

HYERS - ULAM STABILITY OF FIRST AND SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS

By

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(Received : March 13, 2020 ; Revised: August 17, 2020)

DOI: <https://doi.org/10.58250/jnanabha.2020.50204>

Abstract

In this paper, we prove the Hyers-Ulam (HU) stability of the first and second order partial differential equations: $u_x(x, t) + K(x, u(x, t)) = 0$ and $u_{xx}(x, t) + F(x, u)u_x(x, t) + H(x, u) = 0$ respectively.

2010 Mathematics Subject Classifications: 26D10; 35B35; 34K20; 39B52.

Keywords and phrases: Hyers Ulam stability, Partial differential equations, Banach Contraction Principle.

1 Introduction

Hyers Ulam (HU) stability of differential equation has drawn much attention since Ulam's [16] presentation of the problem on stability of group homomorphism in 1940 and Hyers' [5] partial solution to it in 1941. For ordinary differential equations one can refer [3, 15, 6, 7] and [8, 9] for partial differential equations. Its various extensions have been named with additional word. One such extension is Hyers Ulam Rassias (HUR) stability. HUR stability for linear differential operators of n^{th} order with non-constant coefficients is studied in [10] and [11]. HUR stability for special types of non-linear equations has been studied in [1, 2, 12, 13, 14]. In 2011, Gordji et al. [4], proved the HUR stability of non-linear partial differential equations by using Banach's Contraction Principle. In this paper, we prove the HU stability of first and second order partial differential equations:

$$(1.1) \quad u_x(x, t) + K(x, u(x, t)) = 0,$$

where $K : J \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, $u(x, t) \in C^1(J \times J)$, $J = [a, b]$ be a closed interval and

$$(1.2) \quad u_{xx}(x, t) + F(x, u)u_x(x, t) + H(x, u) = 0,$$

where $F, H : J \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions. Here $u(x, t) \in C^2(J \times J)$.

First we define HU stability.

Definition 1.1 The equation (1.1) is said to be HU stable if the following holds:

Let $\epsilon \geq 0$. Assume that, for any function $u(x, t) \in C^1$ satisfying the differential inequality

$$(1.3) \quad |u_x(x, t) + K(x, u(x, t))| \leq \epsilon, \quad \forall x, t \in J,$$

there exists a solution $u_0(x, t) \in C^1$ of equation (1.1) and $M(\epsilon) > 0$ such that

$$|u(x, t) - u_0(x, t)| \leq M(\epsilon), \quad \forall (x, t) \in J \times J.$$

Similarly we can define HU stability for equation (1.2).

We need the following result.

Theorem 1.1 (Banach Contraction Principle) [4] : Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a contraction, that is, there exists $\alpha \in (0, 1)$ such that $d(Tx, Ty) \leq \alpha d(x, y)$, $\forall x, y \in X$. Then \exists a unique $a \in X$ such that $Ta = a$. Moreover, $a = \lim_{n \rightarrow \infty} T^n x$ and $d(a, x) \leq \frac{1}{(1-\alpha)} d(x, Tx)$, $\forall x \in X$.

2 Main Results

In this section we prove the *HU* stability of first and second order partial differential equations (1.1) and (1.2) respectively.

Theorem 2.1 Let $x_0 \in J$ and $K : J \times R \rightarrow R$ be a continuous function such that

$$(2.1) \quad |K(x, v(x, t)) - K(x, w(x, t))| \leq \lambda |v(x, t) - w(x, t)|, \forall x, t \in J,$$

where $\lambda > 0$, $\lambda \in R$ and $v(x, t), w(x, t) \in C^1$.

Let

$$(2.2) \quad M_1 = \sup_{x \in J} \left| \int_{x_0}^x ds \right|,$$

with $0 < \lambda M_1 < 1$. Let $u(x, t) \in C^1$ satisfy

$$(2.3) \quad |u_x(x, t) + K(x, u(x, t))| \leq \epsilon, \forall x, t \in J,$$

then there exists a unique function $u_0(x, t) \in C^1$, such that

$$\left| \frac{\partial u_0}{\partial x}(x, t) + K(x, u_0(x, t)) \right| < \epsilon \quad \text{and} \quad |u(x, t) - u_0(x, t)| \leq \frac{M_1}{1 - \lambda M_1} \epsilon.$$

Proof. Consider the differential equation

$$(2.4) \quad u_x(x, t) + K(x, u(x, t)) = 0, \forall x, t \in J.$$

We define a metric d and an operator P on C^1 , respectively by

$$(2.5) \quad d(\zeta, \eta) = \sup_{x, t \in J} |\zeta(x, t) - \eta(x, t)|$$

and

$$(P\zeta)(x, t) = u(x_0, t) - \int_{x_0}^x K(s, \zeta(s, t)) ds, \forall \zeta \in C^1.$$

Consider,

$$\begin{aligned} d(P\zeta, P\eta) &= \sup_{x, t \in J} |(P\zeta)(x, t) - (P\eta)(x, t)| \\ &= \sup_{x, t \in J} \left| - \int_{x_0}^x K(s, \zeta(s, t)) ds + \int_{x_0}^x K(s, \eta(s, t)) ds \right| \\ &= \sup_{x, t \in J} \left| \int_{x_0}^x K(s, \zeta(s, t)) ds - \int_{x_0}^x K(s, \eta(s, t)) ds \right| \\ &\leq \sup_{x, t \in J} \left| \int_{x_0}^x |K(s, \zeta(s, t)) - K(s, \eta(s, t))| ds \right| \\ &\leq \sup_{x, t \in J} \left| \int_{x_0}^x \lambda |\zeta(s, t) - \eta(s, t)| ds \right| \quad (\text{by equation (2.1)}) \\ &\leq \lambda \sup_{x, t \in J} \left| \int_{x_0}^x \sup_{s, t \in J} |\zeta(s, t) - \eta(s, t)| ds \right| \\ &\leq \lambda d(\zeta, \eta) \times \sup_{x \in J} \left| \int_{x_0}^x ds \right| \\ &\leq \lambda d(\zeta, \eta) \times M_1 \quad (\text{by equation (2.2)}). \end{aligned}$$

Then by using Banach Contraction Principle, there exists a unique $u_0(x, t) \in C^1$ such that $Pu_0(x, t) = u_0(x, t)$. Thus $u_0(x, t)$ satisfy $u(x_0, t) - \int_{x_0}^x K(s, u_0(s, t)) ds = u_0(x, t)$ and

$$(2.6) \quad d(u_0, u) \leq \frac{1}{1 - \lambda M_1} d(u, Pu).$$

Now by inequality (2.3) we get,

$$-\epsilon \leq \frac{\partial u}{\partial x}(x, t) + K(x, u(x, t)) \leq \epsilon, \forall x, t \in J.$$

Integrating from x_0 to x we get,

$$\begin{aligned} -\epsilon \int_{x_0}^x ds &\leq \int_{x_0}^x \left\{ \frac{\partial u}{\partial s}(s, t) + K(s, u(s, t)) \right\} ds \leq \epsilon \int_{x_0}^x ds, \\ \Rightarrow -\epsilon \int_{x_0}^x ds &\leq \{u(x, t) - u(x_0, t) + \int_{x_0}^x K(s, u(s, t))\} ds \leq \epsilon \int_{x_0}^x ds. \end{aligned}$$

$$\begin{aligned}
&\Rightarrow -\epsilon \sup_{x \in J} \int_{x_0}^x ds \leq -\epsilon \int_{x_0}^x ds \leq \{u(x, t) - u(x_0, t) + \int_{x_0}^x K(s, u(s, t))\} ds \leq \epsilon \int_{x_0}^x ds \leq \sup_{x \in J} \int_{x_0}^x ds. \\
&\Rightarrow -\epsilon \sup_{x \in J} \left| \int_{x_0}^x ds \right| \leq -\epsilon \sup_{x \in J} \int_{x_0}^x ds \leq -\epsilon \int_{x_0}^x ds \leq \{u(x, t) - u(x_0, t) + \int_{x_0}^x K(s, u(s, t))\} ds \\
&\quad \leq \epsilon \int_{x_0}^x ds \leq \sup_{x \in J} \int_{x_0}^x ds \leq \epsilon \sup_{x \in J} \left| \int_{x_0}^x ds \right|. \\
&\Rightarrow -\epsilon M_1 \leq \{u(x, t) - u(x_0, t) + \int_x^{x_0} K(s, u(s, t))\} ds \leq \epsilon M_1. \\
&\Rightarrow |u(x, t) - u(x_0, t) + \int_x^{x_0} K(s, u(s, t)) ds| \leq \epsilon M_1. \\
&\Rightarrow |u(x, t) - (Pu)(x, t)| \leq \epsilon M_1. \\
&\Rightarrow \sup_{x, t \in J} |u(x, t) - (Pu)(x, t)| \leq \epsilon M_1. \\
&\Rightarrow d(u, Pu) \leq \epsilon M_1.
\end{aligned}$$

Using this inequality and equation (2.6) we get,

$$\begin{aligned}
|u(x, t) - u_0(x, t)| &= |u_0(x, t) - u(x, t)| \\
&\leq \sup_{x, t \in J} |u_0(x, t) - u(x, t)| \\
&= d(u_0(x, t), u(x, t)) \\
&\leq \frac{1}{1 - \lambda M_1} d(u, Pu) \\
&\leq \frac{M_1}{1 - \lambda M_1} \epsilon.
\end{aligned}$$

We now prove the *HU* stability of equation (1.2).

Theorem 2.2 Let $x_0 \in J$ and $F, H : J \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous functions such that

$$(2.7) \quad |F(x, v(x, t))v_x(x, t) - F(x, w(x, t))w_x(x, t)| \leq \lambda_1 |v(x, t) - w(x, t)|,$$

and

$$(2.8) \quad |H(x, v(x, t)) - H(x, w(x, t))| \leq \lambda_2 |v(x, t) - w(x, t)|, \quad \forall x, t \in J,$$

where $\lambda_1, \lambda_2 > 0, \lambda_1, \lambda_2 \in \mathbb{R}$ and $v(x, t), w(x, t) \in C^2(J \times J)$.

Let

$$(2.9) \quad M_2 = \sup_{x, y \in J} \left| \int_{x_0}^x \int_{x_0}^y ds dy \right|,$$

with $0 < \{\lambda_1 + \lambda_2\} M_2 < 1$. If $u(x, t) \in C^2(J \times J)$ satisfy

$$(2.10) \quad |u_{xx}(x, t) + F(x, u)u_x(x, t) + H(x, u)| \leq \epsilon, \quad \forall x, t \in J,$$

then there exists, a unique function, $u_0(x, t) \in C^2(J \times J)$,

such that $\frac{\partial^2 u_0}{\partial x^2}(x, t) + F(x, u_0(x, t))\frac{\partial u_0}{\partial x}(x, t) + H(x, u_0(x, t)) = 0$ and $|u(x, t) - u_0(x, t)| \leq \frac{M_2}{1 - \{\lambda_1 + \lambda_2\} M_2} \epsilon$.

Proof. Consider the differential equation

$$(2.11) \quad \frac{\partial^2 u}{\partial x^2}(x, t) + F(x, u(x, t))\frac{\partial u}{\partial x}(x, t) + H(x, u(x, t)) = 0, \quad \forall x, t \in J.$$

We define a metric d and an operator P on $C^2(J \times J)$, respectively by

$$(2.12) \quad d(\zeta, \eta) = \sup_{x, t \in J} |\zeta(x, t) - \eta(x, t)|,$$

and

$$(P\zeta)(x, t) = u(x_0, t) - \int_{x_0}^x \int_{x_0}^y F(s, \zeta(s, t))\zeta_s(s, t) ds dy - \int_{x_0}^x \int_{x_0}^y H(s, \zeta(s, t)) ds dy$$

$\forall \zeta \in C^2(J \times J)$.

Consider

$$d(P\zeta, P\eta) = \sup_{x, t \in J} |(P\zeta)(x, t) - (P\eta)(x, t)|$$

$$\begin{aligned}
&= \sup_{x,t \in J} \left| - \int_{x_0}^x \int_{x_0}^y F(s, \zeta(s, t)) \zeta_s(s, t) ds dy - \int_{x_0}^x \int_{x_0}^y H(s, \zeta(s, t)) ds dy \right. \\
&\quad \left. + \int_{x_0}^x \int_{x_0}^y F(s, \eta(s, t)) \eta_s(s, t) ds dy + \int_{x_0}^x \int_{x_0}^y H(s, \eta(s, t)) ds dy \right| \\
&= \sup_{x,t \in J} \left| \int_{x_0}^x \int_{x_0}^y F(s, \zeta(s, t)) \zeta_s(s, t) ds dy + \int_{x_0}^x \int_{x_0}^y H(s, \zeta(s, t)) ds dy \right. \\
&\quad \left. - \int_{x_0}^x \int_{x_0}^y F(s, \eta(s, t)) \eta_s(s, t) ds dy - \int_{x_0}^x \int_{x_0}^y H(s, \eta(s, t)) ds dy \right| \\
&= \sup_{x,t \in J} \left| \int_{x_0}^x \int_{x_0}^y F(s, \zeta(s, t)) \zeta_s(s, t) ds dy - \int_{x_0}^x \int_{x_0}^y F(s, \eta(s, t)) \eta_s(s, t) ds dy \right. \\
&\quad \left. + \int_{x_0}^x \int_{x_0}^y H(s, \zeta(s, t)) ds dy - \int_{x_0}^x \int_{x_0}^y H(s, \eta(s, t)) ds dy \right| \\
&\leq \sup_{x,t \in J} \left| \int_{x_0}^x \int_{x_0}^y F(s, \zeta(s, t)) \zeta_s(s, t) ds dy - \int_{x_0}^x \int_{x_0}^y F(s, \eta(s, t)) \eta_s(s, t) ds dy \right| \\
&\quad + \sup_{x,t \in J} \left| \int_{x_0}^x \int_{x_0}^y H(s, \zeta(s, t)) ds dy - \int_{x_0}^x \int_{x_0}^y H(s, \eta(s, t)) ds dy \right| \\
&\leq \sup_{x,t \in J} \left| \int_{x_0}^x \int_{x_0}^y |F(s, \zeta(s, t)) \zeta_s(s, t) - F(s, \eta(s, t)) \eta_s(s, t)| ds dy \right| \\
&\quad + \sup_{x,t \in J} \left| \int_{x_0}^x \int_{x_0}^y |H(s, \zeta(s, t)) - H(s, \eta(s, t))| ds dy \right| \\
&\leq \sup_{x,t \in J} \left| \int_{x_0}^x \int_{x_0}^y \lambda_1 |\zeta(s, t) - \eta(s, t)| ds dy \right| + \sup_{x,t \in J} \left| \int_{x_0}^x \int_{x_0}^y \lambda_2 |\zeta(s, t) - \eta(s, t)| ds dy \right|
\end{aligned}$$

(by equation (2.7) and (2.8))

$$\begin{aligned}
&\leq \lambda_1 \sup_{x,t \in J} \left| \int_{x_0}^x \int_{x_0}^y \sup_{s,t \in J} |\zeta(s, t) - \eta(s, t)| ds dy \right| + \\
\lambda_2 \sup_{x,t \in J} \left| \int_{x_0}^x \int_{x_0}^y \sup_{s,t \in J} |\zeta(s, t) - \eta(s, t)| ds dy \right| \\
&\leq \lambda_1 \sup_{x,t \in J} \left| \int_{x_0}^x \int_{x_0}^y d(\zeta, \eta) ds dy \right| + \lambda_2 \sup_{x,t \in J} \left| \int_{x_0}^x \int_{x_0}^y d(\zeta, \eta) ds dy \right| \\
&\leq \{\lambda_1 + \lambda_2\} d(\zeta, \eta) \sup_{x,t \in J} \left| \int_{x_0}^x \int_{x_0}^y ds dy \right| \\
&\leq \{\lambda_1 + \lambda_2\} d(\zeta, \eta) M_2 \quad (\text{by equation (2.9)}) \\
&\leq \{\lambda_1 + \lambda_2\} M_2 \times d(\zeta, \eta).
\end{aligned}$$

Therefore by using **Theorem 1.1**, there exists, a unique, $u_0(x, t) \in C^2(J \times J)$ such that $Pu_0(x, t) = u_0(x, t)$. Thus $u_0(x, t)$ satisfies

$$(2.13) \quad u(x_0, t) - \int_{x_0}^x \int_{x_0}^y F(s, u_0(s, t)) u_s(s, t) ds dy - \int_{x_0}^x \int_{x_0}^y H(s, u_0(s, t)) ds dy = u_0(x, t)$$

and

$$d(u_0, u) \leq \frac{1}{1 - (\lambda_1 + \lambda_2) M_2} d(u, Pu).$$

Now by inequality (2.10) we get,

$$-\epsilon \leq \frac{\partial^2 u}{\partial x^2}(x, t) + F(x, u) \frac{\partial u}{\partial x}(x, t) + H(x, u) \leq \epsilon, \quad \forall x, t \in J.$$

Integrating from x_0 to x we derive,

$$-\epsilon \int_{x_0}^x ds \leq \frac{\partial u}{\partial x}(x, t) - \frac{\partial u}{\partial x}(x_0, t) + \int_{x_0}^x F(s, u(s, t)) u_s(s, t) ds + \int_{x_0}^x H(s, u(s, t)) ds \leq \epsilon \int_{x_0}^x ds.$$

Again integrating from x_0 to x we obtain,

$$\begin{aligned}
-\epsilon \int_{x_0}^x \int_{x_0}^y ds dy &\leq u(x, t) - u(x_0, t) - [u(x_0, t) - u(x_0, t)] + \int_{x_0}^x \int_{x_0}^y F(s, u(s, t)) u_s(s, t) ds dy \\
&\quad + \int_{x_0}^x \int_{x_0}^y H(s, u(s, t)) ds dy \leq \epsilon \int_{x_0}^x \int_{x_0}^y ds dy.
\end{aligned}$$

By using the equation (2.12) we get,

$$\begin{aligned}
-\epsilon \int_{x_0}^x \int_{x_0}^y ds dy &\leq u(x, t) - (Pu)(x, t) \leq \epsilon \int_{x_0}^x \int_{x_0}^y ds dy. \\
\Rightarrow -\epsilon \sup_{x,y \in J} \int_{x_0}^x \int_{x_0}^y ds dy &\leq -\epsilon \int_{x_0}^x \int_{x_0}^y ds dy \leq u(x, t) - (Pu)(x, t) \leq \epsilon \int_{x_0}^x \int_{x_0}^y ds dy \leq \epsilon \sup_{x,y \in J} \int_{x_0}^x \int_{x_0}^y ds dy. \\
\Rightarrow -\epsilon \sup_{x,y \in J} \left| \int_{x_0}^x \int_{x_0}^y ds dy \right| &\leq -\epsilon \sup_{x,y \in J} \int_{x_0}^x \int_{x_0}^y ds dy \leq -\epsilon \int_{x_0}^x \int_{x_0}^y ds dy \leq u(x, t) - (Pu)(x, t)
\end{aligned}$$

$$\begin{aligned}
&\leq \epsilon \int_{x_0}^x \int_{x_0}^y dsdy \leq \epsilon \sup_{x,y \in J} \int_{x_0}^x \int_{x_0}^y dsdy \leq \epsilon \sup_{x,y \in J} \left| \int_{x_0}^x \int_{x_0}^y dsdy \right|. \\
&\Rightarrow -\epsilon \sup_{x,y \in J} \left| \int_{x_0}^x \int_{x_0}^y dsdy \right| \leq u(x, t) - (Pu)(x, t) \leq \epsilon \sup_{x,y \in J} \left| \int_{x_0}^x \int_{x_0}^y dsdy \right|. \\
&\Rightarrow -\epsilon M_2 \leq u(x, t) - (Pu)(x, t) \leq \epsilon M_2. \\
&\Rightarrow |u(x, t) - (Pu)(x, t)| \leq \epsilon M_2. \\
&\Rightarrow \sup_{x,t \in J} |u(x, t) - (Pu)(x, t)| \leq \epsilon M_2. \\
&\Rightarrow d(u, Pu) \leq \epsilon M_2,
\end{aligned}$$

which with equation (2.13) yields

$$\begin{aligned}
|u(x, t) - u_0(x, t)| &= |u_0(x, t) - u(x, t)| \\
&\leq \sup_{x,t \in J} |u_0(x, t) - u(x, t)| \\
&= d(u_0(x, t), u(x, t)), \\
&\leq \frac{1}{1 - \{\lambda_1 + \lambda_2\}M_2} d(u, Pu), \\
&\leq \frac{M_2}{1 - \{\lambda_1 + \lambda_2\}M_2} \epsilon.
\end{aligned}$$

Hence the result.

3 Conclusion

In this paper, we have proved the Hyers - Ulam stability of first and second order partial differential equations (1.1) and (1.2) respectively by employing Banach's Contraction Principle.

Acknowledgement. The authors are very much grateful to the Editor and Reviewer for their valuable suggestion's for the improvements of the paper in its present form.

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