A NEW APPROACH TO GENERATE FORMULAE FOR PYTHAGOREANS TRIPLES, QUADRUPLES AND THEIR GENERALISATION TO N-TUPLES

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Abstract

In this paper, innovative methods have been devised to generate formulae for Pythagorean’s Triples, Quadruples and these are finally generalised to generate Pythagorean’s n-tuples. First method utilises formula for solution of a quadratic equation and generate two sets of Pythagorean’s Triples. Second method determines universal identities that satisfy Pythagorean’s Triples, Quadruples so on up to n-tuples. These methods are unprecedented, easy to derive and hence are comprehensible to students and scholars alike.

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1 Introduction

Integers X, Y and Z are said to be Pythagorean’s Triple if these satisfy the relation

\[(1.1) \quad X^2 + Y^2 = Z^2.\]

When \(X = Y\), in that event, Equation (1.1) yields \(Z = \sqrt{2}X\) which is an irrational quantity and can not be Pythagorean’s triple. It is amply explained in basic books of number theory [3], [5]. On the other hand, when \(X\) and \(Y\) are unequal say \(X = x, Y = x + a, Z = x + b\), \(a\) and \(b\) are real and rational quantities then algebraic Equation (1.1) takes the form

\[(x)^2 + (x + a)^2 = (x + b)^2.\]

Such algebraic equations have also been used [6] while deriving identities for Pythagorean’s quadruples. On expansion

\[(1.2) \quad x^2 - 2x(b - a) - (b^2 - a^2) = 0.\]

Equation (1.2) being a quadratic has two roots given by

\[(1.3) \quad x = (b - a) \pm \sqrt{(b - a)^2 + (b^2 - a^2)}.\]

2 Theory and Concept

**Lemma 2.1** Integers \(x, (x+a)\) and \((x+b)\) will be unnormalised Pythagorean’s Triples if \(x = (b-a)\pm \sqrt{(b-a)^2 + (b^2 - a^2)}\) and \(a\) and \(b\) are rational quantities such that \(\pm \sqrt{(b-a)^2 + (b^2 - a^2)}\) is real, perfect square and hence rational. If \(x, (x + a)\) and \((x + b)\) so determined are fractions (unnormalised), then multiplying them with their lowest common multiplier LCM will give normalised Pythagorean’s Triples in integer form.

**Lemma 2.2** If real and rational quantities \(x, (x + a)\) and \((x + b)\) have \(x = (b-a)\pm \sqrt{(b-a)^2 + (b^2 - a^2)} = p/q, a = p_1/q_1, b = p_2/q_2\) and \(p, q, p_1, q_1, p_2, q_2\) are integers then \(p, q, p_1, q_1, p_2, q_2\) will be Pythagorean’s Triples.

2.1 Values of \(x, a \) and \(b\) Satisfying Equation (1.3)

We will assume different values of \(a\) and \(b\) so that Equation (1.3) has real and rational solutions. That is term \(\pm \sqrt{(b-a)^2 + (b^2 - a^2)}\) should be a perfect square. After assumption of different values of \(a\) and \(b\), values of \(x\) are calculated by Equation (1.3). Two cases then arise, one when \(x\) is found integer, two when \(x\) found is a fraction as \(p/q\). If \(x\) is found to be an integer, no further operation is required.
2.1a. Normalisation: When $x$ Found is a Fraction $p/q$.

If $x$ is found to be a fraction of form $p/q$ where $p,q,a$ and $b$ are integers then by Equation (1.1)

$$(p/q)^2 + (p/q + a)^2 = (p/q + b)^2.$$ 

On expansion and simplification

$$(2.1) \quad (p)^2 + (p + q/a)^2 = (p + q/b)^2.$$ 

Since $p,q,a$ and $b$ are all integers then $p,(p + q/a)$and $(p + q/b)$ being sum and product of integers are always integers. Such explanation can be found in basic books of number theory [3], [5]. Equation (2.1) is obtained on multiplying $(p/q)^2 + (p/q + a)^2 = (p/q + b)^2$ with lowest common multiplier $LCM$. On putting different integer values of $a$ and $b$, values of $x$ are calculated from Equation (1.3) and if values of $x$ are found to be fractions, these are normalised as discussed above. Explanation of $LCM$ and conversion of fraction to integers are given in books on number theory [3], [5].

If $x$ found is of form $p/q$ , $a$ of form $p_1/q_1$ and $b$ of form $p_2/q_2$ then

$$(p/q)^2 + (p/q + p_1/q_1)^2 = (p/q + p_2/q_2)^2$$

or

$$(2.2) \quad (p/q_1.q_2)^2 + (p/q_1.q_2 + p_1.q_2)^2 = (p.q_1.q_2 + p_2.q_2)^2.$$ 

Values of $x$ calculated with different values of $a$ and $b$ are given in Table 1.1 after normalisation. It is also worth mentioning if value of $x$ is found to be negative and as its square will always be positive, therefore, negative value of $x$ will not have any effect on Pythagorean’s triple. In the Table 1.1 , at one place, $x$ is calculated negative $11/2$, after normalisation it is taken as positive $11$ on account of the fact already discussed.

<table>
<thead>
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<th>$a$</th>
<th>$b$</th>
<th>Normalised $x$</th>
<th>Normalised $y = x + a$</th>
<th>Normalised $z = x + b$</th>
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<td>305</td>
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</tbody>
</table>

2.2 Pythagorean’s Quadruples

Pythagorean’s quadruples $x,y,w$ and $z$ are given by equation given below. Studies for derivation of identifies for Pythagorean’s quadruples from the point of view of area of one face of tetrahedron that equals areas of three opposite faces has been done and also Wikipedia provides an analytic account of it [1], [9].

$$(2.3) \quad x^2 + y^2 + w^2 = z^2.$$ 

Again by putting $y = x + a$, $z = x + b$ and $w = x + c$ where$x, a, b, and c$ are all rational quantities, In this regards, theory of representation of rational quantities in algebraic is given in basic books on number theory [3], [5]. Equation (2.3) transforms to $(x)^2 + (x + a)^2 + (x + b)^2 = (x + c)^2$ or

$$(2.4) \quad x^2 - x(c - b - a) - \frac{1}{2}(c^2 - b^2 - a^2) = 0.$$
Equation (2.4) being quadratic has roots given by

\[
(2.5) \quad x = \frac{1}{2}(c - b - a) \pm \frac{1}{2}\sqrt{(c - b - a)^2 + 2(c^2 - b^2 - a^2)}
\]

**Lemma 2.3** Integers \(x, y, z\) and \(w\) are Pythagorean’s quadruples if

\[
x = \frac{1}{2}(c - b - a) \pm \frac{1}{2}\sqrt{(c - b - a)^2 + 2(c^2 - b^2 - a^2)}\]

and \(z = (x + c)\) where \(a, b, c\) are rational quantities such that quantity \(\pm \frac{1}{2}\{(c - b - a)^2 + 2(c^2 - b^2 - a^2)\}\) is rational, in other words quantity \(\pm \frac{1}{2}\{(c - b - a)^2 + 2(c^2 - b^2 - a^2)\}\) is real and a perfect square. If \(x, y, z\) and \(w\) so determined are fractions, then multiplying \(x, y, z\) and \(w\) with their lowest common multiplier \(\text{LCM}\) will give normalised Pythagorean’s quadruples in integer form.

**Lemma 2.4** If \(x, y, z\) and \(w\) are rational quantities and \(x\) is is given by

\[
x = \frac{1}{2}(c - b - a) \pm \frac{1}{2}\sqrt{(c - b - a)^2 + 2(c^2 - b^2 - a^2)}\]

and \(z = (x + c)\) where \(a, b, c\) are rational quantities such that quantity \(\pm \frac{1}{2}\{(c - b - a)^2 + 2(c^2 - b^2 - a^2)\}\) is rational, in other words quantity \(\pm \frac{1}{2}\{(c - b - a)^2 + 2(c^2 - b^2 - a^2)\}\) is real and perfect square and if \(x\) is of form \(p/q\), \(a\) of form \(p_1/q_1\), \(b\) of form \(p_2/q_2\) and \(c\) of form \(p_3/q_3\) then \((p/q_1,q_2,q_3),(p,q_1,q_2,q_3),(p,q_1,q_2,q_3+p,q_1,q_2,q_3),(p,q_1,q_2,q_3+p,q_1,q_2,q_3+p,q_1,q_2,q_3)\) and \((p,q_1,q_2,q_3+p,q_1,q_2,q_3)\) will be Pythagorean’s Quadruples.

**2.2a. Values of \(x, a\) and \(b\) Satisfying Equation (2.5)**

Different values of \(a, b\) and \(c\) are assumed so that Equation (2.5) has rational solutions i.e. \(\{(c - b - a)^2 + 2(c^2 - b^2 - a^2)\}\) is rational and is a perfect square. After assumption of values of \(a, b\) and \(c\), values of \(x\) are calculated by equations (2.5). On calculations two cases arise, one when \(x\) is found integer, no further operation is required, two if \(x\) is found to be a fraction of the form \(p/q\), it requires normalisation.

**2.2b. Normalisation: When \(x\) on Calculation is a Fraction of the Form \(p/q\)**

When \(x\) is a fraction say of kind \(p/q\) where \(p, q, a\) and \(b\) are integers then by Equation (2.3),

\[
(p/q)^2 + (p/q + a)^2 + (p/q + b)^2 = (p/q + c)^2.
\]

After normalisation by multiplying with \(\text{LCM}\),

\[
(2.6) \quad p^2 + (p + q.a)^2 + (p + q.b)^2 = (p + q.c)^2.
\]

Since \(p, q, a\) and \(b\) are all integers, therefore, \(p, (p + q.a), (p + q.b)\) and \((p + q.c)\) being sum and product of integers are always integers. Such explanation is given in books [3], [5]. On putting different integer values of \(a, b\) and \(c\), values of \(x\) are calculated from equation (2.5) and if values of \(x\) are found to be fractions, these are normalised as discussed above. If \(x\) calculated is of form \(p/q\) then \(a\) of form \(p_1/q_1\), \(b\) of form \(p_2/q_2\) and \(c\) of the form \(p_3/q_3\) then

\[
(2.7) \quad (p/q_1,q_2,q_3)^2 + (p,q_1,q_2,q_3)^2 + (p,q_1,q_2,q_3)^2(p,q_1,q_2,q_3) = (p/q_1,q_2,q_3)^2.
\]

Values of \(x\) are calculated by putting different values of \(a\) and \(b\) in equations (2.5). These values of \(x, a, b\) and \(c\) are given in Table 2.2. It is also worth mentioning that if value of \(x\) is found to be negative and as its square will always be positive, therefore, assuming this negative value of \(x\) or \((x + a)\) or \((x + b)\) or \((x + c)\) as positive will not have any effect on Pythagorean’s quadruples. In the Table 2.2, at some places, \(x\) is calculated negative, but after normalisation, it is taken as positive on account of the facts as have already been discussed.
2.2c. Pythagorean Quadruples of the Form \( x^2 + y^2 = z^2 + w^2 \)

Let \( y = x + a, z = x + b \) and \( w = x + c \) then \( x^2 + (x + a)^2 = (x + b)^2 + (x + c)^2 \).

Therefore,

\[
(2.8) \quad x = \frac{b^2 + c^2 - a^2}{2(a - b - c)},
\]

where \( a, b, \text{and} \ c \) are rationals. Amongst others, Wikipedia describes the method of generation of Pythagorean’s triples [8]. Let \( x \) is of form \( p/q \), \( a \) of form \( p_1/q_1 \), \( b \) of form \( p_2/q_2 \), \( c \) of form \( p_3/q_3 \) where \( p, q, p_1q_1, p_2, q_2, p_3, q_3 \) are all integers then after normalisation,

\[
(2.9) \quad (p.q_1.q_2.q_3)^2 + (p.q_1.q_2.q_3 + p_1.q_2.q_3)^2 = (p.q_1.q_2.q_3 + p_2.q_1.q_3)^2 + (p.q_1.q_2.q_3 + p_3.q_1.q_2)^2.
\]

Based on the above said formula, some Pythagorean’s quadruples are given in Table 2.3.

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
x & a & b & c & \text{Normalised} & \text{Normalised} & \text{Normalised} \\
& & & & x = x + a & z = x + b & w = x + c \\
\hline
-7/6 & 1 & 2 & 2 & 7 & 1 & 5 \\
-17/10 & 1 & 3 & 3 & 17 & 7 & 13 \\
-31/4 & 1 & 4 & 4 & 31 & 17 & 25 \\
-19/2 & 1 & -2 & 4 & 19 & 17 & 23 \\
-19/14 & -1 & 2 & 4 & 19 & 33 & 9 \\
19/6 & 1 & 2 & -4 & 19 & 25 & 31 \\
1/6 & 2 & 1 & -2 & 1 & 13 & 7 \\
-1/10 & -2 & 1 & 2 & 1 & 21 & 9 \\
7/2 & 2 & 3 & -3 & 7 & 11 & 13 \\
7/8 & 2 & -3 & -3 & 7 & 23 & 17 \\
-1/3 & 3 & -2 & -1 & 1 & 8 & 7 \\
-1/10 & 3 & -3 & 1 & 1 & 31 & 29 \\
1/14 & 3 & -3 & -1 & 1 & 43 & 41 \\
-9/2 & 1 & -1 & 3 & 9 & 7 & 11 \\
\hline
\end{array}
\]
2.2d. Pythagorean Quadruples of the Form \( x^2 + y^2 = v^2 \) where \( x^2 + y^2 = z^2 \) and \( z^2 + w^2 = v^2 \)

Pythagorean’s triples of the form \( x^2 + y^2 = z^2 \) have already been formulated in paragraphs 1 to 2.1a and are not repeated here. After finding value of \( z \), value of \( c \) and \( d \) are to be found so that these may satisfy the equation \( z^2 + w^2 = v^2 \). This equation is same as \( x^2 + y^2 = z^2 \) and values of \( z \) can be found by Equation (1.3). Values of \( x, y, z \) and \( w \) are given in Table 2.4.

| Table 2.4: Pythagorean Quadruples of the Form \( x^2 + y^2 = z^2 + w^2 \) |
|---|---|---|---|
| \( x \) | \( y \) | \( z = \left(x^2 + y^2\right)^{1/2} \) | \( w \) | \( v = \left(z^2 + w^2\right)^{1/2} \) |
| 3  | 4  | 5  | 12 | 13  |
| 5  | 12 | 13 | 84 | 85  |
| 9  | 12 | 15 | 8  | 17  |
| 13 | 84 | 85 | 132| 157 |
| 15 | 36 | 39 | 80 | 89  |
| 28 | 96 | 100| 105| 145 |
| 33 | 56 | 65 | 420| 425 |
| 33 | 56 | 65 | 72 | 97  |

2.2e. Pythagorean Quintuples of the Form \( x^2 + y^2 + z^2 + w^2 = v^2 \)

Let \( y = (x + a), z = (x + b), w = (x + c) \) and \( v = (2x + d) \) [8] then \( (x + a)^2 + (x + b)^2 + (x + c)^2 = (2x + d)^2 \).

Therefore,

\[
(2.10) \quad x = \frac{d^2 - a^2 - b^2 - c^2}{2(a + b + c - 2d)}
\]

On putting different integer values of \( a, b, c \) and \( d \), different values of \( x \) are obtained. From those values of \( x, y, z, w, \) and \( v \) are calculated and normalised. Wikipedia describes the method of generation of Pythagorean’s quintuples [8]. Table 2.5 gives Pythagorean’s Quintuples based on the above said formula.

| Table 2.5: Pythagorean Pentagonal Numbers of the Form \( x^2 + y^2 + z^2 + w^2 = v^2 \) |
|---|---|---|---|---|---|
| \( a \) | \( b \) | \( c \) | \( d \) | \( x \) | Norm.x | Norm.y = \( x + a \) | Norm.z = \( x + b \) | Norm.w = \( x + c \) | Norm.v = \( 2x + d = (x^2 + y^2 + z^2 + w^2)^{1/2} \) |
| 1  | 2  | 3  | 4  | -1/2 | 1  | 1  | 3  | 5  | 6  |
| 1  | 3  | 4  | 5  | 1/4  | 1  | 5  | 13 | 17 | 22 |
| 1  | 2  | 3  | -1/4| 1/14 | 1  | 15 | 29 | 43 | 54 |
| 1  | 2  | -1/3| 4  | -1/8 | 1  | 7  | 15 | 25 | 30 |
| -1 | 2  | 3  | 4  | -1/4 | 1  | 5  | 7  | 11 | 14 |
| -1 | 2  | -3 | 4  | -1/8 | 1  | 9  | 17 | 23 | 30 |
| -1 | 2  | 3  | -4 | 1/10 | 1  | 11 | 19 | 31 | 38 |
| 1  | -3 | 4  | 5  | 1/16 | 1  | 11 | 25 | 37 | 46 |
| 1  | -3 | 4  | -4 | 1/20 | 1  | 21 | 61 | 79 | 102 |

2.2f. Pythagorean \( N \)-Tuples of the Form \( x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = y^2 \)

Let \( x_2 = (x_1 + a_2), x_3 = (x_1 + a_3), x_4 = (x_1 + a_4), \ldots, x_n = (x_1 + a_n) \) and \( y = (x_1 + a_1) \) then putting these values in equation \( x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = y^2 \) it will take the form

\[
x_1^2 + (x_1 + a_2)^2 + (x_1 + a_3)^2 + (x_1 + a_4)^2 + \ldots + (x_1 + a_n)^2 = (x_1 + a_1)^2.
\]

Methods of generating such generalised Pythagorean’s n-tuples has been described by many mathematicians [4], [10], [1], [2]. On expansion and rearrangement,

\[
(n-1)x_1^2 + 2x_1(a_2 + a_3 + a_4 + \ldots + a_n - a_1) + a_1^2 + a_2^2 + a_3^2 + a_4^2 + \ldots a_n^2 - a_1^2 = 0
\]
This is a quadratic in $x_1$ and its roots are

$$x_1 = \frac{-Q \pm \sqrt{Q^2 - 4PR}}{2P},$$

where $P = (n-1), Q = 2(a_2 + a_3 + a_4 + \ldots + a_n - a_1)$ and $R = a_2^2 + a_3^2 + a_4^2 + \ldots a_n^2 - a_1^2$. Values of $a_1, a_2, a_3, \ldots a_n$ can be assumed arbitrarily such that part under square root is rational and that determines $x_1$. Once $x_1$ is determined, $a_1, a_2, a_3, \ldots, a_n$ are known, $x_2, x_3, \ldots, x_n$ and $y$ can be calculated. By assuming different set of $a_1, a_2, a_3, \ldots, a_n$, different set of $x_1, x_2, x_3, \ldots, x_n$ can be found out, hence n-tuples are generated.

### 2.3 Second Method: To Generate Pythagorean’s Numbers

#### 2.3a. Pythagorean’s Triples

Pythagorean’s Triples are given by the equation $a^2 + b^2 = c^2$. This can also be written as $a^2 = c^2 - b^2$ or $a^2 = (c-b)(c+b)$ or $(\frac{c-b}{a})(\frac{c+b}{a}) = 1$. Let $x = (\frac{c-b}{a})$ then from above equation, $\frac{1}{x} = (\frac{c+b}{a})$. On adding and subtracting,

$$\begin{align*}
(2.11) & \quad x + \frac{1}{x} = 2c/a \\
(2.12) & \quad x - \frac{1}{x} = 2b/a.
\end{align*}$$

Since $c^2/a^2 - b^2/a^2 = 1$, therefore using Equations (2.11) and (2.12),

$$\begin{align*}
(2.13) & \quad \frac{1}{4}(x + \frac{1}{x})^2 - \frac{1}{4}(x - \frac{1}{x})^2 = 1, \\
(2.14) & \quad (x - \frac{1}{x})^2 + 2 = (x + \frac{1}{x})^2
\end{align*}$$

identity (2.14) generates Pythagorean’s Triples $(x-1/x), 2$ and $(x+1/x)$ satisfying $a^2+b^2=c^2$. Let $x = \frac{3}{2}$ then $(\frac{3}{2} - \frac{2}{3}), 2$ and $(\frac{3}{2} + \frac{2}{3})$ on normalisation are Pythagorean’s triple as 5, 12 and 13. Procedure analogous to this has also been adopted by some mathematicians and it also makes mention in Wikipedia.[7], [8]. Some Pythagorean’s triples using Equation (2.14) are given in Table 2.6.

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<th>Normalised $b$</th>
<th>Normalised $c$</th>
</tr>
</thead>
<tbody>
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<td>5/2</td>
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<td>4</td>
<td>5</td>
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#### 2.3b. Pythagorean’s Quadruples of the Form $a^2 + b^2 + c^2 = d^2$ where $a^2+b^2 = e^2$ and $e^2 + c^2 = d^2$

Let $a^2 + b^2 = e^2$ then as proved in paragraph 2.3a, $x - \frac{1}{x} = 2\frac{b}{a}$ and $x + \frac{1}{x} = 2\frac{e}{a}$. That makes $a^2 + b^2 = e^2$ as $\frac{1}{4}(x-\frac{1}{x})^2 = 1 = \frac{1}{4}(x+\frac{1}{x})^2$. On dividing left hand side and right hand side by $\frac{1}{4}(x+\frac{1}{x})^2$

$$\begin{align*}
(2.15) & \quad (x - \frac{1}{x})^2(x + \frac{1}{x})^2 - 4(x + \frac{1}{x})^2 = 1. \\
\text{Since } a^2 + b^2 + c^2 = d^2 \text{ and it is assumed } a^2 + b^2 = e^2, \text{ therefore, } e^2 + c^2 = d^2. \text{ That makes } y - \frac{1}{y} = 2\frac{c}{d} \text{ and } y + \frac{1}{y} = 2\frac{d}{c}. \text{ Therefore,}
\end{align*}$$

$$\begin{align*}
(2.16) & \quad 1 + \frac{1}{4}(y - \frac{1}{y})^2 = \frac{1}{4}(y + \frac{1}{y})^2.
\end{align*}$$

On putting value of 1 from equation (2.15) in Equation (2.16),

$$\begin{align*}
(2.17) & \quad (x - \frac{1}{x})^2(x + \frac{1}{x})^2 - 4(x + \frac{1}{x})^2 + \frac{1}{4}(y - \frac{1}{y})^2 = \frac{1}{4}(y + \frac{1}{y})^2.
\end{align*}$$
Equation (2.17) is an equation of Pythagorean’s Quadruples for all real and rational values of x and y except \( x = y = 1 \). An example where \( x = 2, y = 3 \) is taken up below,
\[
(x - \frac{1}{2})^2(x + \frac{1}{2})^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}.
\]
\[
4(x + \frac{1}{2})^2 = 4\left(\frac{3}{2}\right)^2 = \frac{16}{9},
\]
\[
4(x + \frac{1}{2})^2 = 4\left(\frac{3}{2}\right)^2 = \frac{16}{9},
\]
\[
\frac{1}{2}(y - \frac{1}{2})^2 = \frac{1}{2}\left(\frac{1}{4}\right)^2 = \frac{1}{16},
\]
\[
\frac{1}{2}(y + \frac{1}{2})^2 = \frac{1}{2}\left(\frac{3}{4}\right)^2 = \frac{9}{16}.
\]
That makes \( \frac{9}{4} + \frac{16}{9} + \frac{16}{9} = \frac{25}{9} \) or \( \left(\frac{3}{2}\right)^2 + \left(\frac{4}{3}\right)^2 + \left(\frac{5}{4}\right)^2 = \left(\frac{2}{3}\right)^2 \) and after normalisation, \( 9^2 + 12^2 + 20^2 = 25^2 \).

In this way, using equation (2.17), Pythagorean’s Quadruples are generated and are given in Table 2.7.

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<th>((\frac{x-y}{(x+y)}))</th>
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<th>(\frac{1}{2}(y - \frac{1}{2}))</th>
<th>(\frac{1}{2}(y + \frac{1}{2}))</th>
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2.3c. Pythagorean’s Quintuples of the Form \( a^2 + b^2 + c^2 + d^2 = g^2 \) where \( a^2 + b^2 = e^2 \), \( e^2 + c^2 = f^2 \) and \( f^2 + d^2 = g^2 \).

Pythagorean’s Quadruples, generating equation has already been derived at (2.17), we shall proceed further from this equation which can also be written as
\[
4(x - \frac{1}{2})^2(x + \frac{1}{2})^2(y + \frac{1}{2})^2 + 4\left(x + \frac{1}{2}\right)^2(y + \frac{1}{2})^2 + \left(y - \frac{1}{2}\right)^2(y + \frac{1}{2})^2 = 1
\]
or
\[
(2.18) \left\{2(x - \frac{1}{2})(x + \frac{1}{2})(y + \frac{1}{2}) + 4(x + \frac{1}{2})(y + \frac{1}{2}) + \left(y - \frac{1}{2}\right)(y + \frac{1}{2})\right\} = 1,
\]

since \( a^2 + b^2 = e^2 \), \( e^2 + c^2 = f^2 \) and \( f^2 + d^2 = g^2 \).

From discussion made in the paragraph related to generation of Quadruples, \( f, d \) and \( g \) can be given by relations, \( \frac{f}{d} = \frac{1}{2}(z - \frac{1}{2}) \) and \( \frac{g}{f} = \frac{1}{2}(z + \frac{1}{2}) \) where \( z \) is real rational number. Equation \( f^2 + d^2 = g^2 \) can be written as \( \frac{1}{2}(z - \frac{1}{2})^2 + 1 = \frac{1}{4}(z + \frac{1}{2})^2 \) and substituting 1 as given by equation (2.18), this equation takes the form
\[
(2.19) \left\{2(x - \frac{1}{2})(x + \frac{1}{2})(y + \frac{1}{2}) + 4(x + \frac{1}{2})(y + \frac{1}{2}) + \left(y - \frac{1}{2}\right)(y + \frac{1}{2})\right\} = \left(\frac{1}{2}(z + \frac{1}{2})\right)^2.
\]

Equation (2.19) after normalisation generates Pythagorean’s Quintuples for all real rational values of \( x, y \) and \( z \). Let \( A = \left\{2(x - \frac{1}{2})(x + \frac{1}{2})(y + \frac{1}{2})\right\}, B = \left\{4(x + \frac{1}{2})(y + \frac{1}{2})\right\}, C = \left\{(y - \frac{1}{2})(y + \frac{1}{2})\right\}, D = \left\{\frac{1}{2}(z - \frac{1}{2})\right\} \) and \( G = \left\{\frac{1}{2}(z + \frac{1}{2})\right\}. \) Un-normalised Pythagorean’s Quintuples are generated by equation \( A^2 + B^2 + C^2 + D^2 = G^2 \) and after normalisation, we get \( a^2 + b^2 + c^2 + d^2 = g^2 \).

Let us take an example where \( x = 2, y = 3 \) and \( z = 4 \). Then \( A^2 = \left\{2(\frac{3}{2})(\frac{5}{7})(\frac{1}{15})\right\}^2 = \left(\frac{2}{25}\right)^2, B^2 = \left\{4(\frac{3}{2})(\frac{15}{9})\right\}^2 = \left(\frac{12}{25}\right)^2, C^2 = \left\{\left(\frac{3}{2}\right)(\frac{15}{7})\right\}^2 = \left(\frac{15}{7}\right)^2, D^2 = \left\{\left(\frac{3}{2}\right)(\frac{5}{4})\right\}^2 = \left(\frac{15}{8}\right)^2 \) and \( G^2 = \left\{\left(\frac{3}{2}\right)(\frac{3}{2})\right\}^2 = \left(\frac{9}{4}\right)^2 \). Since \( A^2 + B^2 + C^2 + D^2 = G^2 \), therefore \( \left(\frac{2}{25}\right)^2 + \left(\frac{12}{25}\right)^2 + \left(\frac{15}{7}\right)^2 = \left(\frac{9}{4}\right)^2 \). After normalisation, \( a^2 + b^2 + c^2 + d^2 = g^2 \) or \( 72^2 + 96^2 + 160^2 + 375^2 = 425^2 \).

In this way, using equation (2.19), Pythagorean’s Quintuples are generated and are given in the Table 2.8.
### Table 2.8: Pythagorean’s Quintuples Of The Form $a^2 + b^2 + c^2 + d^2 + g^2 = h^2$ where $a^2 + b^2 = e^2$, $e^2 + c^2 = f^2$, $f^2 + d^2 = k^2$ and $k^2 + g^2 = h^2$

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#### 2.3d. Pythagorean’s Sextuplets of the Form $a^2 + b^2 + c^2 + d^2 + g^2 = h^2$ where $a^2 + b^2 = e^2$, $e^2 + c^2 = f^2$, $f^2 + d^2 = k^2$ and $k^2 + g^2 = h^2$

Equation (2.19) can also be written as

$$
(2.20) \left\{ 2^3(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1}(z + \frac{1}{z})^{-1} \right\}^2 + \left\{ 2^2(x - \frac{1}{x})(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1}(z + \frac{1}{z})^{-1} \right\}^2 \\
+ \left\{ 2(y - \frac{1}{y})(y + \frac{1}{y})^{-1}(z + \frac{1}{z})^{-1} \right\}^2 + \left\{ 2^0(z - \frac{1}{z})(z + \frac{1}{z})^{-1} \right\}^2 = 1.
$$

Also $k^2 + g^2 = h^2$ can be written as $\left\{ \frac{1}{2}(w - \frac{1}{w}) \right\}^2 + 1 = \left\{ \frac{1}{2}(w + \frac{1}{w}) \right\}^2$ where $w + \frac{1}{w} = 2\frac{x}{y}$ and $w - \frac{1}{w} = 2\frac{y}{x}$ from equations (2.11) and (2.12). On substituting 1 as given in equation (2.20) in the above equation, we get

$$
(2.21) \left\{ 2^3(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1}(z + \frac{1}{z})^{-1} \right\}^2 + \left\{ 2^2(x - \frac{1}{x})(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1}(z + \frac{1}{z})^{-1} \right\}^2 \\
+ \left\{ 2(y - \frac{1}{y})(y + \frac{1}{y})^{-1}(z + \frac{1}{z})^{-1} \right\}^2 + \left\{ 2^0(z - \frac{1}{z})(z + \frac{1}{z})^{-1} \right\}^2 + \left\{ 2^{-1}(w - \frac{1}{w}) \right\}^2 = \left\{ 2^{-1}(w + \frac{1}{w}) \right\}^2.
$$

Equation (2.21) after normalisation generates Pythagorean’s Sextuplets for all real rational values of $x, y, z$ and $w$.

Let

$$
A^2 = \left\{ 2^3(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1}(z + \frac{1}{z})^{-1} \right\}^2 \\
B^2 = \left\{ 2^2(x - \frac{1}{x})(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1}(z - \frac{1}{z})^{-1} \right\}^2 \\
C^2 = \left\{ 2(y - \frac{1}{y})(y + \frac{1}{y})^{-1}(z + \frac{1}{z})^{-1} \right\}^2 \\
D^2 = \left\{ (z - \frac{1}{z})(z + \frac{1}{z})^{-1} \right\}^2 \\
G^2 = \left\{ \frac{1}{2}(w - \frac{1}{w}) \right\}^2 \\
H^2 = \left\{ \frac{1}{2}(w + \frac{1}{w}) \right\}^2,
$$

then unnormalised Pythagorean’s Sextuplets are generated by $A, B, C, D, G$ and $H$ by the equation (2.21) where

$$
$$

After normalisation, we get $a^2 + b^2 + c^2 + d^2 + g^2 = h^2$.

Let us take an example where $x = 2, y = 2, z = 2$ and $w = 3$ then

$$
A^2 = \left\{ 2^3(\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) \right\}^2 = \left( \frac{64}{125} \right)^2,
$$

$$
B^2 = \left\{ 2^2(\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) \right\}^2 = \left( \frac{48}{125} \right)^2,
$$

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\[ C^2 = \left(2\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\right)^2 = \left(\frac{1}{3}\right)^2, \]
\[ D^2 = \left(\frac{1}{2}\left(\frac{1}{3}\right)\right)^2 = \left(\frac{1}{3}\right)^2, \]
\[ G^2 = \left(\frac{1}{2}\left(\frac{3}{8}\right)\right)^2 = \left(\frac{3}{8}\right)^2, \]
\[ H^2 = \left(\frac{1}{2}\left(\frac{10}{5}\right)\right)^2 = \left(\frac{5}{4}\right)^2. \]

Therefore,
\[ \left(\frac{64}{125}\right)^2 + \left(\frac{48}{125}\right)^2 + \left(\frac{12}{25}\right)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{1}{3}\right)^2 = \left(\frac{5}{3}\right)^2. \]

After normalisation
\[ 192^2 + 144^2 + 180^2 + 225^2 + 500^2 = 625^2. \]

In this way, using Equation (2.21), Pythagorean’s Sextuples are generated and unnormalised are given in the Table 2.9 A and normalised in Table 2.9 B.

**Table 2.9 A: Unnormalised Pythagorean’s Sextuples**

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**Table 2.9 B: Pythagorean’s Sextuples of the Form \( a^2 + b^2 + c^2 + d^2 + e^2 = h^2 \)**

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**2.3e. Pythagorean’s N-Tuples of the Form \( x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = y^2 \)**

Equations 2.17), (2.19) and (2.20) on changing \( x_{2n+1}, y_{2n+2}, z_{2n+3} \) so on can be written in the following way for number of Pythagorean’s terms \( n = 4 \),
\[
\left\{ 2\left(x_1 + \frac{1}{x_1}\right)^{-1}\right\} \left\{ 2\left(x_1 - \frac{1}{x_1}\right)\left(x_1 + \frac{1}{x_1}\right)^{-1}\right\} \left\{ 2\left(x_2 - \frac{1}{x_2}\right)\right\} \left\{ 2\left(x_2 + \frac{1}{x_2}\right)\right\} = \left\{ 2\left(x_2 + \frac{1}{x_2}\right)\right\}^2.
\]

For number of Pythagorean’s terms \( n = 5 \),
\[
\left\{ 2\left(x_1 + \frac{1}{x_1}\right)^{-1}x_2 + \frac{1}{x_2}\right\} \left\{ 2\left(x_1 - \frac{1}{x_1}\right)\left(x_1 + \frac{1}{x_1}\right)^{-1}\left(x_2 + \frac{1}{x_2}\right)^{-1}\right\} + \left\{ 2\left(x_2 - \frac{1}{x_2}\right)\right\} \left\{ 2\left(x_2 + \frac{1}{x_2}\right)\right\} = \left\{ 2\left(x_2 + \frac{1}{x_2}\right)\right\}^2.
\]

For number of Pythagorean’s terms \( n = 6 \),
\[
\left\{ 2\left(x_1 + \frac{1}{x_1}\right)^{-1}x_2 + \frac{1}{x_2}\right\} \left\{ 2\left(x_1 - \frac{1}{x_1}\right)\left(x_1 + \frac{1}{x_1}\right)^{-1}\left(x_2 + \frac{1}{x_2}\right)^{-1}\left(x_3 + \frac{1}{x_3}\right)^{-1}\right\} + \left\{ 2\left(x_1 - \frac{1}{x_1}\right)\left(x_1 + \frac{1}{x_1}\right)^{-1}\left(x_2 + \frac{1}{x_2}\right)^{-1}\left(x_3 + \frac{1}{x_3}\right)^{-1}\right\} = \left\{ 2\left(x_2 + \frac{1}{x_2}\right)\right\}^2.
\]
\[
+\left(2^1 (x_2 - \frac{1}{x_2})(x_2 + \frac{1}{x_2})^{-1}(x_3 + \frac{1}{x_3})^{-1}\right)^2 + \left(2^0 (x_3 - \frac{1}{x_3})(x_3 + \frac{1}{x_3})^{-1}\right)^2 + \left(2^{-1} (x_4 - \frac{1}{x_4})\right)^2 = \left(2^{-1} (x_4 + \frac{1}{x_4})\right)^2.
\]

By mathematical induction for Pythagorean’s terms \(n\),

\[(2.22) \left(2^{-1} (x_{n-2} + \frac{1}{x_{n-2}})\right)^2 = 2^{n-3} (x_1 + \frac{1}{x_1})^{-1}(x_2 + \frac{1}{x_2})^{-1}(x_3 + \frac{1}{x_3})^{-1} \ldots (x_{n-3} + \frac{1}{x_{n-3}})^{-1}\right)^2
+\left(2^{-(n-4)}(x_1 - \frac{1}{x_1})(x_2 + \frac{1}{x_2})^{-1} \ldots (x_{n-4} + \frac{1}{x_{n-4}})^{-1}\right)^2
+\left(2^{-(n-5)}(x_2 - \frac{1}{x_2})(x_3 + \frac{1}{x_3})^{-1} \ldots (x_{n-5} + \frac{1}{x_{n-5}})^{-1}\right)^2
+\left(2^{-(n-6)}(x_3 - \frac{1}{x_3})(x_4 + \frac{1}{x_4})^{-1} \ldots (x_{n-6} + \frac{1}{x_{n-6}})^{-1}\right)^2
+ \ldots + \left(\left(2^0 (x_{n-3} - \frac{1}{x_{n-3}})(x_{n-2} + \frac{1}{x_{n-2}})^{-1}\right)^2 + \left(2^{-1} (x_{n-2} - \frac{1}{x_{n-2}})\right)^2\right)\]
3 Results And Conclusions

Pythagorean’s triples \(x, y, z\) satisfying equation \(x^2 + y^2 = z^2\) can be given by equation
\[(x)^2 + (y)^2 = (z)^2\] or \((x^2 - 2x(y + z))(b^2 - a^2) = 0\) where \(x = (x + a)\) and \(z = (x + b)\). Above quadratic equation has two roots as \(x = (b - a) \pm \left\{ (b - a)^2 + (b^2 - a^2) \right\}^{\frac{1}{2}}\) where, \(a\) and \(b\) are real and rational quantities. Values of \(a\) and \(b\) are so assumed that roots are real and rationals. If the roots found are real but fractions or \(a\) or \(b\) or both \(a\) and \(b\) are real and rationals but fractions, then quantities \(x, (x + a)\) and \((x + b)\) are normalised by multiplying with \(LCM\) to make these integers. Since different values of \(a\) and \(b\) can be chosen, therefore, a number of values of \(x\) can be generated, hence a number of Pythagorean’s triples.

Pythagorean’s quadruples \(x, y, z\) and \(w\) satisfying equation \(x^2 + y^2 + w^2 = z^2\) can be found by equation \((x + a)^2 + (y + b)^2 = (z + c)^2\) where \(y = (x + a), z = (x + b)\) and \(w = (x + c)\). This quadratic has roots as
\[x = \frac{b^2 + c^2 - a^2}{2(a - b - c)}\].

Assigning different real and rational values of \(a\), \(b\), and \(c\) will yield different values of \(x\) and hence different quadruples after normalisation.

Pythagorean Quintuples of form \(x^2 + y^2 + z^2 + w^2 = v^2\) can also be reduced to linear equation \((x + a)^2 + (y + b)^2 + (z + c)^2 + (w + d)^2 = (2x + d)^2\) where \(y = (x + a), z = (x + b), w = (x + c)\) and \(v = (2x + d)\). The above equation on simplification reduces to
\[x = \frac{d^2 - a^2 - b^2 - c^2}{2(a + b + c - 2d)}\].

Different real rational values are assigned to \(a, b, c\) and \(d\) so that real rational values of \(x\) are obtained. After normalisation, Pythagorean’s Quintuples are generated.

Pythagorean \(n\)-tuples of form \(x_1^2 + x_2^2 + x_3^2 + \ldots + x_n^2 = y^2\) can be reduced to quadratic equation \((n - 1)x_1^2 + 2x_1(a_2 + a_3 + a_4 + \ldots + a_{n-1}) + a_2^2 + a_3^2 + a_4^2 + \ldots + a_{n-1}^2 - a_1^2 = 0\) where \(x_2 = (x_1 + a_2), x_3 = (x_1 + a_3), x_4 = (x_1 + a_4)\ldots x_n = (x_1 + a_n)\) and \(y = (x_1 + a_1)\). Different real rational values of \(a_1, a_2, a_3, \ldots, a_n\) are assumed so that real rational values of \(x_1\) are obtained as
\[x_1 = -Q \pm \sqrt{Q^2 - 4PR}\]
\[2P\]
where \(P = (n - 1), Q = 2(a_2 + a_3 + a_4 + \ldots + a_n - a_1)\) and \(R = a_2^2 + a_3^2 + a_4^2 + \ldots + a_n^2 - a_1^2\).

Apart from this, there is a second method to generate Pythagorean’s \(n\)-tuples. In this method, an identity is derived satisfying Pythagorean’s \(n\)-tuples and then assigning different real rational values to variables in the identity, Pythagorean’s numbers are generated.

Pythagorean’s triples \(a, b, c\) given by the equation \(a^2 + b^2 = c^2\) can be generated from identity
\[\left\{ \frac{1}{2}(x - \frac{1}{x}) \right\}^2 + 1^2 = \left\{ \frac{1}{2}(x + \frac{1}{x}) \right\}^2\],
where \(\frac{1}{2}(x - \frac{1}{x}) = \frac{y}{a}\), and \(\frac{1}{2}(x + \frac{1}{x}) = \frac{z}{a}\) and \(x\) is real rational quantity. It may be integer or fraction of the form \(p/q\) where \(p\) and \(q\) are integers.

Pythagorean’s Quadruples of the form \(a^2 + b^2 + c^2 = d^2\) can be generated after normalisation, from the identity
\[(x - \frac{1}{x})^2(x + \frac{1}{x})^2 = 4(x + \frac{1}{x})^2 + \frac{1}{4}(y - \frac{1}{y})^2 = \frac{1}{4}(y + \frac{1}{y})^2\]
where \(x\) and \(y\) are real rational quantities and these may be integers or fractions of kind \(p/q\).

Pythagorean’s Quintuples can be generated by the identity
\[\left\{ (x - \frac{1}{x})(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1} \right\}^2 + \left\{ 2^x(x + \frac{1}{x})^{-1}(y + \frac{1}{y})^{-1} \right\}^2 + \left\{ 2y(y - \frac{1}{y})(y + \frac{1}{y})^{-1} \right\}^2 + \left\{ 2z(z - \frac{1}{z}) \right\}^2 = \left\{ 2z(z + \frac{1}{z}) \right\}^2\]
after normalisation where \( x, y \) and \( z \) are real rational quantities and these may be integers or fractions of kind \( p/q \).

Pythagorean’s Sextuples can be generated by the identity
\[
\left\{ 2^3 \left( \frac{x}{y} + \frac{1}{y} \right)^{-1} (y + \frac{1}{y}) (y + \frac{1}{y}) \right\}^2 + \left\{ 2^2 \left( \frac{x}{y} \right) \left( y + \frac{1}{y} \right)^{-1} (y + \frac{1}{y}) \right\}^2 \\
+ \left\{ 2 \left( \frac{x}{y} \right) (y + \frac{1}{y}) \right\}^2 + \left\{ 2^0 (y + \frac{1}{y}) \right\}^2 + \left\{ 2^{-1} (w - \frac{1}{w}) \right\}^2 = \left\{ 2^{-1} (w + \frac{1}{w}) \right\}^2 ,
\]

after normalisation where \( x, y, z \) and \( w \) are real rational quantities and these may be integers or fractions of kind \( p/q \).

Pythagorean’s \( n \)-tuples can be generated by the identity
\[
\left\{ 2^{-1} \left( x_{n-2} + \frac{1}{x_{n-2}} \right)^2 \\
eq 2^{n-3} \left( \frac{x_1 + \frac{1}{x_1}}{x_1} \right)^{-1} (x_2 + \frac{1}{x_2})^{-1} (x_3 + \frac{1}{x_3})^{-1} \ldots (x_{n-3} + \frac{1}{x_{n-3}})^{-1} \right\}^2 \\
+ \left\{ 2^{n-4} \left( \frac{x_1 + \frac{1}{x_1}}{x_1} \right)^{-1} (x_2 + \frac{1}{x_2})^{-1} (x_3 + \frac{1}{x_3})^{-1} \ldots (x_{n-3} + \frac{1}{x_{n-3}})^{-1} \right\}^2 \\
+ \left\{ 2^{n-5} (x_2 + \frac{1}{x_2})^{-1} (x_3 + \frac{1}{x_3})^{-1} \ldots (x_{n-3} + \frac{1}{x_{n-3}})^{-1} \right\}^2 \\
+ \left\{ 2^{n-6} (x_3 + \frac{1}{x_3})^{-1} \ldots (x_{n-3} + \frac{1}{x_{n-3}})^{-1} \right\}^2 \\
+ \ldots + \left\{ 2^0 (x_{n-3} + \frac{1}{x_{n-3}})^{-1} \right\}^2 + \left\{ 2^{-1} (x_{n-2} + \frac{1}{x_{n-2}}) \right\}^2 \\
\]

after normalisation where \( x_1, x_2, \ldots, x_{n-2} \) are real rational quantities and these may be integers or fractions of kind \( p/q \).

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References

Internet Resources: