rho$ are defined as follows:

\[ \text{Int}_\rho(A) = \bigcup\{G \in I^X : G \subseteq A \text{ and } G \in \tau_\rho\} \]
\[ \text{Cl}_\rho(A) = \bigcap\{K \in I^X : A \subseteq K \text{ and } K^c \in \tau_\rho\} \]
3 Fuzzy-$\rho$-Pre Open (Closed) Sets

In this section, we define fuzzy-$\rho$-pre open sets and fuzzy-$\rho$-pre closed sets in So-fuzzy topological space and investigate their properties.

**Definition 3.1** Let $(X, \tau)$ be a So-fuzzy topological space and $A \in I^X$ be a fuzzy set. Then for any $\rho \in I$, a fuzzy set $A$ is said to be

(i) Fuzzy-$\rho$-pre open set in $X$ iff $A \subseteq \text{Int}_\rho(Cl_\rho(A))$,

(ii) Fuzzy-$\rho$-pre closed set in $X$ iff $A \supseteq Cl_\rho(\text{Int}_\rho(A))$.

Clearly fuzzy sets 0 and 1 are both trivially fuzzy $\rho$-pre open as well as fuzzy $\rho$-pre closed sets in $X$.

**Remark 3.1** It is clear that every fuzzy-$\rho$-open (closed) set is a fuzzy-$\rho$-pre open (closed) set, but converse of these may not be true in general.

**Example 3.1** Let $X = \{a, b\}$ and $A, B, C \in I^X$ be fuzzy sets defined as follows:

$A = \{(a, 0.6), (b, 0.3)\}; \quad B = \{(a, 0.4), (b, 0.2)\}; \quad C = \{(a, 0.8), (b, 0.5)\}.$

Define a map $\tau : I^X \rightarrow I$ as follows:

$\tau(F) = \begin{cases} 
1, & \text{if } F = 0, 1 \\
0.3, & \text{if } F = A \\
0.2, & \text{if } F = B \\
0, & \text{otherwise}.
\end{cases}$

Suppose $\rho = 0.1$. We see that fuzzy set $C$ is a fuzzy-$\rho$-pre open set because $\text{Int}_\rho(Cl_\rho(C)) = 1 \supseteq C$. But it is not a fuzzy-$\rho$-open set (because $\tau(C) = 0 \neq 0.1$).

**Theorem 3.1** Let $(X, \tau)$ be a So-fuzzy topological space. Then for any $\rho \in I$,

(a) Any union of fuzzy-$\rho$-pre open sets is a fuzzy-$\rho$-pre open set;

(b) Any intersection of fuzzy-$\rho$-pre closed sets is a fuzzy-$\rho$-pre closed set.

**Proof.** (a) Let $\{A_i : i \in J\}$ be an arbitrary collection of fuzzy-$\rho$-pre open sets in So-fuzzy topological space $(X, \tau)$. Then for each $i \in J$, we have $A_i \subseteq \text{Int}_\rho(Cl_\rho(A_i))$. Hence

$\bigcup_{i \in J} A_i \subseteq \bigcup_{i \in J} \text{Int}_\rho(Cl_\rho(A_i)) \subseteq \text{Int}_\rho(\bigcup_{i \in J} Cl_\rho(A_i)) \subseteq \text{Int}_\rho(Cl_\rho(\bigcup_{i \in J} A_i)).$

Thus $\bigcup_{i \in J} A_i$ is a fuzzy-$\rho$-pre open set. We can prove (b) similarly.

**Definition 3.2** Let $(X, \tau)$ be a So-fuzzy topological space and $A \in I^X$ be a fuzzy set. Then for each $\rho \in I$, fuzzy-$\rho$-pre interior and fuzzy-$\rho$-pre closure of fuzzy set $A$ denoted as $P-\text{int}_\rho(A)$ and $P-\text{cl}_\rho(A)$ are defined as follows:

$P-\text{int}_\rho(A) = \bigcup\{G \in I^X : G \subseteq A \text{ and } G \text{ is a fuzzy } \rho \text{-pre open set in } X\},$

$P-\text{cl}_\rho(A) = \cap\{K \in I^X : K \supseteq A \text{ and } K \text{ is a fuzzy } \rho \text{-pre closed set in } X\}.$

**Theorem 3.2** Let $(X, \tau)$ be a So-fuzzy topological space and $A \in I^X$ be a fuzzy set. Then for any $\rho \in I$,

(i) $P-\text{cl}_\rho(1 - A) = 1 - P-\text{int}_\rho(A),$

(ii) $P-\text{int}_\rho(1 - A) = 1 - P-\text{cl}_\rho(A).$

**Proof.** (i) Suppose $\{G_i\}_{i \in J}$ is the family of all fuzzy-$\rho$-preopen sets in $X$ contained in $A$. Then

$P-\text{int}_\rho(A) = \bigcup_{i \in J} G_i = 1 - \bigcap_{i \in J} G_i^c.$

Since $G_i \subseteq A$, we have $G_i^c \supseteq A^c$, $\forall i \in J$. Thus $\{G_i^c\}_{i \in J}$ is the collection of all fuzzy-$\rho$-preclosed sets containing $A^c$. Hence $\bigcap_{i \in J} G_i^c = P-\text{cl}_\rho(A^c) = P-\text{cl}_\rho(1 - A)$. Thus $P-\text{int}_\rho(A) = 1 - P-\text{cl}_\rho(1 - A)$. Hence $P-\text{cl}_\rho(1 - A) = 1 - P-\text{int}_\rho(A)$.

(ii) It can be proved in a similar manner.

**Theorem 3.3** Let $(X, \tau)$ be a So-fuzzy topological space. Then for any $\rho \in I$, a fuzzy set $A \in I^X$ is a

(a) Fuzzy-$\rho$-pre open set iff $P-\text{int}_\rho(A) = A$;

(b) Fuzzy-$\rho$-pre closed set iff $P-\text{cl}_\rho(A) = A$. 

45
In this section, we define a fuzzy-

\[ P \cap \rho = \bigcup_{i \in J} G_i \subseteq A \] 

Since each \( G_i \subseteq A \), \( i \in J \), we have \( \bigcup_{i \in J} G_i \subseteq A \). Therefore

\[ A \subseteq \bigcup_{i \in J} G_i = P - \text{pre} \rho(A). \]

From equations (3.3) and (3.2), \( A = P - \text{int} \rho(A). \)

Conversely; suppose \( A \) is a fuzzy set in So-fuzzy topological space \( X, \tau \) such that \( A = P - \text{int} \rho(A). \) Then

\[ A = P - \text{int} \rho(A) = \bigcup\{ G_i \in I^X : G_i \subseteq A \text{ and } G_i \text{ is a fuzzy - } \rho - \text{preopen set} \}. \]

Since any union of fuzzy-preopen sets is a fuzzy-preopen set, in view of (3.3), set \( A \) is a fuzzy-\( \rho \)-pre open set in \( X \).

**Proof.** (b) This can be proved in a similar manner.

**Theorem 3.4** Let \( (X, \tau) \) be a So-fuzzy topological space. Then for any \( \rho \in I \), the following properties hold for fuzzy-pre closure:

(i) \( P - \text{cl}_\rho(0) = 0 \);

(ii) \( P - \text{cl}_\rho(A) \) is a fuzzy-\( \rho \)-pre closed set in \( X \);

(iii) \( P - \text{cl}_\rho(A) \subseteq P - \text{cl}_\rho(B) \) if \( A \subseteq B \);

(iv) \( P - \text{cl}_\rho(P - \text{cl}_\rho(A)) = P - \text{cl}_\rho(A) \);

(v) \( P - \text{cl}_\rho(A \cup B) \supseteq P - \text{cl}_\rho(A) \cup P - \text{cl}_\rho(B) \);

(vi) \( P - \text{cl}_\rho(A \cap B) \subseteq P - \text{cl}_\rho(A) \cap P - \text{cl}_\rho(B) \).

**Proof.** It is easy to prove.

**Theorem 3.5** Let \( (X, \tau) \) be a So-fuzzy topological space and \( A, B \in I^X \) be fuzzy sets. Then for any \( \rho \in I \),

(i) \( P - \text{int}_\rho(1) = 1 \);

(ii) \( P - \text{int}_\rho(A) \) is a fuzzy-\( \rho \)-pre open set in \( X \);

(iii) \( P - \text{int}_\rho(A) \subseteq P - \text{int}_\rho(B) \) if \( A \subseteq B \);

(iv) \( P - \text{int}_\rho(P - \text{int}_\rho(A)) = P - \text{int}_\rho(A) \);

(v) \( P - \text{int}_\rho(A \cup B) \supseteq P - \text{int}_\rho(A) \cup P - \text{int}_\rho(B) \);

(vi) \( P - \text{int}_\rho(A \cap B) \subseteq P - \text{int}_\rho(A) \cap P - \text{int}_\rho(B) \).

4 Fuzzy-\( \rho \)-Pre Continuous Map

In this section, we define a fuzzy-\( \rho \)-pre continuous map from one So-fuzzy topological space to another and investigate its characteristic properties. We know fuzzy-\( \rho \)-continuous map is defined (see [3]) as follows:

**Definition 4.1** Let \( (X, \tau) \) and \( (Y, \sigma) \) be two So-fuzzy topological spaces. A map \( f : X \to Y \) is said to be a fuzzy-\( \rho \)-continuous map if \( \tau(f^{-1}(B)) \geq \sigma(B) \), for each fuzzy set \( B \in I^Y \) such that \( \sigma(B) \geq \rho \).

Now we define fuzzy-\( \rho \)-pre continuous map as follow:

**Definition 4.2** Let \( (X, \tau) \) and \( (Y, \sigma) \) be two So-fuzzy topological spaces. A map \( f \) from \( X \) to \( Y \) is called a fuzzy-\( \rho \)-pre continuous map if \( f^{-1}(B) \) is a fuzzy-\( \rho \)-pre open set for any fuzzy set \( B \in I^Y \) such that \( \sigma(B) \geq \rho \).

**Remark 4.1** It is obvious that every fuzzy-\( \rho \)-pre continuous map is a fuzzy-\( \rho \)-pre continuous map, but converse may not be true.

**Example 4.1** Let \( X = [a, b], Y = [u, v] \) and \( A, B \in I^X, C \in I^Y \) be fuzzy sets defined as follows:

\[
A = \{(a, 0.7), (b, 0.2)\}; \quad \quad B = \{(a, 0.5), (b, 0.6)\}; \quad \quad C = \{(a, 0.7), (b, 0.6)\};
\]

\[
D = \{(a, 0.5), (b, 0.2)\}; \quad \quad E = \{(u, 0.8), (v, 0.7)\}.
\]

We define fuzzy topologies \( \tau : I^X \to I \) and \( \sigma : I^Y \to I \) as follows:

\[ \tau(F) = \begin{cases} 
1, & \text{if } F = 0, 1 \\
0.2, & \text{if } F = A, D \\
0.5, & \text{if } F = B \\
0.6, & \text{if } F = C \\
0, & \text{otherwise}, 
\end{cases} \]

\[ \sigma(G) = \begin{cases} 
1, & \text{if } G = 0, 1 \\
0.2, & \text{if } G = D \\
0.5, & \text{if } G = B \\
0.6, & \text{otherwise}, 
\end{cases} \]
Consider a map \( f : (X, \tau) \rightarrow (Y, \sigma) \) defined as \( f(A) = u, \ f(b) = v \). Suppose \( \rho = 0.1 \). We see that \( f^{-1}(E) \subseteq \text{Int}_\rho(\text{Cl}_\rho(f^{-1}(E))) \). Hence \( f^{-1}(E) \) is a fuzzy-\( \rho \)-pre open set. Similarly \( f^{-1}(0) \equiv 0 \) and \( f^{-1}(1) \equiv 1 \) are also fuzzy-\( \rho \)-pre open sets. Thus \( f \) is a fuzzy-\( \rho \)-pre continuous map. But \( f \) is not a fuzzy-\( \rho \)-continuous map because \( f^{-1}(E) \) is not a fuzzy-\( \rho \)-open set.

**Theorem 4.1** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a map from one So-fuzzy topological space to another such that \( \tau^*(f^{-1}(B)) \geq \rho \), for each \( B \in I^Y \) with \( \sigma^*(B) \geq \rho \), then \( f \) is a fuzzy-\( \rho \)-pre continuous map.

**Proof.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a map such that \( \tau^*(f^{-1}(B)) \geq \rho \), for each \( B \in I^Y \) for which \( \sigma^*(B) \geq \rho \). Since \( f^{-1}(B) \in I^X \) and \( \tau^*(f^{-1}(B)) = \tau((f^{-1}(B))^\rho) = \tau(f^{-1}(B^\rho)) \geq \rho \), we conclude that \( f^{-1}(B^\rho) \) is a fuzzy-\( \rho \)-open set in \( X \). Since every fuzzy \( \rho \)-open set is a fuzzy \( \rho \)-pre open set, \( f^{-1}(B^\rho) \) is a fuzzy-\( \rho \)-pre open set in \( X \). Further \( \sigma(B^\rho) = \sigma(B) \geq \rho \). Thus \( f^{-1}(B^\rho) \) is a fuzzy-\( \rho \)-pre open set in \( X \) for each \( B^\rho \in I^X \) such that \( \sigma(B^\rho) \geq \rho \). Therefore \( f \) is a fuzzy-\( \rho \)-pre continuous map.

**Theorem 4.2** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a map from one So-fuzzy topological space to another. Then for any \( \rho \in I \), following statements are equivalent:

(a) \( f \) is a fuzzy-\( \rho \)-pre continuous map;
(b) \( f^{-1}(B) \) is a fuzzy-\( \rho \)-pre closed set for each fuzzy-\( \rho \)-closed set \( B \) in \( Y \);
(c) \( \text{Cl}_\rho(\text{Int}_\rho(f^{-1}(B))) \subseteq f^{-1}(\text{Cl}_\rho(B)) \), for each fuzzy set \( B \) in \( Y \);
(d) \( f(\text{Cl}_\rho(\text{Int}_\rho(A))) \subseteq \text{Cl}_\rho(f(A)) \), for each fuzzy set \( A \) in \( X \).

**Proof.** Let \( (X, \tau) \) and \( (Y, \sigma) \) be two So-fuzzy topological spaces. We will prove this theorem in following steps:

(i) (a)\( \rightarrow \)(b): Let \( f : X \rightarrow Y \) be a fuzzy-\( \rho \)-pre continuous map for any \( \rho \in I \). Let \( B \) be a fuzzy-\( \rho \)-closed set in \( Y \). Then \( B^\rho \) is a fuzzy-\( \rho \)-open set in \( Y \) so that \( \sigma(B^\rho) \geq \rho \). Since \( f \) is a fuzzy-\( \rho \)-continuous map, we find that \( f^{-1}(B^\rho) \) is a fuzzy-\( \rho \)-pre open set in \( X \). Therefore \( f^{-1}(B^\rho)^\rho = f^{-1}(B) \) is a fuzzy-\( \rho \)-pre open set in \( X \). Similarly we can prove (b)\( \rightarrow \)(a).

(ii) (b)\( \rightarrow \)(c): Let \( B \) be a fuzzy set in \( Y \), then \( \text{Cl}_\rho(B) \) is a fuzzy-\( \rho \)-closed set in \( Y \) and hence by (b), \( f^{-1}(\text{Cl}_\rho(B)) \) is a fuzzy-\( \rho \)-pre closed set in \( X \). Therefore by definition, \( f^{-1}(\text{Cl}_\rho(B)) \subseteq \text{Cl}_\rho(\text{Int}_\rho(f^{-1}(\text{Cl}_\rho(B)))) \subseteq \text{Cl}_\rho(\text{Int}_\rho(f^{-1}(B))) \).

Thus \( \text{Cl}_\rho(\text{Int}_\rho(f^{-1}(B))) \subseteq f^{-1}(\text{Cl}_\rho(B)) \).

(iii) (c)\( \rightarrow \)(d): Let \( A \in I^X \) be any fuzzy set, then \( f(A) \in I^Y \). Now by (c), \( \text{Cl}_\rho(\text{Int}_\rho(f^{-1}(f(A)))) \subseteq f^{-1}(\text{Cl}_\rho(f(A))) \).

It implies that \( \text{Cl}_\rho(\text{Int}_\rho(A)) \subseteq f^{-1}(\text{Cl}_\rho(f(A))) \). Hence \( f(\text{Cl}_\rho(\text{Int}_\rho(A))) \subseteq f(f^{-1}(\text{Cl}_\rho(f(A)))) \subseteq \text{Cl}_\rho(f(A)) \).

(iv) (d)\( \rightarrow \)(b): can be proved easily.

**Theorem 4.3** Let \( (X, \tau), (Y, \sigma) \) and \( (Z, \delta) \) be three So-fuzzy topological spaces and let \( \rho \in I \) be any real number. If \( f : X \rightarrow Y \) is a fuzzy-\( \rho \)-pre continuous map and \( g : Y \rightarrow Z \) is a fuzzy-\( \rho \)-continuous map, then \( g \circ f : X \rightarrow Z \) is a fuzzy-\( \rho \)-pre continuous map.

**Proof.** Let \( C \) be a fuzzy-\( \rho \)-open set in \( Z \) so that \( \delta(C) \geq \rho \), then \( \sigma(g^{-1}(C)) \geq \delta(C) \geq \rho \). Thus by hypothesis \( g^{-1}(C) \) is a fuzzy-\( \rho \)-open set in \( Y \). Since \( f \) is a fuzzy-\( \rho \)-pre continuous map, we get that \( f^{-1}(g^{-1}(C)) \) is a fuzzy-\( \rho \)-pre open set in \( X \). Now \( f^{-1}(g^{-1}(C)) = (g \circ f)^{-1}(C) \).

Hence \( g \circ f \) is a fuzzy-\( \rho \)-pre open set in \( X \). Now \( g \circ f : (X, \tau) \rightarrow (Z, \delta) \) is a map and we have derived that for any fuzzy-\( \rho \)-open set \( C \) in \( Z \), fuzzy set \( (g \circ f)^{-1}(C) \) is a fuzzy-\( \rho \)-pre open set in \( X \). Hence \( (g \circ f) \) is a fuzzy-\( \rho \)-pre continuous map.

5 Conclusion

In the present paper, we have defined fuzzy pre open (closed) sets and fuzzy pre-continuity in Sostak fuzzy topological space. The concept is introduced as an extension of concepts of fuzzy preopen sets introduced in [6]. Several significant results have been obtained.

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