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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75th Birth Anniversary Celebrations)

NONLINEAR ABSTRACT MEASURE HYBRID DIFFERENTIAL EQUATIONS WITH A LINEAR PERTURBATION OF SECOND TYPE

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Abstract

In this paper, an existence result for perturbed abstract measure differential equations is proved via hybrid fixed point theorems of Dhage [4] under the mixed generalized Lipschitz and Carathéodory conditions. The existence of the extremal solutions is also proved under certain monotonicity conditions and using a hybrid fixed point theorem of Dhage [4] on ordered Banach spaces. Our existence results include the existence results of Sharma [23], Joshi [19] and Shendge and Joshi [25] as special cases under weaker continuity condition.
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1 Introduction

Sharma [23, 24] introduced the abstract measure differential equations as the generalizations of the ordinary differential equations in which ordinary derivative is replaced with the Radon-Nykodym derivative of vector measures in abstract spaces. The basic results concerning the existence and uniqueness of solutions for such equations in the above papers via fixed point techniques from nonlinear functional analysis. Later, such abstract measure differential equations are studied by various authors for different aspects of the solutions (see Joshi [19], Shendge and Joshi [25], Dhage [1, 2, 3], Dhage et al. [13], Dhage and Graef [14], Dhage and Reddy [16] and the references therein).

It is quite familiar that if a nonlinear differential equation is not solvable, but when we perturb it, we obtain very interesting results along with existence of solution. The classifications of different types of perturbations appear in Dhage [5]. The perturbed differential equation of any type is called a hybrid differential equations and studied extensively in the literature for different aspects of the solutions via hybrid fixed point theory initiated by Krasnoseslskii [20] and Dhage [4, 5, 7]. In the present paper, we consider a nonlinear abstract measure differential equation with linear perturbation of second type and deal with a variant of Krasnoselskii [20] fixed point theorem due to Dhage [4]. The results of this paper complement and generalize the results of Sharma [23, 24], Joshi [19], Shendge and Joshi [25], Dhage [1, 2, 3] on abstract measure differential equations under suitable conditions.

The rest of the paper is organized as follows. Section 2 deals with the statement of the problem of abstract measure differential equation and Section 3 deals with the auxiliary results needed in the subsequent sections of the paper. The main existence result is presented in Section 4 and an existence result concerning the extremal solutions is given in Section 5. Finally, a comparison of our AMDE with the ordinary differential equations along with a couple of examples are presented in Section 6.

2 Statement of the Problem

Let X be a real Banach space with a convenient norm $\|\cdot\|$ and let $x, y \in X$ be any two elements. Then the line segment $\overline{xy}$ in X is defined by

(2.1) $\overline{xy} = \{z \in X \mid z = x + r(y - x), 0 \leq r \leq 1\}$.

Let $x_0 \in X$ be a fixed point and $z \in X$. Then for any $x \in \overline{x_0 z}$, we define the sets S_x and $\overline{S}_x$ in X by

(2.2) $S_x = \{rx \mid -\infty < r < 1\}$,

and

(2.3) $\overline{S}_x = \{rx \mid -\infty < r \leq 1\}$.

Let $x_1, x_2 \in \overline{xy}$ be arbitrary. We say $x_1 < x_2$ if $S_{x_1} \subset S_{x_2}$, or equivalently, $\overline{x_0 x_1} \subset \overline{x_0 x_2}$. In this case we also write $x_2 > x_1$.

Let M denote the σ-algebra of all subsets of X such that (X, M) is a measurable space. Let $ca(X, M)$ be the space of all vector measures (real signed measures) and define a norm $|\cdot|$ on $ca(X, M)$ by

(2.4) $\|p\| = |p|(X)$,

where $|p|$ is a total variation measure of p and is given by

(2.5) $|p|(X) = \sup\limits_{\sigma} \sum_{i=1}^{\infty} |p(E_i)|, \ E_i \subset X$,

where the supremum is taken over all possible partitions $\sigma = \{E_i : i \in \mathbb{N}\}$ of measurable subsets of X. It is known that $ca(X, M)$ is a Banach space with respect to the norm $\|\cdot\|$ given by (2.4).

Let μ be a σ-finite positive measure on X, and let $p \in ca(X, M)$. We say p is absolutely continuous with respect to the measure μ if $\mu(E) = 0$ implies $p(E) = 0$ for $E \in M$. In this case we also write $p << \mu$.

Let $x_0 \in X$ be fixed and let M_0 denote the σ- algebra on S_{x_0}. Let $z \in X$ be such that $z > x_0$ and let M_z denote the σ-algebra of all sets containing M_0 and the sets of the form S_x, $x \in \overline{x_0 z}$.

Given a $p \in ca(X, M)$ with $p << \mu$, consider the abstract measure differential equation (AMDE) with a linear perturbation of second type of the form

(2.6) $\dfrac{d}{d\mu}\Big[p(\overline{S}_x) - f(x, p(\overline{S}_x))\Big] = g(x, p(\overline{S}_x))$ a.e. $[\mu]$ on $\overline{x_0 z}$,

and

(2.7) $p(E) = q(E), \quad E \in M_0$,

where q is a given known vector measure, $\dfrac{dp}{d\mu}$ is a Radon-Nikodym derivative of p with respect to μ, $f : S_z \times \mathbb{R} \to \mathbb{R}$ is a continuous function with the map $E \mapsto [p(E) - f(x, p(E))] = \lambda(E)$ is an absolutely continuous measure w.r.t. the measure μ for each $x \in S_z$ and the function $g : S_z \times \mathbb{R} \to \mathbb{R}$ is such that the map $x \mapsto g(x, p(\overline{S}_x))$ is μ-integrable for each $p \in ca(S_z, M_z)$.

***Remark* 2.1** *Let* $\lambda(E) = \big[p(E) - f(x, p(E))\big]$ *for* $x \in S_z$ *and* $E \in M_z$. *If* $p << \mu$ *and* $\lambda << \mu$, *then* $f(x, 0) = 0$ *for each* $x \in S_z$.

***Definition* 2.1** Given an initial real measure q on M_0, a vector $p \in ca(S_z, M_z)$ $(z > x_0)$ is said to be a *solution* of AMDE (2.6)-(2.7) if

(i) $p(E) = q(E)$, $E \in M_0$,
(ii) $p << \mu$ on $\overline{x_0 z}$, and
(iii) p satisfies (2.6) a.e. $[\mu]$ on $\overline{x_0 z}$.

The following result from measure theory is often times used for transforming the abstract measure differential equation into an equivalent abstract measure integral equation.

***Theorem* 2.1** (Radon-Nikkodym theorem) *Let λ and μ be two σ-finite measures defined on a measurable space* (X, M) *such that $\lambda << \mu$. Then there exists a M-measurable function* $f : X \to [0, \infty)$ *such that*

(2.8) $\lambda(E) = \int_E f \, d\mu$

for any $E \in M$. The function f is unique upto the set of measure zero.

Note that the function f in the expression (2.8) is called the Radon-Nikodym derivative of the measure λ with respect to the measure μ and in this case we write

(2.9) $\dfrac{d\lambda}{d\mu} = f$ a.e. $[\mu]$ on X.

The details of Radon-Nikodym derivative and its integral representation appear in Ruddin [22], Sharma [23, 24], Dhage [1] and the references therein.

***Remark* 2.2** *By an application of Radon-Nikodym theorem given in **Theorem 2.1**, the AMDE* (2.6)*-*(2.7) *is equivalent to the abstract measure integral equation (in short AMIE)*

(2.10) $p(E) = f(x, p(E)) + \int_E g(x, p(\overline{S}_x)) \, d\mu$,

if $E \in M_z$, $E \subset \overline{x_0 z}$. *and*

(2.11) $p(E) = q(E)$ *if* $E \in M_0$.

A solution p of the AMDE (2.6)*-*(2.7) *on* $\overline{x_0 z}$ *will be denoted by* $p(\overline{S}_{x_0}, q)$.

Note that our AMDE (2.6)-(2.7) includes the abstract measure differential equation considered in the previous papers as special case. To see this, define $f(x, y) = 0$ for all $x \in \overline{x_0 z}$ and $y \in \mathbb{R}$, then AMDE (2.6)-(2.7) reduces to

(2.12) $\dfrac{dp}{d\mu} = g(x, p(\overline{S}_x))$ a.e. $[\mu]$ on $\overline{x_0 z}$.

and

(2.13) $p(E) = q(E)$ if $E \in M_0$.

The AMDE (2.11)-(2.12) has been studied in Joshi [19] and Dhage et. al [13] which further includes the abstract measure differential equations studied by Sharma [23, 24] and Leela [21] as special cases. Thus our AMDE (2.6)-(2.7) is more general and we claim that it is a new to the literature on measure differential equations. As a result the results of the present study are new and original contribution to the theory of nonlinear measure differential equations. In the following section we shall prove the existence and monotonicity theorems for AMDE (2.6)-(2.7).

3 Auxiliary Results

***Definition* 3.1** (Dhage [4, 5, 6, 8]) *An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function if $\psi(0) = 0$. The class of all $\mathcal{D}$-functions on $\mathbb{R}+$ is denoted by $\mathfrak{D}$.*

***Remark* 3.1** It is clear that if ϕ, ψ are $\mathcal{D}$-functions, then (i) $\phi + \psi$, (ii) $\lambda\phi$, $\lambda > 0$, and (iii) $\phi \circ \psi$ are also $\mathcal{D}$-functions, where " $\circ$ " is the composite operation of two functions on $\mathbb{R}_+$.

***Definition* 3.2** (Dhage [4, 5, 6, 8, 9]) *Let $\mathfrak{X}$ be a Banach space. An operator $\mathcal{T} : \mathfrak{X} \to \mathfrak{X}$ is called $\mathcal{D}$-Lipschitz if there exists a $\mathcal{D}$-function $\psi_{\mathcal{T}} \in \mathfrak{D}$ such that*

(3.1) $\|\mathcal{T}x - \mathcal{T}y\| \le \psi_{\mathcal{T}}(\|x - y\|)$

for all elements $x, y \in \mathfrak{X}$. If $\psi_{\mathcal{T}}(r) = k\,r$, $k > 0$, $\mathcal{T}$ is called a Lipschitz operator on $\mathfrak{X}$ with the Lipschitz constant k. Again, if $0 \le k < 1$, then $\mathcal{T}$ is called a contraction on $\mathfrak{X}$ with contraction constant k. Furthermore, if $\psi_{\mathcal{T}}(r) < r$ for $r > 0$, then $\mathcal{T}$ is called a nonlinear $\mathcal{D}$-contraction on $\mathfrak{X}$. The class of all $\mathcal{D}$-functions satisfying the condition of nonlinear $\mathcal{D}$-contraction is denoted by $\mathfrak{D}\mathfrak{N}$.

An operator $\mathcal{T} : \mathfrak{X} \to \mathfrak{X}$ is called compact if $\overline{\mathcal{T}(\mathfrak{X})}$ is a compact subset of $\mathfrak{X}$. $\mathcal{T}$ is called totally bounded if for any bounded subset S of $\mathfrak{X}$, $\mathcal{T}(S)$ is a totally bounded subset of $\mathfrak{X}$. T is called completely continuous if $\mathcal{T}$ is continuous and totally bounded on $\mathfrak{X}$. Every compact operator is totally bounded, but the converse may not be true, however, two notions are equivalent on bounded subsets of $\mathfrak{X}$. The details of different types of nonlinear contraction, compact and completely continuous operators appear in Granas and Dugundji [17].

To prove the main existence result of this section, we need the following variant of Krasnoselskii fixed point theorem proved in Dhage [4, 9, 11, 12] for the sum of two operators in a Banach space $\mathfrak{X}$. Also see Dhage [10] for related results and applications.

***Theorem* 3.1** *Let S be a closed convex and bounded subset of a Banach space $\mathfrak{X}$ and let $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ and $\mathcal{B} : S \to \mathfrak{X}$ be two operators satisfying the following conditions.*

(a) $\mathcal{A}$ is nonlinear $\mathcal{D}$-contraction,

(b) $\mathcal{B}$ is completely continuous, and

(c) $\mathcal{A}x + \mathcal{B}y = x \implies x \in S$ for all $y \in S$.

Then the operator equation

(3.2) $\mathcal{A}x + \mathcal{B}x = x$

has a solution in S.

An interesting corollary to Theorem 3.1 in the applicable form is

***Corollary* 3.1** *Let S be a closed convex and bounded subset of a Banach space $\mathfrak{X}$ and let $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ and $\mathcal{B} : S \to \mathfrak{X}$ be two operators satisfying the following conditions.*

(a) $\mathcal{A}$ is linear contraction,

(b) $\mathcal{B}$ is compact and continuous, and

(c) $\mathcal{A}x + \mathcal{B}y = x \implies x \in S$ for all $y \in S$.

Then the operator equation (3.2) *has a solution in S.*

In the following section we state our perturbed abstract measure differential equations to be discussed qualitatively in the subsequent part of this paper.

4 Existence Theorem

We need the following definition in the sequel.

***Definition* 4.1** *A function $\beta : S_z \times \mathbb{R} \to \mathbb{R}$ is called Carathéodory if*
(i) *$x \to \beta(x,y)$ is μ-measurable for each $y \in \mathbb{R}$, and*
(ii) *$y \to \beta(x,y)$ is continuous almost everywhere $[\mu]$ on $\overline{x_0 z}$.*
Further a Carathéodory function $\beta(x,y)$ is called $L^{\mu}_{\mathbb{R}}$-Carathéodory if
(iii) *there exists a μ-integrable function $h : S_z \to \mathbb{R}$ such that*

$$|\beta(x,y)| \leq h(x) \quad a.\,e.\ [\mu], \quad x \in \overline{x_0 z}$$

for all $y \in \mathbb{R}$.

We consider the following set of assumptions.

(A_0) For any $z > x_0$, the σ-algebra M_z is compact with respect to the topology generated by the pseudo-metric d defined on M_z by

$$d(E_1, E_2) = |\mu|(E_1 \Delta E_2)$$

for all $E_1, E_2 \in M_z$.
(A_1) $\mu(\{x_0\}) = 0$.
(A_2) There exist real numbers $L > 0$ and $M > 0$ such that

$$|f(x,y_1) - f(x,y_2)| \leq \frac{L\,|y_1 - y_2|}{M + |y_1 - y_2|} \quad \text{a.e. } [\mu], \quad x \in \overline{x_0 z},$$

for all $y_1, y_2 \in \mathbb{R}$. Moreover, $L \leq M$.
(B_0) q is continuous on M_z with respect to the pseudo-metric d defined in (A_0).
(B_1) The function $g(x,y)$ is $L^{\mu}_{\mathbb{R}}$-Carathéodory.

***Theorem* 4.1** *Suppose that the assumptions $(A_0) - (A_2)$ and $(B_0) - (B_1)$ hold. Then the AMDE (2.6) - (2.7) has a solution on $\overline{x_0 z}$.*

Proof. By expressions (2.2) and (2.3), we have a decreasing sequence $\{r_n\}$ of positive real numbers such that $r_n \to 1$ as $n \to \infty$ and $S_{r_n x_0} \supset S_{x_0}$. Then, from hypothesis (H_1), it follows that

$$\bigcap_{r \to 1} \left(\overline{S}_{r x_0} - S_{x_0}\right) = \{x_0\}$$

and so,

$$\mu\left(\overline{S}_{r x_0} - S_{x_0}\right) = \mu(\{x_0\}) = 0 \quad \text{as} \quad r \to 1.$$

Therefore, we can choose a real number $r^* > 1$ such that $S_{r^* x_0} \supset S_{x_0}$ and

$$\mu(\overline{S}_{r^* x_0} - S_{x_0}) < 1 \quad \text{and} \quad \int\limits_{\overline{S}_{r^* x_0} - S_{x_0}} h(x)\, d\mu < 1.$$

Let $z^* = r^* x_0$ and consider the measure p_0 on M_{z^*} which is a continuous extension of the measure q on M_0 defined by

$$p_0(E) = \begin{cases} q(E) & \text{if} \quad E \in M_0, \\ 0 & \text{if} \quad E \notin M_0. \end{cases}$$

Now define a subset $S(\rho)$ of $ca(S_{z^*}, M_{z^*})$ by

(4.1) $S(\rho) = \{p \in ca(S_{z^*}, M_{z^*}) \mid \|p - p_0\| \leq \rho\}$

where $\rho = F_0 + L + 1$. Clearly, $S(\rho)$ is a closed convex ball in $ca(S_{z^*}, M_{z^*})$ centred at p_0 of radius ρ and $q \in S(\rho)$.

Define two operators $\mathcal{A} : ca(S_{z^*}, M_{z^*}) \to ca(S_{z^*}, M_{z^*})$ and $\mathcal{B} : S(\rho) \to ca(S_{z^*}, M_{z^*})$ by

(4.2) $$\mathcal{A}p(E) = \begin{cases} f(x, p(\overline{S}_x)), & \text{if } E \in M_z, E \subset \overline{x_0 z^*}, \\ 0, & \text{if } E \in M_0, \end{cases}$$

and

(4.3) $$\mathcal{B}p(E) = \begin{cases} \int_E g(x, p(\overline{S}_x))\, d\mu & \text{if} \in M_{z^*}, \text{if } E \subset \overline{x_0 z^*}, \\ q(E) & \text{if } E \in M_0. \end{cases}$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of ***Corollary* 2.1** on S.

Step I: First we show that $\mathcal{A}$ is a contraction on $ca(S_{z^*}, M_{z^*})$. Let $p_1, p_2 \in ca(S_{z^*}, M_{z^*})$ be arbitrary. Then by assumption (A_2),

$$\begin{aligned}|\mathcal{A}p_1(E) - \mathcal{A}p_2(E)| &= |f(x, p_1(E)) - f(x, p_2(E))| \\ &\leq \frac{L\,|p_1(E) - p_2(E)|}{M + |p_1(E) - p_2(E)|} \\ &\leq \frac{L\,|p_1 - p_2|(E)}{M + |p_1 - p_2|(E)}\end{aligned}$$

for all $E \in M_{z^*}$. Hence by definition of the norm in $ca(S_{z^*}, M_{z^*})$ one has

$$\|\mathcal{A}p_1 - \mathcal{A}p_2\| \leq \frac{L\,\|p_1 - p_2\|}{M + \|p_1 - p_2\|}$$

for all $p_1, p_2 \in ca(S_{z^*}, M_{z^*})$. As a result $\mathcal{A}$ is a nonlinear $\mathcal{D}$-contraction on $ca(S_{z^*}, M_{z^*})$ with the $\mathcal{D}$-function ψ given by $\psi(r) = \dfrac{Lr}{M + r}$.

Step II : We show that $\mathcal{B}$ is continuous on S. Let $\{p_n\}$ be a sequence of vector measures in S converging to a vector measure p. Then by dominated convergence theorem,

$$\begin{aligned}\lim_{n\to\infty} \mathcal{B}p_n(E) &= \lim_{n\to\infty} \int_E g(x, p_n(\overline{S}_x))\, d\mu \\ &= \int_E \left[\lim\nolimits_{n\to\infty}\, g(x, p_n(\overline{S}_x))\right] d\mu \\ &= \int_E g(x, p(\overline{S}_x))\, d\mu \\ &= \mathcal{B}p(E)\end{aligned}$$

for all $E \in M_{z^*}$, $E \subset \overline{x_0 z}$. Similarly, if $E \in M_{z^*}$, then

$$\lim_{n\to\infty} \mathcal{B}p_n(E) = q(E) = \mathcal{B}p(E),$$

and so $\mathcal{B}$ is a pointwise continuous operator on S.

Next we show that $\{\mathcal{B}p_n : n \in \mathbb{N}\}$ is a equi-continuous sequence in $ca(S_{z^*}, M_{z^*})$. Let $E_1, E_2 \in M_{z^*}$. Then there exist subsets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_z$, $G_1 \subset \overline{x_0 z^*}$, $G_2 \subset \overline{x_0 z^*}$ such that

$$E_1 = F_1 \cup G_1 \text{ with } F_1 \cap G_1 = \emptyset$$

and

$$E_2 = F_2 \cup G_2 \text{ with } F_2 \cap G_2 = \emptyset.$$

We know the identities

(4.4) $G_1 = (G_1 - G_2) \cup (G_2 \cap G_1)$,

and

(4.5) $G_2 = (G_2 - G_1) \cup (G_1 \cap G_2)$.

Therefore, we have

$$\mathcal{B}p_n(E_1) - \mathcal{B}p_n(E_2) \leq q(F_1) - q(F_2) + \int_{G_1 - G_2} g(x, p_n(\overline{S}_n))d\mu + \int_{G_2 - G_1} g(x, p_n(\overline{S}_x))d\mu.$$

Since $g(x, y)$ is L^l_μ- Carathéodory, we have that

$$\begin{aligned}|\mathcal{B}p_n(E_1) - \mathcal{B}p_n(E_2)| &\leq |q(F_1) - q(F_2)| + \int_{G_1 \Delta G_2} |g(x, p_n(\overline{S}_x))| d\mu \\ &\leq |q(F_1) - q(F_2)| + \int_{G_1 \Delta G_2} h(x)\, d\mu.\end{aligned}$$

Assume that

$$d(E_1, E_2) = |\mu|(E_1 \Delta E_2) \to 0.$$

Then we have that $E_1 \to E_2$. As a result $F_1 \to F_2$ and $|\mu|(G_1 \Delta G_2) \to 0$. As q is continuous on compact M_z , it is uniformly continuous and so

$$|\mathcal{B}p_n(E_1) - \mathcal{B}p_n(E_2)| \leq |q(F_1) - q(F_2)| + \int_{G_1 \Delta G_2} h_r(x) d\mu$$

$\to 0$ as $E_1 \to E_2$

uniformly for all $n \in \mathbb{N}$. This shows that $\{\mathcal{B}p_n : n \in \mathbb{N}\}$ is a equi-continuous set in $ca(S_{z^*}, M_{z^*})$. As a result, $\{\mathcal{B}p_n\}$ converges to $\mathcal{B}p$ uniformly on M_{z^*} and a fortipri $\mathcal{B}$ is a continuous operator on $S(\rho)$.

Step III: Next we show that $\mathcal{B}(S)$ is a totally bounded set in $ca(S_{z^*}, M_{z^*})$, where $S = S(\rho)$. We shall show that the set is uniformly bounded and equi-continuous set in $ca(S_{z^*}, M_{z^*})$. Firstly, we show that $\mathcal{B}(S)$ is a uniformly bounded set in $ca(S_{z^*}, M_{z^*})$.

Let $\lambda \in \mathcal{B}(S)$ be an arbitrary element. Then, there is a member $p \in S$ such that $\lambda(E) = \mathcal{B}p(E)$ for all $E \in M_{z^*}$. Let $E \in M_{z^*}$. Then there exist two subsets $F \in M_0$ and $G \in M_{z^*}$, $G \subset \overline{x_0 z^*}$ such that

$E = F \cup G$ and $F \cap G = \phi$.

Hence by definition of $\mathcal{B}$,

$$
\begin{aligned}
(4.6)\quad |\lambda(E)| &= |\mathcal{B}p(E)| \\
&\le |q(F)| + \int_G |g(x, p(\overline{S}_x))| d\mu \\
&\le \|q\| + \int_G h(x) d\mu \\
&\le \|q\| + \int_E h(x) d\mu \\
&= \|q\| + \|h\|_{L^l_\mu}
\end{aligned}
$$

for all $E \in M_{z^*}$.

From (4.6) it follows that

$$
\begin{aligned}
\|\lambda\| &= \|\mathcal{B}p\| \\
&= |\mathcal{B}p|(E) \\
&= \sup_{\sigma} \textstyle\sum_{i=1}^{\infty} |Bp(E_i)| \\
&= \|q\| + \|h\|_{L^l_\mu}
\end{aligned}
$$

for all $\lambda \in \mathcal{B}(S)$.

Hence the sequence $\mathcal{B}(S)$ is uniformly bounded set in $ca(S_{z^*}, M_{z^*})$.

Next we show that $\mathcal{B}(S)$ is a equi-continuous set of measures in $ca(S_{z^*}, M_{z^*})$. Let $E_1, E_2 \in M_{z^*}$. Then there exist subsets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_{z^*}$, $G_1 \subset \overline{x_0 z^*}$, $G_2 \subset \overline{x_0 z^*}$ such that

$E_1 = F_1 \cup G_1$ with $F_1 \cap G_1 = \emptyset$

and

$E_2 = F_2 \cup G_2$ with $F_2 \cap G_2 = \emptyset$.

We know the identities

$$(4.7)\quad G_1 = (G_1 - G_2) \cup (G_2 \cap G_1),$$

and

$$(4.8)\quad G_2 = (G_2 - G_1) \cup (G_1 \cap G_2).$$

Therefore, we have

$$|\lambda(E_1) - \lambda(E_2)| = |Bp(E_1) - Bp(E_2)| \le |q(F_1) - q(F_2)| + \int\limits_{G_1 - G_2} |g(x, p(\overline{S}_n))|\, d\mu + \int\limits_{G_2 - G_1} |g(x, p(\overline{S}_x))|\, d\mu.$$

Since $g(x, y)$ is L^l_μ- Carathéodory, we have that

$$
\begin{aligned}
|\lambda(E_1) - \lambda(E_2)| &\le |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} |g(x, p(\overline{S}_x))|\, d\mu \\
&\le |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} h(x)\, d\mu.
\end{aligned}
$$

Assume that

$d(E_1, E_2) = |\mu|(E_1 \Delta E_2) \to 0.$

Then we have that $E_1 \to E_2$. As a result $F_1 \to F_2$ and $|\mu|(G_1 \Delta G_2) \to 0$. As q is continuous on compact M_{z^*}, it is uniformly continuous and so

$$|\lambda(E_1) - \lambda(E_2)| \le |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} h_r(x) d\mu$$

$$\to 0 \text{ as } E_1 \to E_2$$

uniformly for all $\lambda \in \mathcal{B}(S)$. This shows that $\mathcal{B}(S)$ is a equi-continuous set in the banach space $ca(S_{z^*}, M_{z^*})$. Now an application of the Arzela-Ascolli theorem yields that $\mathcal{B}$ is a totally bounded operator on S. Now $\mathcal{B}$ is continuous and totally bounded, it is completely continuous operator on S.

Step IV: Finally, we show that the hypothesis (c) of ***Theorem* 3.1** is satisfied. Let $p \in S$ be arbitrary and let there is an element $u \in ca(S_{z^*}, M_{z^*})$ such that $\mathcal{A}u + \mathcal{B}p = u$. We show that $u \in S$. Now, by definitions of the operators $\mathcal{A}$ and $\mathcal{B}$,

$$u(E) = \begin{cases} f(x, u(E)) + \int_E g(x, p(\overline{S}_x))\, d\mu, & \text{if } E \in M_z,\ E \subset \overline{x_0 z^*} \\ q(E), & \text{if } E \in M_0. \end{cases}$$

for all $E \in M_{z^*}$.

If $E \in M_{z^*}$, then there exist sets $F \in M_0$ and $G \in M_{z^*}$, $G \subset \overline{x_0 z^*}$ such that $E = F \cup G$ and $F \cap G = \emptyset$. Then we have

$$u(E) = q(F) + f(x, u(G)) + \int_G g(x, p(\overline{S}_x))\, d\mu.$$

Hence,

$$\begin{aligned} |u(E) - p_0(E)| &\le \left| f(x, u(G)) - f(x, 0) \right| + \int_G |g(x, p(\overline{S}_x))|\, d\mu \\ &\le \frac{L|u(G)|}{M + |u(G)|} + \int_G h(x)\, d\mu \\ &\le L + \int_{\overline{x_0 z^*}} h(x)\, d\mu \\ &\le L + 1 \\ &= \rho \end{aligned}$$

which further implies that

$$\|u - p_0\| \le L + 1 = \rho.$$

As a result, we have $u \in S(\rho)$ and so hypothesis (c) of ***Theorem* 3.1** is satisfied. In consequence, the operator equation $\mathcal{A}p(E) + \mathcal{B}p(E) = p(E)$ has a solution $p(S_{x_0}, q)$ in $ca(S_{z^*}, M_{z^*})$. This further implies that the AMDE (2.6)-(2.7) has a solution on $\overline{x_0 z}$. This completes the proof.

5 Existence of Extremal Solutions

In this section we prove the existence of the extremal solutions for the AMDE (2.6)-(2.7) on $\overline{x_0 z}$ under certain monotonicity conditions. We define an order relation $\preceq$ in $ca(S_z, M_z)$ with the help of the cone K in $ca(S_z, M_z)$ given by

(5.1) $K = \{p \in ca(S_z, M_z) \mid p(E) \ge 0 \text{ for all } E \in M_z\}.$

Thus for any $p_1, p_2 \in ca(S_z, M_z)$, one has

(5.2) $p_1 \preceq p_2 \iff p_2 - p_1 \in K$

or, equivalently,

$$p_1 \preceq p_2 \iff p_1(E) \le p_2(E)$$

for all $E \in M_z$.

A cone K in $ca(S_z, M_z)$ is called normal if the norm is semi-monotone on K. The details of different properties of cones in Banach spaces appear in Heikkilä and Lakshmikantham [18].

The following lemma follows immediately from the definition of the cone K in $ca(S_z, M_z)$.

***Lemma* 5.1** *The cone K is normal in the Banach space $ca(S_z, M_z)$.*

Proof. To finish, it is enough to prove that the norm $\|\cdot\|$ is semi-monotone on K. Let $p_1, p_2 \in K$ be such that $p_1 \preceq p_2$ on M_z. Then we have

$$0 \le p_1(E) \le p_2(E)$$

for all $E \in M_z$. Now, for a countable partition $\sigma = \{E_n : n \in \mathbb{N}\}$ of measurable subsets of S_z, by definition of the norm in $ca(S_z, M_z)$, one has

$$\|p_1\| = |p_1|(S_z) = \sup_\sigma \sum_{i=1}^\infty |p_1(E_i)| = \sup_\sigma \sum_{i=1}^\infty p_1(E_i)$$

$$\leq \sup_{\sigma} \sum_{i=1}^{\infty} p_2(E_i) = \sup_{\sigma} \sum_{i=1}^{\infty} |p_2(E_i)| = |p_2|(S_z) = \|p_2\|.$$

This shows that $\|\cdot\|$ is semi-monotone on K and consequently the order cone K is normal in $ca(S_z, M_z)$. The proof of the lemma is complete.

We need the following fixed point theorem of Dhage [5] involving the sum of two operators in a ordered Banach space.

***Theorem* 5.1** *Let K be a cone in a real Banach space $\mathfrak{X}$ and let $\mathcal{A}, \mathcal{B} : \mathfrak{X} \to \mathfrak{X}$ be two nondecreasing operators such that*

(a) *$\mathcal{A}$ is linear contraction,*

(b) *$\mathcal{B}$ is completely continuous, and*

(c) *there exist elements $u, v \in \mathfrak{X}$ such that $u \leq v$ satisfying $u \leq \mathcal{A}u + \mathcal{B}u$ and $\mathcal{A}v + \mathcal{B}v \leq v$.*

Further if the cone K is normal, then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a minimal and a maximal solution in $[u, v]$.

We need the following definitions in the sequel.

***Definition* 5.1** *A vector measure $u \in ca(S_z, M_z)$ is called a lower solution of AMDE (2.6)-(2.7) if*

$$\frac{d}{d\mu}\Big[u(\overline{S}_x) - f(x, u(\overline{S}_x))\Big] \leq g(x, u(\overline{S}_x)) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z},$$

and

$$u(E) \leq q(E), \quad E \in M_0.$$

Similarly, a vector measure $v \in ca(S_z, M_z)$ is called an upper solution to AMDE (2.6)-(2.7) if

$$\frac{d}{d\mu}\Big[v(\overline{S}_x) - f(x, v(\overline{S}_x))\Big] \geq g(x, v(\overline{S}_x)) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z},$$

and

$$v(E) \geq q(E), \quad E \in M_0.$$

A vector measure $p \in ca(S_z, M_z)$ is a solution to AMDE (2.6)-(2.7) if it is upper as well as lower solution to AMDE (2.6)-(2.7) on $\overline{x_0 z}$.

***Definition* 5.2** *A solution p_M is called a maximal solution for the AMDE (2.6)-(2.7) if for any other solution $p(\overline{S}_{x_0}, q)$ of the AMDE (2.6)-(2.7) we have that*

$$p(E) \leq p_M(E) \quad \forall E \in M_z.$$

Similarly, a minimal solution $p_m(\overline{S}_{x0}, q)$ for the AMDE (2.6)-(2.7) is defined on $\overline{x_0 z}$.

We consider the following assumptions:

(C_1) The functions $f(x, y)$ and $g(x, y)$ are nondecreasing in y a.e. $[\mu]$ for $x \in \overline{x_0 z}$.

(C_3) AMDE (2.6)-(2.7) has a lower solution u and an upper solution v such that $u \leq v$ on M_z.

***Theorem* 5.2** *Suppose that the assumptions (A_0) - (A_2), (B_1)-(B_2) and (C_1)-(C_2) hold. Then the AMDE (2.6)-(2.7) has a minimal and a maximal solution defined on $\overline{x_0 z}$.*

Proof. Now, AMDE (2.6)-(2.7) is equivalent to the abstract measure integral equation (in short AMIE)

(5.3) $p(E) = f(x, p(E))\, d\mu + \int_E g(x, p(\overline{S}_x))\, d\mu, \quad E \in M_z, E \subset \overline{x_0 z},$

and

(5.4) $p(E) = q(E), \quad E \in M_0.$

Define two operators $\mathcal{A}, \mathcal{B} : [u, v] \to ca(S_z, M_z)$ by

(5.5) $$\mathcal{A}p(E) = \begin{cases} f(x, p(E)) & \text{if } E \in M_z,\ E \subset \overline{x_0 z}, \\ 0 & \text{if } E \in M_0. \end{cases}$$

and

(5.6) $$\mathcal{B}p(E) = \begin{cases} \int_E g(x, p(\overline{S}_x))d\mu, & \text{if } E \in M_z,\ E \subset \overline{x_0 z}, \\ q(E) & \text{if } E \in M_0. \end{cases}$$

Then the AMIE (2.6)-(2.7) is equivalent to the operator equation

(5.7) $\mathcal{A}p(E) + \mathcal{B}p(E) = p(E), \quad E \in M_z.$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of ***Theorem* 5.1** on $ca(S_z, M_z)$. Since μ is a positive measure , from assumption (C_1) it follows that $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing operators on $ca(S_z, M_z)$. To show this , let $p_1, p_2 \in ca(S_z, M_z)$ be such that $p_1 \leq p_2$ on M_z. From hypothesis (C_2), it follows that

$$\begin{aligned}\mathcal{A}p_1(E) &= f(x, p_1(E)\, d\mu \\ &\leq f(x, p_2(E))\, d\mu \\ &= \mathcal{A}p_2(E)\end{aligned}$$

for all $E \in M_z$, $E \subset \overline{x_0 z}$ and

$$\mathcal{A}p_1(E) = 0 = \mathcal{A}p_2(E)$$

for $E \in M_0$. Hence $\mathcal{A}$ is nondecreasing on $ca(S_z, M_z)$.

Similarly, we have

$$\begin{aligned}\mathcal{B}p_1(E) &= \int_E g(x, p_1(\overline{S}_x))\, d\mu \\ &\leq \int_E g(x, p_2(\overline{S}_x))\, d\mu \\ &= \mathcal{B}p_2(E)\end{aligned}$$

for all $E \in M_z$, $E \subset \overline{x_0 z}$. Again if $E \in M_0$, then

$$\mathcal{B}p_1(E) = q(E) = \mathcal{B}p_2(E).$$

Therefore, the operator $\mathcal{B}$ is also nondecreasing on $ca(S_z, M_z)$. Now it can be shown as in the proof of ***Theorem* 4.1** that the operators $\mathcal{A}$ is a nonlinear $\mathcal{D}$-contraction on $ca(S_z, M_z)$ with the $\mathcal{D}$-function ψ given by $\psi(r) = \frac{Lr}{M+r}$ and the operator $\mathcal{B}$ is completely continuous on S. Since u is a lower solution of AMDE (2.6)-(2.7), we have

(5.8) $u(E) \leq f(x, u(E)) + \int_E g(x, u(\overline{S}_x)) d\mu, \quad E \in M_z,\ E \subset \overline{x_0 z},$

and

(5.9) $u(E) \leq q(E), \quad E \in M_0.$

From (6.8) and (6.9) it follows that

$$u(E) \leq \mathcal{A}u(E) + \mathcal{B}u(E) \quad \text{if } E \in M_z$$

and so, $u \leq \mathcal{A}u + \mathcal{B}u$. Similarly since $v \in ca(S_z, M_z)$ is an upper solution of AMDE (2.6)-(2.7), it can be proved that $\mathcal{A}v(E) + \mathcal{B}v(E) \leq v(E)$ for all $E \in M_z$ and consequently $\mathcal{A}v + \mathcal{B}v \leq v$ on M_z. Thus hypotheses (a)-(c) of ***Theorem* 5.1** are satisfied.

Thus the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of ***Theorem* 5.1** and so an application of it yields that the operator equation $\mathcal{A}p + \mathcal{B}p = p$ has a maximal and a minimal solution in $[u, v]$. This further implies that AMDE (2.6)-(2.7) has a maximal and a minimal solution on $\overline{x_0 z}$. This completes the proof.

6 Special Case

In this section it is shown that, in a certain situation, the AMDE (2.6)-(2.7) reduces to an ordinary perturbed differential equation, viz.,

(6.1) $$\frac{d}{dx}\Big[y(x) - f(x, y(x))\Big] = g(x, y(x)), \quad x \geq x_0,$$
$$y(x_0) = y_0,$$

where f is a continuous real-valued function on $[x_0, x_0 + T] \times \mathbb{R}$, and the function g satisfies Carathéodory conditions on $[x_0, x_0 + T] \times \mathbb{R}$. Note that the hybrid ordinary differential equation (6.1) is discussed first time in Dhage and Jadhav [15] for some basic results related to its solution.

Let $X = \mathbb{R}$, $\mu = m$, the Lebesgue measure on $\mathbb{R}$, $\overline{S}_x = (-\infty, x]$, $x \in \mathbb{R}$, and q a given real Borel measure on M_0. Then equations (2.6)-(2.7) take the form

(6.2) $$\frac{d}{dm}\Big[p((-\infty, x]) - f(x, p((-\infty, x]))\Big] = g(x, p((-\infty, x])),$$
$$p(E) = q(E), \quad E \in M_0.$$

It will now be shown that, the equations (6.1) and (6.2) are equivalent in the sense of the following theorem.

***Theorem* 6.1** *Let $q : M_0 \to \mathbb{R}$ be a given initial measure such that $q(E) = 0$ for all $E \in M_0$ and $q(\{x_0\}) = 0$. Then*

(a) to each solution $p = p(\overline{S}_{x_0}, q)$ of (6.2) existing on $[x_0, x_1)$, there corresponds a solution y of (6.1) satisfying $y(x_0) = y_0$.

(b) Conversely, to every solution $y(x)$ of (6.1), there corresponds a solution $p(\overline{S}_{x_0}, q)$, of (6.2) existing on $[x_0, x_1)$ with a suitable initial measure q provided f satisfies the relation $f(x_0, 0) = 0$.

Proof. (a) Let $p = p(\overline{S}_{x_0}, q)$ be a solution of (6.2), existing on $[x_0, x_1)$. Define a real Borel measure p_1 on $\mathbb{R}$ as follows.

$$(6.3)\quad p_1((-\infty, x)) = \begin{cases} 0, & \text{if } x \le x_0, \\ p((-\infty, x]) - p((-\infty, x_0]), & \text{if } x_0 < x < x_1 \\ p((-\infty, x_1)), & \text{if } x \ge x_1, \end{cases}$$

and

$$p_1(-\infty, x_0]) = p(-\infty, x_0]).$$

Define the functions $y_1(x)$ and $y(x)$ by

$$(6.4)\quad y_1(x) = p_1((-\infty, x)),\ x \in \mathbb{R}$$

$$y(x) = y_1(x) + p((-\infty, x_0]), x \in [x_0, x_1).$$

The condition $q(\{x_0\}) = 0$, the definition of the solution p, and the definitions of $y(x)$ imply that

$$p_1(\{x_0\}) = p(\{x_0\}) = 0.$$

Now for each $x \in [x_0, x_1)$ we obtain from (6.2) and the definition of $y(x)$

$$(6.5)\quad \begin{aligned} y(x) &= y_1(x) + p((-\infty, x_0]) \\ &= p_1((-\infty, x)) + p((-\infty, x_0]) \\ &= p(\overline{S}_x). \end{aligned}$$

Since p is a solution of (6.2) we have $p << m$ on $[x_0, x_1)$. Hence $y(x)$ is absolutely continuous on $[x_0, x_1)$. The details concerning these arguments appear in Rudin [22, pages 163-165]. This shows that $y'(x)$ exists a.e. on $[x_0, x_1)$. Now for each $x \in [x_0, x_1)$, we have, by virtue of (6.3) and (6.4)

$$p([x_0, x]) - f(x, p([x_0, x]) = \int_{[x_0,x]} \tfrac{d}{dm}[p((-\infty, t]) - f(t, p((-\infty, t]))]\, dm.$$

Therefore,

$$[p((-\infty, x]) - p((-\infty, x_0]) - f(x, p((-\infty, x]) - p((-\infty, x_0]))$$

$$= \int_{[x_0,x]} \tfrac{d}{dm}[p((-\infty, t]) - f(t, p((-\infty, t]))]\, dm.$$

This further implies that

$$p(\overline{S}_x) - f(x, p(\overline{S}_x)) = p(\overline{S}_{x_0}) + \textstyle\int_{x_0}^{x} g(t, p(\overline{S}_t))\, dm.$$

That is,

$$y(x) - f(x, y(x)) = y(x_0) + \textstyle\int_{x_0}^{x} g(t, y(t))dt.$$

Hence,

$$\frac{d}{dx}[y(x) - f(x, y(x))] = g(x, y(x)) \quad \text{a.e on } [x_0, x_1).$$

This proves that $y(x)$ is a solution of (6.1) on $[x_0, x_1)$ satisfying

$$y(x_0) = y_0.$$

(b) Conversely, suppose that $y(x)$ be a solution of (6.1) existing on $[x_0, x_1]$. Then, y is absolutely continuous on $[x_0, x_1]$. Now, corresponding to the absolutely continuous function $y(x)$ which is a solution of (6.1) on $[x_0, x_1)$, we can construct a absolutely continuous real Borel measure p on M_{x_1} such that,

$$(6.6)\quad p(E) = 0, \quad \text{for all } E \in M_0,$$

$$p(\overline{S}_x) = y(x), \text{ if } x \in [x_0, x_1).$$

The details concerning these arguments appear in Rudin [22, pages 163-165]. Since $y(x)$ is a solution of (6.1) we have for $x \in [x_0, x_1)$,

$$y(x) - f(x, y(x)) = y(x_0) + f(x_0, y(x_0)) + + \int_{x_0}^{x} f(t, y(t))\, dt.$$

Now, $y(x_0) = p(S_{x_0}) = 0$ and so, $f(x_0, y_0) = 0$. Hence by (6.6) it follows that

$$[p(\overline{S}_x) - p(\overline{S}_{x_0})] - f(x, p(\overline{S}_x) - p(\overline{S}_{x_0})) = \int_{[x_0,x]} g(t, p(\overline{S}_t))\, dm.$$

That is,

$$p([x_0, x]) = f(x, p([x_0, x])) + \int_{[x_0,x]} f(t, p(\overline{S}_t))\, dm.$$

In general, if $E \in M_{x_1}$, $E \subset \overline{x_0 x_1}$, then

$$p(E) = f(x, p(E)) + \int_E g(x, p((-\infty, x]))\, dm.$$

By definition of Radon-Nykodym derivative, we obtain

$$\frac{d}{dm}\Big[p((-\infty, x]) - f(x, p((-\infty, x]))\Big] = g(x, p((-\infty, x])) \;\; a.e. \; [\mu] \;\; \text{on } \overline{x_0 z}$$
$$p(E) = 0 \;\text{ for }\; E \in M_0.$$

This shows that p is a solution of (6.2) on $[x_0, x_1)$ and the proof of (b) is complete.

***Remark* 6.1** ***Theorem* 6.1** *shows that our results for the equation* (6.2) *are general in the sense that they include the corresponding results for the equation* (6.1).

***Example* 6.1** *Given a* $p \in \mathrm{ca}(X, M)$ *with* $p << \mu$, *consider the abstract measure differential equation (AMDE) with a linear perturbation of second type of the form*

(6.7) $$\frac{d}{d\mu}\left[p(\overline{S}_x) - \frac{|p(\overline{S}_x)|}{1 + |p(\overline{S}_x)|}\right] = \frac{\ln(1 + |p(\overline{S}_x)|)}{1 + |p(\overline{S}_x)|} \quad a.e. \; [\mu] \; on \; \overline{x_0 z}.$$

and

(6.8) $p(\overline{S}_{x_0})) = 1,$

where $\dfrac{dp}{d\mu}$ *is a Radon-Nikodym derivative of p with respect to* μ.

Here, $f(x, y) = \dfrac{|y|}{1 + |y|}$ *and* $g(x, y) = \dfrac{\ln(1 + |y|)}{1 + |y|}$ *for all* $x \in \overline{x_0 z}$ *and* $y \in \mathbb{R}$. *Clearly, the function* $(x, y) \mapsto \dfrac{|y|}{1 + |y|} = f(x, y)$ *is continuous and bounded on* $S_z \times \mathbb{R}$ *with ound* $M_f = 1$. *Also, after simple computation it can be shown that f satisfies the assumption* (A_3) *with* $L = 1$ *and* $M = 1$. *Next, the function g is continuous and bounded on* $S_z \times \mathbb{R}$ *with bound* $M_g = 1$. *Therefore, if the assumptions* (A_0)-(A_1) *hold, then the AMDE* (6.7) - (6.8) *has a solution defined on* $\overline{x_0 z}$.

***Example* 6.2** *Given a* $p \in \mathrm{ca}(X, M)$ *with* $p << \mu$, *consider the abstract measure differential equation (AMDE) with a linear perturbation of second type of the form*

(6.9) $$\frac{d}{d\mu}\Big[p(\overline{S}_x) - \gamma \sin p(\overline{S}_x)\Big] = \frac{1 + |p(\overline{S}_x)|}{2 + p^2(\overline{S}_x)} \quad a.e. \; [\mu] \; on \; \overline{x_0 z}.$$

and

(6.10) $p(\overline{S}_{x_0})) = 1,$

where $\dfrac{dp}{d\mu}$ *is a Radon-Nikodym derivative of p with respect to* μ *and* $0 \le \gamma < 1$.

Here, $f(x, y) = \gamma \sin y$ and $g(x, y) = \dfrac{1 + |y|}{2 + y^2}$ for all $x \in \overline{x_0 z}$ and $y \in \mathbb{R}$. Clearly, the function $(x, y) \mapsto \gamma \sin y = f(x, y)$ is continuous and bounded on $S_z \times \mathbb{R}$ with bound $M_f = 1$. Also, after simple computation it can be shown that f satisfies the assumption (A_3) with $L = \gamma M \le M$. Next, the function g is continuous and bounded on $S_z \times \mathbb{R}$ with bound $M_g = 1$. Therefore, if the assumptions (A_0)-(A_1) hold, then the AMDE (6.9) - (6.10) has a solution defined on $\overline{x_0 z}$.

***Remark* 6.2** If we define the initial vector measure q on M_0 by

$$q(\overline{S}_{x_0}) = \alpha, \;\text{ and }\; q(E) = 0 \;\text{ if }\; E \neq \overline{S}_{x_0},$$

where α is a real number and $f \equiv 0$ on $S_z \times \mathbb{R}$, then the equations (2.6)-(2.7) is reduced to the form

(6.11) $$\frac{dp}{d\mu} = g(x, p(\overline{S}_x)), \quad p(\overline{S}_{x_0}) = \alpha$$

which is the AMDE studied in Sharma [23, 24]. Thus our existence results of this paper include as particular cases, the results in Sharma [23, 24] under weaker Carathoédory conditions.

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References

[1] B.C. Dhage, On abstract measure integro-differential equations, *J. Math. Phy. Sci.,* **20** (1986), 367- 380.

[2] B.C. Dhage, On system of abstract measure integro-differential inequalities and applications, *Bull. Inst. Math. Acad. Sinica* **18** (1989), 65-75.

[3] B.C. Dhage, Mixed monotonicity theorems for a system of abstract measure delay integro-differential equations, *An. Stint. Univ.,* "Al. I. Cuza" Iasi **XLII** (1996), 355-366.

[4] B.C. Dhage, On a fixed point theorem of Krasnoselskii-Schaefer type, *Electronic J. Qualitative Theory of Differential Equations,* **6** (2002), 19.

[5] B.C. Dhage, Remarks on two fixed point theorems involving the sum and product of two operators, *Compt. Math. Appl.,* **46(12)** (2003), 1779-1785.

[6] B.C. Dhage, Local fixed point theory for the sum of two operators in Banach spaces, *Fixed Point Theory,* **4** (2003), 49-60.

[7] B.C. Dhage, On some nonlinear alternatives of Leray-Schauder type and functional integral equations, *Arch. Math. (Brno),* **42** (2006), 11-23.

[8] B.C. Dhage, A general multi-valued hybrid fixed point theorem and perturbed differential inclusions, *Nonlinear Anal.,* **64** (2006), 2747-2772.

[9] B.C. Dhage, Existence theory for quadratic perturbations of abstract measure integrodifferential equations, *Differ. Equ. Appl.,* **3** (2009), 307-323.

[10] B.C. Dhage, Some variants of two basic hybrid fixed point theorems of Krasnoselskii and Dhage with applications, *Nonlinear Studies,* **25(3)** (2018), 559-573.

[11] B.C. Dhage, On existence and uniqueness theorems for generalized nonlinear abstract measure integrodifferential equations, *Nonlinear Studies,* **27**(2) (2020), 337-355.

[12] B.C. Dhage, Local fixed point theory involving three operators in Banach algebras, *Topological Methods in Nonlinear Anal*., **24** (2004), 377-386.

[13] B.C. Dhage, D. N. Chate and S. K. Ntouyas, Abstract measure differential equations, *Dynamic Systems & Appl.,* **13** (2004), 105-108.

[14] B.C. Dhage, J.R. Graef, On stability of abstract measure delay integrodifferential equations, *Dynamic Systems and Appl.,* **19** (2010), 323-334.

[15] B.C. Dhage, N.S, Jadhav, Basic results in the theory of hybrid differential equations with linear perturbations of second type, Tamkang J. Math. **44**(2) (2013), 171-186.

[16] B.C. Dhage, P.R.M. Reddy, On existence and uniqueness theorems for abstract measure integrodifferential equtions, *Jñānābha*, **49(2)** (2019), 82-93.

[17] A. Granas and J. Dugundji, Fixed Point Theory, *Springer Verlag*, 2003.

[18] S. Hekkilä and V. Lakshikantham, Monotone Iterative Technique for Discontinuous Nonlinear Differential Equations, *Marcel Dekker Inc.,* New York 1994.

[19] S.R. Joshi, A system of abstract measure delay differential equations, *J. Math. Phy. Sci.,* **13** (1979), 497-506.

[20] M.A. Krasnoselskii, Topological Methods in the Theory of Nonlinear Integral Equations, *Pergamon Press,* New York, 1964.

[21] S. Leela, Stability of measure differential equations, *Pacific Jour. Math.,* **52(2)** (19740, 489-498.

[22] W. Rudin, *Real and Complex Analysis*, McGraw-hill Inc. New York 1966.

[23] R.R. Sharma, An abstract measure differential equation, *Proc. Amer. Math. Soc.,* **32** (1972), 503-510.

[24] R.R. Sharma, Existence of solution of abstract measure differential equation, *Proc. Amer. Math. Soc.,* **35(1)** (1972), 129-136.

[25] G.R. Shendge and S.R. Joshi, Abstract measure differential inequalities and applications, *Acta Math. Hung.,* **41** (1983), 53-54.