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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

NUMERICAL STUDY OF MHD BOUNDARY LAYER FLOW OF WILLIAMSON FLUID WITH VARIABLE FLUID PROPERTIES

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Abstract

This numerical study investigates the *MHD* boundary layer flow and heat transfer of the Williamson fluid over a permeable nonlinearly stretching sheet. The partial differential equations corresponding to the momentum and energy are converted into ordinary differential equations with the help of similarity transformations. The numerical solution is computed by Runge-Kutta fourth order method with shooting technique. The effects of various parameters such as viscosity variation parameter, thermal conductivity variation parameter, magnetic field parameter, suction parameter, Williamson fluid parameter , radiation parameter and Eckert number on velocity and temperature profiles are discussed through graphs.
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1 Introduction

The heat transfer of boundary layer flow over a stretching sheet has been attracted many researchers in last two decades due to its wide range of applications in many standard manufacturing and chemical engineering processes such as polymer manufacturing, crystal manufacturing, petroleum filtering operation,paper production and food preserving processes. Williamson [21] represented the model of the flow of pseudoplastic materials. Chen and Char [1] discussed the effects of suction or blowing on the heat transfer of a continuous stretching surface. Free convection on a vertical stretching surface with suction or blowing has been studied by Gorla and Sidawi [2]. Rapits [15] described the effects of radiation on the flow of a micropolar fluid past a continuously moving plate. Raptis [16] described the radiation and viscoelastic flow. Mohammadein and Gorla [10] studied the heat transfer in a micropolar fluid over a stretching sheet with viscous dissipation and internal heat generation. Mukhopadhyay et al. [11] studied of the effects of variable viscosity on *MHD* boundary layer flow over a heated stretching sheet. Vajravelu et al. [20] analyzed the peristaltic transport of a Williamson fluid in asymmetric channels with permeable walls. Jat and Chand [4] presented the *MHD* flow and heat transfer over an exponentially stretching sheet with viscous dissipation and radiation effects. Mukhopadhyay [12] investigated the slip effect on *MHD* boundary layer flow over an exponentially stretching sheet with suction/blowing and thermal radiation. Nadeem et al. [13] studied the flow of a Williamson fluid over a stretching sheet. Hayat et al. [3] investigated the effects of joule heating and thermal radiation in flow of third-grade fluid over radiative surface. Nadeem and Hussain [14] discussed the flow and heat transfer analysis of Williamson nanofluid. Rashidi et al. [17] studied the mixed convective heat transfer for *MHD* viscoelastic fluid flow over a porous wedge with thermal radiation. Yadav and Sharma [22] analyzed the effects of radiation and viscous dissipation on *MHD* boundary layer flow due to an exponentially moving stretching sheet in porous medium. Megahed [8] examined the effects of heat flux and thermal radiation on heat transfer of a non-Newtonian power-law fluid over a non-linearly stretching vertical surface.Reddy et al. [18] presented the *MHD* flow and heat transfer characteristics of Williamson nanofluid over a stretching sheet with variable thickness and variable thermal conductivity. Shafiq and Sindhu [19] discussed the statistical study of hydromagnetic boundary layer flow of Williamson fluid regarding a radiative surface. Kumar et al. [5] investigated the Slip effect on *MHD* stagnation point flow and heat transfer of cross fluid with heat generation in a porous medium. Lund et al. [6] analyzed the dual solution for *MHD* flow of Williamson fluid with slippage. Williamson fluid flow over a nonlinearly stretching sheet with viscous dissipation and thermal radiation by Megahed [9].

In this paper we use the Williamson fluid flow model which was given by Williamson [21]. The objective of present paper is to analyze the *MHD* Williamson fluid flow over a nonlinearly permeable stretching sheet with thermal radiation, viscous dissipation and variable fluid properties.

2 Formulation of the Problem

We assume moving fluid as a Williamson fluid with a time constant Γ. The fluid is flowing on nonlinearly permeable stretching sheet in the presence of radiation, viscous dissipation phenomena and magnetic field. Here x-axis is along the sheet whereas y-axis is taken perpendicular to stretching sheet. The stretching may create the velocity $U_w = cx^m$ for the fluid, where c is a constant and m is a exponent. Here it is assumed that both thermal conductivity and the fluid viscosity are varying with temperature while remaining properties are constant.

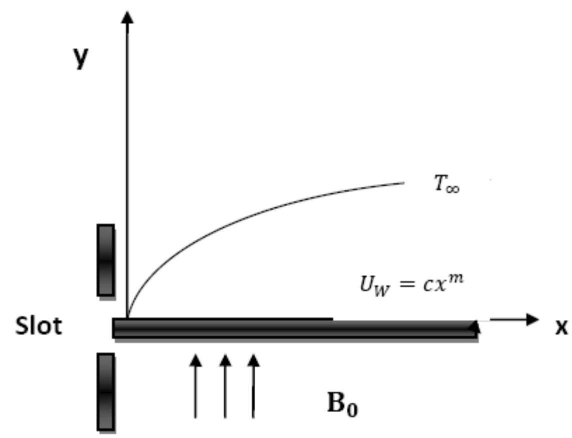

Figure **2.1:** Sketch of problem.

The governing equations of present fluid flow can be introduced in the following form [13]:

$$(2.1)\quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$(2.2)\quad u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \frac{1}{\rho_\infty}\frac{\partial}{\partial y}\left(\mu(T)\frac{\partial u}{\partial y} + \mu(T)\frac{\Gamma}{\sqrt{2}}\left(\frac{\partial u}{\partial y}\right)^2\right) - \frac{\sigma B_0^2}{\rho_\infty}u,$$

$$(2.3)\quad u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\rho_\infty c_p}\frac{\partial}{\partial y}\left(\kappa(T)\frac{\partial T}{\partial y}\right) + \frac{\mu(T)}{\rho_\infty c_p}\left(1 + \frac{\Gamma}{\sqrt{2}}\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial y}\right)^2 - \frac{1}{\rho_\infty c_p}\frac{\partial q_r}{\partial y},$$

with boundary conditions

$$(2.4)\quad u = cx^m, \quad v = -V_w, \quad T_w(x) = T_\infty + Ax^r \quad at \quad y = 0,$$

$$(2.5)\quad u \to 0, \quad T_w(x) \to T_\infty \quad at \quad y \to 0,$$

where u and v are the components of velocity in the x and y directions, respectively. Also, T is the temperature for the Williamson fluid, ρ_∞ refers to the fluid density at the ambient, q_r is the radiation heat flux and c_p is the specific heat at constant pressure. T_∞ is the constant ambient temperature, A and r are constants.

By taking Rosseland approximation [15], q_r can be described in the following form:

$$(2.6)\quad q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y},$$

where the constant σ^* is the Stefan-Boltzmann and k^* is the absorption coefficient.The term T^4 can be simplified by using Taylor expansion about the constant value T_∞ as $T^4 \cong 4T_\infty^3 T - 3T_\infty^4$, after neglecting all higher order terms [16].

Using the similarties transformation

$$(2.7)\quad \eta = \left(\frac{cx^{m-1}}{\nu_\infty}\right)^{1/2} y, \quad \psi(x,y) = \left(cx^{m+1}\nu_\infty\right)^{1/2} f(\eta), \quad \theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty},$$

(2.8) $u = \frac{\partial\psi}{\partial y}, \qquad v = -\frac{\partial\psi}{\partial x},$

where ν_∞ is the kinematic viscosity at the ambient and $\psi(x, y)$ is the stream function.

The important assumptions in this research is that the viscosity is altering exponentially with the temperature and the thermal conductivity is changing linearly with temperature for these equations [7]:

(2.9) $\mu = \mu_\infty e^{-\alpha\theta}, \qquad \kappa = \kappa_\infty (1 + \epsilon\theta),$

where μ_∞ is the viscosity at the ambient, κ_∞ represents the thermal conductivity at the ambient, α is the viscosity variation parameter and ϵ is the thermal conductivity variation parameter.

Equations (2.2)-(2.3) converted into

(2.10) $e^{-\alpha\theta}\left(\left(1 + \delta f^{''}\right) f^{'''} - \alpha\theta^{'} f^{''}\left(1 + \frac{\delta}{2}\right)\right) - Mf^{'} + \left(\frac{m+1}{2}\right) f f^{''} - m f^{'2} = 0,$

(2.11) $\frac{1}{Pr}\left(\epsilon\theta^{'2} + (1 + R + \epsilon\theta)\,\theta^{''}\right) + \left(\frac{m+1}{2}\right) f\theta^{'} - r f^{'}\theta + Ec\left(1 + \frac{\delta}{2}\right) f^{''2} e^{-\alpha\theta} = 0,$

and reduced boundary conditions are

(2.12) $f(0) = S, \quad f^{'}(0) = 1, \quad \theta(0) = 1,$

(2.13) $f^{'} \to 0, \quad \theta \to 0, \quad at \quad \eta \to \infty,$

where $\delta = \left(\frac{\sqrt{2}c^{3/2}x^{\frac{3m-1}{2}}}{\sqrt{\nu_\infty}}\right)\Gamma$ is the local Williamson fluid parameter, M is magnetic field parameter, S is the suction parameter, R is the radiation paramter, $Ec = \frac{U_w^2}{c_p(T_w - T_\infty)} = \frac{c^2 x^{2m-r}}{Ac_p}$ is the Eckert number, $Pr = \frac{\mu_\infty c_p}{\kappa_\infty}$ is the Prandtl number. Here we take $r = 2m = \frac{2}{3}$, so,these parameters take the form $\delta = \left(\frac{\sqrt{2}c^{3/2}}{\nu_\infty}\right)\Gamma$ and $Ec = \frac{c^2}{Ac_p}$.

The local skin - friction coefficient Cf_x and the local Nusselt number Nu_x are given as

(2.14) $Cf_x = -2Re_x^{-1/2}\left(1 + \frac{\delta}{2} f^{''}(0)\right) f^{''}(0) e^{-\alpha\theta(0)},$

(2.15) $Nu_x = -Re_x^{-1/2}\left(1 + R + \epsilon\theta(0)\right)\theta^{'}(0),$

where $Re_x = \frac{U_w x}{\nu_\infty}$ is the local Reynolds number.

3 Method of Solution

Numerical shooting technique with Runge-Kutta fourth order method was adopted to solve the problem. Equations (2.10)-(2.11) subject to the boundary conditions (2.12)-(2.13) are transformed into the following system of first-order differential equations:

(3.1) $f_1^{'} = f_2,$

(3.2) $f_2^{'} = f_3,$

(3.3) $f_3^{'} = \frac{e^{\alpha f_4}}{(1 + \delta f_3)}\left(m f_2^2 - \left(\frac{m+1}{2}\right) f_1 f_3 + M f_2\right) + \frac{\alpha f_5 f_3}{(1 + \delta f_3)}\left(1 + \frac{\delta}{2} f_3\right),$

(3.4) $f_4^{'} = f_5,$

(3.5) $f_5^{'} = \frac{1}{(1 + R + \epsilon f_4)}\left(Pr\left(r f_2 f_4 - \left(\frac{m+1}{2}\right) f_1 f_5 - Ec\left(1 + \frac{\delta}{2} f_3\right) f_3^2 e^{-\alpha f_4}\right) - \epsilon f_5^2\right),$

where $f = f_1$, $f^{'} = f_2$, $f^{''} = f_3$, $\theta = f_4$ and $\theta^{'} = f_5$
and initial conditions are $f_1(0) = S$, $f_2(0) = 1$,$f_4(0) = 1$.

This system can not be solved with the infinite conditions which appear in eq.(2.13). So, these conditions are replaced by appropriate finite guessing values $f^{''}(0)$ and $\theta^{'}(0)$.

4 Results and Discussion

The influence of viscosity variation parameter α on velocity and temperature profile is shown in Fig. **4.1** and Fig. **4.2**. From Fig. **4.1** and Fig. **4.2**, it is clear that velocity decrease with respect to increasing value of α and temperature profile increase with respect to increasing value of α.

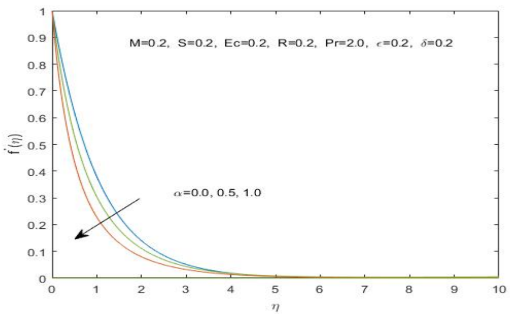

Figure **4.1:** Velocity profiles for different values of viscosity variation parameter α.

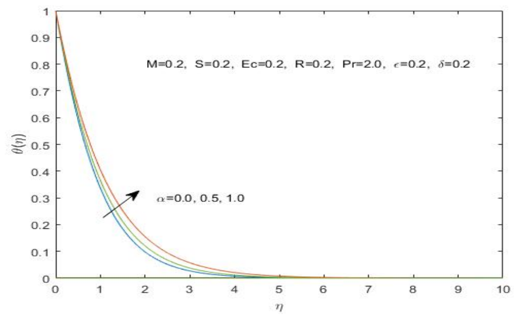

Figure **4.2:** Temperature profiles for different values of viscosity variation parameter α.

The varaition of local Williamson fluid parameter δ on velocity and temperature distributions is shown in Fig. **4.3** and Fig. **4.4**. From Fig. **4.3**, it is noticed that increase in δ leads to decrease in velocity. From Fig. **4.4**, it is observed that increase in δ tends to increase in temperature.

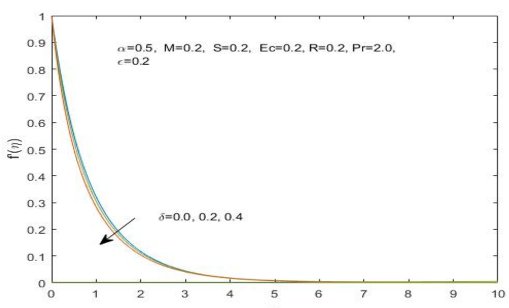

Figure **4.3:** Velocity profiles for different values of Williamson fluid parameter δ.

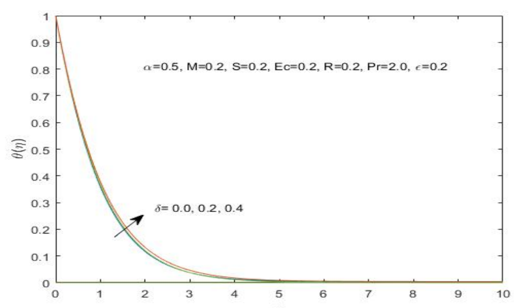

Figure **4.4:** Temperature profiles for different values of Williamson fluid parameter δ.

The influence of magnetic field parameter M on both velocity and temperature profiles are shown in Fig. **4.5** and Fig. **4.6**. From Fig. **4.5**, it is observed that velocity of fluid decreases with increase in M. The magnetic field causes a amount of resistance to its motion due to Lorentz force, which reduces the fluid velocity. From Fig. **4.6**, it is clear that increase in M leads to increase in temperature. Fig. **4.7** and Fig. **4.8** depict the variation of suction parameter S on velocity and temperature profile. From Fig. **4.7** and Fig. **4.8**, it is observed that velocity and temperature decrease with increase in S.

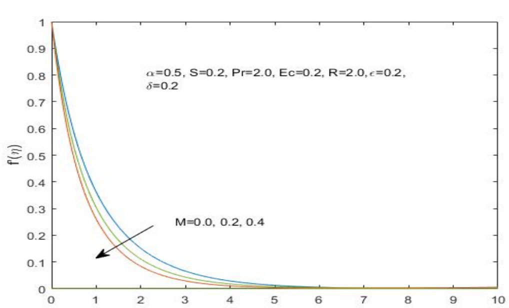

Figure **4.5:** Velocity profiles for different values of magnetic field parameter M.

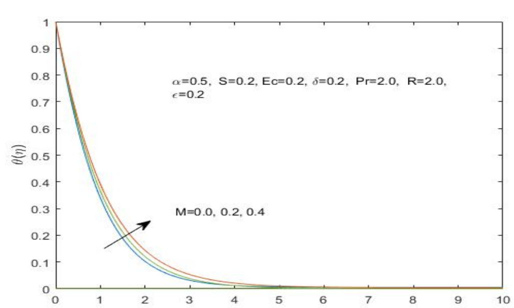

Figure **4.6:** Temperature profiles for different values of magnetic field parameter M.

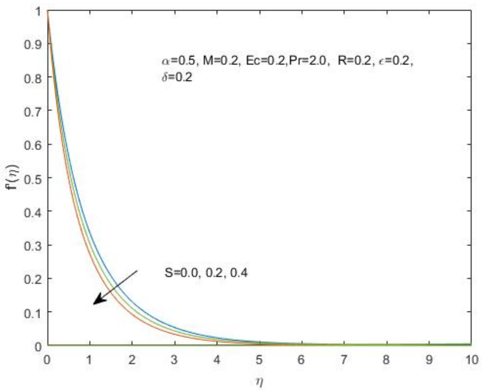

Figure **4.7:** Velocity profiles for different values of suction parameter S.

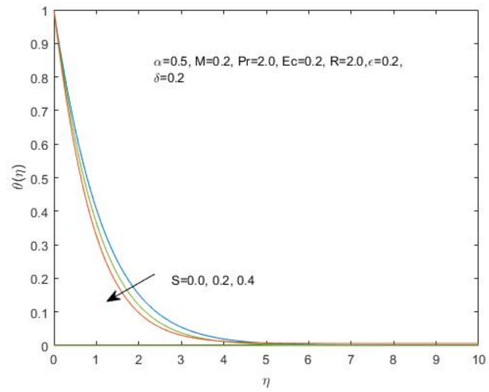

Figure **4.8:** Temperature profiles for different values of suction parameter S.

The effect of thermal conductivity parameter ϵ and radiation parameter R on temperature profile are given in Fig. **4.9** and Fig. **4.10** respectively, it is observed that increase in ϵ and R, leads to increase in temperature. Because increase in radiation parameter R provides more heat to fluid that leads increase in temperature profile.

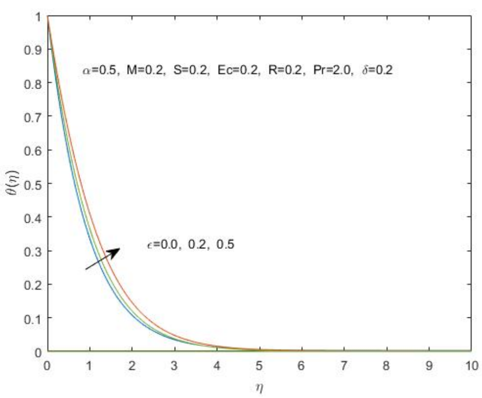

Figure **4.9:** Temperature profiles for different values of thermal conductivity parameter ϵ.

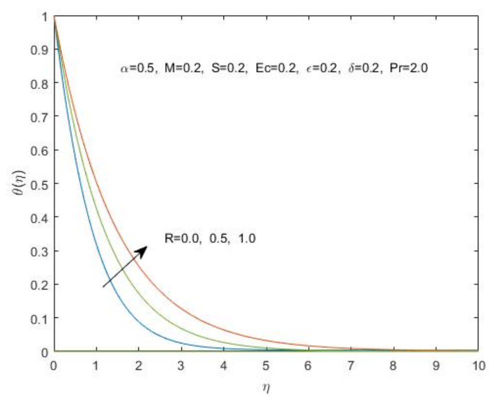

Figure **4.10:** Temperature profiles for different values of radiation parameter R.

The effect of Eckert number Ec on temperature profile is shown in Fig. **4.11**. From Fig. **4.11**, it is clear that increase in Ec leads to increase in temperature profile because viscosity of fluid converts the energy from motion into the internal energy of fluid which results in increasing of temperature.

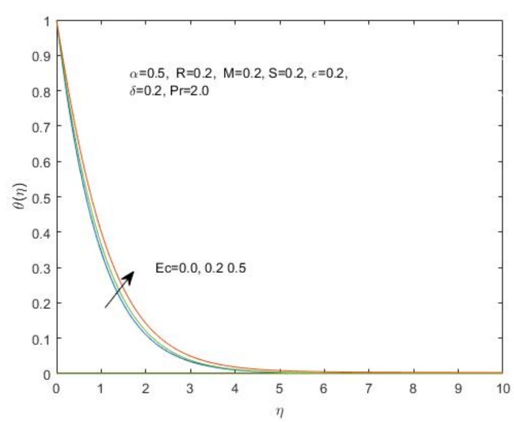

Figure **4.11:** Temperature profiles for different values of Eckert number Ec.

Table **4.1:** Comparison of Nusselt number $Re_x^{-1/2}Nu_x$ for various values of Pr when $\delta = \alpha = \epsilon = R = Ec = M = S = r = 0$ and $m = 1$

Pr	Gorla and Sidawi (1994)	Megahed (2019)	Present study
2.0	0.91142	0.911358	0.911355
7.0	1.89046	1.895453	1.894460
20.0	3.35391	3.353902	3.353904

The comparison of present problem with previous research works done by Megahed[9] and Gorla and Sidawi[2] are shown in Table **4.1**. The present problem can be transformed into the previous published work when $\delta = \alpha = \epsilon = R = Ec = M = S = r = 0$ and $m = 1$.

Table **4.2** presents the numerical values of both the local Nusselt number and the local skin friction parameter for various values of Eckert number Ec, magnetic field parameter M, suction parameter S, Williamson fluid parameterδ, radiation parameter R, viscosity variation parameter α and thermal conductivity parameter ϵ. Analysis of table shows that the Eckert number, the Williamson fluid parameter, the viscosity parameter lead to decresing behaviour for both the local Nusselt number and the local skin friction coeffıcent and the suction parameter leads to a increasing behaviour for both. Magnetic field parameter leads a diminishing behaviour for the local Nusselt number and increasing behaviour for the the local skin friction.

Table **4.2:** Values of $1/2Re_x^{1/2}Cf_x$ and $Re_x^{-1/2}Nu_x$ for various values of α, δ, ϵ , M, S, R, Ec with $m = 1/3$, $r = 2/3$, $Pr = 2.0$

α	δ	ϵ	M	S	R	Ec	$\frac{1}{2}Re_x^{1/2}Cf_x$	$Re_x^{-1/2}Nu_x$
0.0	0.2	0.2	0.2	0.2	0.2	0.2	0.8541	1.3875
0.5	0.2	0.2	0.2	0.2	0.2	0.2	0.7168	1.3122
1.0	0.2	0.2	0.2	0.2	0.2	0.2	0.5862	1.2173
0.5	0.0	0.2	0.2	0.2	0.2	0.2	0.7491	1.3298
0.5	0.2	0.2	0.2	0.2	0.2	0.2	0.7168	1.3122
0.5	0.4	0.2	0.2	0.2	0.2	0.2	0.6744	1.2789
0.5	0.2	0.0	0.2	0.2	0.2	0.2	0.7201	1.2639
0.5	0.2	0.2	0.2	0.2	0.2	0.2	0.7168	1.3122
0.5	0.2	0.4	0.2	0.2	0.2	0.2	0.7128	1.3738
0.5	0.2	0.2	0.0	0.2	0.2	0.2	0.6195	1.3881
0.5	0.2	0.2	0.2	0.2	0.2	0.2	0.7168	1.3122
0.5	0.2	0.2	0.4	0.2	0.2	0.2	0.7981	1.2415
0.5	0.2	0.2	0.2	0.0	0.2	0.2	0.6443	1.1907
0.5	0.2	0.2	0.2	0.2	0.2	0.2	0.7168	1.3122
0.5	0.2	0.2	0.2	0.4	0.2	0.2	0.7952	1.4412
0.5	0.2	0.2	0.2	0.2	0.0	0.2	0.7208	1.2412
0.5	0.2	0.2	0.2	0.2	0.5	0.2	0.7123	1.3981
0.5	0.2	0.2	0.2	0.2	1.0	0.2	0.7068	1.5092
0.5	0.2	0.2	0.2	0.2	0.2	0.0	0.7189	1.4377
0.5	0.2	0.2	0.2	0.2	0.2	0.2	0.7168	1.3122
0.5	0.2	0.2	0.2	0.2	0.2	0.5	0.7156	1.1139

5 Conclusions

In this paper we have studied the boundary layer flow and heat transfer of Williamson fluid over a permeable nonlinearly stretching sheet in the presence of magnetic field. We draw the following conclusions from our study

(1) Both the thermal radiation parameter and Eckert number enhance the temperature distribution, thicken thermal region.
(2) An increase in both the viscosity parameter and the Williamson parameter results in a rise in the temperature distribution.
(3) Thermal conductivity leads to an increasing behaviour for the local Nusselt number.

(4) Magnetic field parameter leads to decreasing behaviour for the local Nusselt number and increasing behaviour for the local skin-friction coefficient.

(5) Velocity and temperature profiles are decreasing as the value of suction parameter is increasing

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References

[1] C. K. Chen and M. I. Char, Heat transfer of a continuous stretching surface with suction or blowing, *J. Math. Anal. Appl.*, **135** (1988), 568580.

[2] R. S. R. Gorla and I. Sidawi, Free convection on a vertical stretching surface with suction and blowing, *Appl. Sci. Res.*, **52** (1994), 247257.

[3] T. Hayat, A. Shafiq and A. Alsaedi, Effect of Joule heating and thermal radiation in flow of third-grade fluid over radiative surface, *PLOS One*, **9(1)** (2014), e83153.

[4] R. N. Jat and G. Chand, *MHD* flow and heat transfer over an exponentially stretching sheet with viscous dissipation and radiation effects, *Appl. Math. Sci.*, **7** (2013), 167-180.

[5] N. Kumar, R. N. Jat and S. Choudhary, Slip effect on *MHD* stagnation point flow and heat transfer of cross fluid with heat generation in a porous medium, *Jñānābha*, **49(2)** (2019), 94-104.

[6] A. L. Lund, Z. Omar and I. Khan, Analysis of dual solution for *MHD* flow of Williamson fluid with slippage, *Heliyon*, **5** (2019), e01345.

[7] M. A. A. Mahmoud and A. M. Megahed, *MHD* flow and heat transfer in a non-Newtonian liquid film over an unsteady stretching sheet with variable fluid properties, *Can. J. Phys.*, **87** (2009), 1065-1071.

[8] A. M. Megahed, Flow and heat transfer of a non-Newtonian power-law fluid over a non-linearly stretching vertical surface with heat flux and thermal radiation, *Meccanica*, **50** (2015), 16931700.

[9] A.M. Megahed, Williamson fluid flow due to a nonlinearly stretching sheet with vicsous dissipation and thermal radiation, *Journal of the Egyptian Mathematical Society*, **27(12)** (2019), 1-10.

[10] A. A. Mohammadein and R. S. R. Gorla, Heat transfer in a micropolar fluid over a stretching sheet with viscous dissipation and internal heat generation, *Int. J. Numer. Meth. Heat Fluid Flow*, **11** (2001), 50-58.

[11] S. Mukhopadhyay, G. C. Layek and S. A. Samad, Study of *MHD* boundary layer flow over a heated stretching sheet with variable viscosity, *Int. Journal Heat and Mass trans.*, **48(21-22)** (2005), 4460-4466.

[12] S. Mukhopadhyay, Slip effects on *MHD* boundary layer flow over an exponentially stretching sheet with suction/blowing and thermal radiation, *Ain Shams Eng. J.*, **4(3)** (2013), 485-491.

[13] S. Nadeem, S. T. Hussain and C. Lee, Flow of a Williamson fluid over a stretching sheet, *Braz. J. Chem. Eng.*, **30** (2013), 619625.

[14] S. Nadeem and S. T. Hussain, Flow and heat transfer analysis of Williamson nanofluid, *Appl. Nanosci.*, **4** (2014), 1005-1012.

[15] A. Raptis, Flow of a micropolar fluid past a continuously moving plate by the presence of radiation, *Int. J. Heat Mass Tran.*, **41** (1998), 28652866.

[16] A. Raptis, Radiation and viscoelastic flow, *Int. Commun. Heat Mass Trans.*, **26** (1999), 889-895.

[17] M. M. Rashidi , M. Ali, N. Freidoonimehr, B. Rostami, and M. A. Hossain, Mixed convective heat transfer for *MHD* viscoelastic fluid flow over a porous wedge with thermal radiation, *Adv. Mech. Eng.*, **10** (2014), 1-10.

[18] S. Reddy, K. Naikoti and M. M. Rashidi, *MHD* flow and heat transfer characteristics of Williamson nanofluid over a stretching sheet with variable thickness and variable thermal conductivity, *Trans. A. Razmadze Math. Ins.*, **171(2)** (2017), 195-211.

[19] A. Shafiq and T. N. Sindhu, Statistical study of hydromagnetic boundary layer flow of Williamson fluid regarding a radiative surface, *Results in Physics*, **7**(2017), 30593067.

[20] K. Vajravelu, S. Sreenadh, K. Rajanikanth and C. Lee, Peristaltic transport of a Williamson fluid in asymmetric channels with permeable walls, *Nonlinear Anal. R. World Appl.*, **13(6)** (2012), 2804-2822.

[21] W. W. Williamson, The flow of pseudoplastic materials, *Ind. Eng. Chem.*, **21** (1929), 11081111.

[22] R. S. Yadav and P. R. Sharma, Effects of radiation and viscous dissipation on *MHD* boundary layer flow due to an exponentially moving stretching sheet in porous medium, *Asian J. Multidiscip. Stud.*, **2(8)** (2014), 119-124.