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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75th Birth Anniversary Celebrations)

HYPERGEOMETRIC FORMS OF SOME MATHEMATICAL FUNCTIONS VIA DIFFERENTIAL EQUATION APPROACH

By

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Abstract

In this paper, by changing the independent and dependent variables in the suitable ordinary differential equations of first and second order, and comparing the resulting ordinary differential equations with standard ordinary differential equations of Leibnitz and Gauss, we obtain the hypergeometric forms of following functions:

$$-\frac{4}{x}\,\ell n\left(\frac{1+\sqrt{1-x}}{2}\right),\ \tan^{-1}(x),\ \ \ell n(1+x),\ \ \sin(b\sin^{-1}x),\ \ \cos(b\sin^{-1}x).$$

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1 Introduction and Preliminaries

In our investigations, we shall use the following standard notations:

$\mathbb{N} :=\ \{1, 2, 3, \cdots\}\,; \mathbb{N}_0 := \mathbb{N}\cup\{0\}\,; \mathbb{Z}_0^- := \mathbb{Z}^-\cup\{0\} = \{0, -1, -2, -3, \cdots\}\,.$

The symbols $\mathbb{C}, \mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.

Pochhammer symbol

The Pochhammer symbol (or the *shifted* factorial) $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$[16, p.22 Eqn.(1), p.32 Q.N.(8) and Q.N.(9)], see also [18, p.23, Eqn.(22) and Eqn.(23)], is defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\backslash\{0\}),\\ \prod_{j=0}^{n-1}(\lambda+j) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}),\\ \frac{(-1)^k n!}{(n-k)!} & (\lambda = -n;\ \nu = k;\ n, k \in \mathbb{N}_0;\ 0 \leqq k \leqq n),\\ 0 & (\lambda = -n;\ \nu = k;\ n, k \in \mathbb{N}_0;\ k > n),\\ \frac{(-1)^k}{(1-\lambda)_k} & (\nu = -k;\ k \in \mathbb{N};\ \lambda \in \mathbb{C}\backslash\mathbb{Z}), \end{cases}$$

it being assumed tacitly that the Gamma quotient exists.

Generalized hypergeometric function of one variable

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha, \beta; \gamma; z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$$(1.1)\quad {}_pF_q\left[\begin{array}{c}(\alpha_p);\\(\beta_q);\end{array} z\right] = {}_pF_q\left[\begin{array}{c}\alpha_1, \alpha_2, \ldots, \alpha_p;\\ \beta_1, \beta_2, \ldots, \beta_q;\end{array} z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\frac{z^n}{n!},$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$ take on complex values, provided that

$\beta_j \neq 0, -1, -2, \ldots\ ;\ \ j = 1, 2, \ldots, q.$

Here, none of the denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by Eqn.(1.1):

(i) converges for $|z| < \infty$, if $p \leq q$,
(ii) converges for $|z| < 1$, if $p = q + 1$,
(iii) diverges for all z, $z \neq 0$, if $p > q + 1$,
(iv) converges absolutely for $|z| = 1$, if $p = q + 1$ and $\Re(\omega) > 0$,

(v) converges conditionally for $|z| = 1 (z \neq 1)$, if $p = q + 1$ and $-1 < \Re(\omega) \leqq 0$,

(vi) diverges for $|z| = 1$, if $p = q + 1$ and $\Re(\omega) \leqq -1$,

where by convention, a product over an empty set is interpreted as 1 and

(1.2) $\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j$,

$\Re(\omega)$ being the real part of complex number ω.

(I) When $x^2 = -t$ or $x = i\sqrt{t}$ where $i = \sqrt{-1}$, then

(1.3) $\dfrac{dx}{dt} = \dfrac{i}{2\sqrt{t}}$ or $\dfrac{dt}{dx} = -2i\sqrt{t}$,

(1.4) $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = -2i\sqrt{t}\dfrac{dy}{dt}$,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(-2i\sqrt{t}\frac{dy}{dt}\right)\frac{dt}{dx}$$

after simplification, we get

(1.5) $\dfrac{d^2y}{dx^2} = -4t\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt}$.

(II) When $y = i\sqrt{t}z$, where z is the function of t then

(1.6) $\dfrac{dy}{dt} = \dfrac{iz}{2\sqrt{t}} + i\sqrt{t}\dfrac{dz}{dt}$,

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{iz}{2\sqrt{t}} + i\sqrt{t}\frac{dz}{dt}\right)$$

after simplification, we get

(1.7) $\dfrac{d^2y}{dt^2} = i\left[\sqrt{t}\dfrac{d^2z}{dt^2} + \dfrac{1}{\sqrt{t}}\dfrac{dz}{dt} - \dfrac{z}{4t^{\frac{3}{2}}}\right]$.

(III) When $x = -t$, then

(1.8) $\dfrac{dt}{dx} = -1$,

(1.9) $\dfrac{dy}{dx} = \dfrac{dy}{dt} \times \dfrac{dt}{dx} = -\dfrac{dy}{dt}$,

(1.10) $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(-\dfrac{dy}{dt}\right)\dfrac{dt}{dx} = \dfrac{d^2y}{dt^2}$.

(IV) When $y = -tz$, where z is the function of t then

(1.11) $\dfrac{dy}{dt} = -z - t\dfrac{dz}{dt}$,

(1.12) $\dfrac{d^2y}{dt^2} = -2\dfrac{dz}{dt} - t\dfrac{d^2z}{dt^2}$.

(V) When $x^2 = t$ or $x = \sqrt{t}$, then

(1.13) $\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{t}}$ or $\dfrac{dt}{dx} = 2\sqrt{t}$,

(1.14) $\dfrac{dy}{dx} = 2\sqrt{t}\dfrac{dy}{dt}$,

(1.15) $\dfrac{d^2y}{dx^2} = 2\dfrac{dy}{dt} + 4t\dfrac{d^2y}{dt^2}$.

(VI) Gauss' ordinary differential equation [15, Ch.(6), pp.144-148 and pp.157-158]

When $\gamma \neq 0, \pm 1, \pm 2, \pm 3, \cdots$ and $|t| < 1$, then two linearly independent power series solutions of following Gauss'ordinary homogeneous linear differential equation of second order with variable coefficients

(1.16) $t(1-t)\dfrac{d^2z}{dt^2} + [\gamma - (1+\alpha+\beta)t]\dfrac{dz}{dt} - \alpha\beta z = 0$, are given by

(1.17) $z_1 = {}_2F_1\left[\begin{array}{c} \alpha, \beta; \\ \gamma; \end{array} t\right]$,

and

(1.18) $z_2 = t^{1-\gamma}\ {}_2F_1\left[\begin{array}{c} \alpha+1-\gamma, \beta+1-\gamma; \\ 2-\gamma; \end{array} t\right]$,

when γ is an integer then one solution may or may not, depending on the values of α and β, become logarithmic.

If any one solution of given differential equation is $y(x)$ then $Ay(x)$ will be the solution of same differential equation, where A is any suitable constant.

The present article is organized as follows:

In *Sections* 2, we have derived the hypergeometric forms of some mathematical functions by using differential equation approach. For hypergeometric forms of other mathematical functions and functions of mathematical physics, we refer the literature [1],[2],[3],[4],[5],[6],[7],[8],[9],[10],
[11],[12],[13],[14], [15], [17] and [19], where the proof of hypergeometric forms of related functions are not given. So we are interested to give the proof of hypergeometric forms of the functions mentioned in ***Section* 2**.

2 Hypergeometric forms of mathematical functions

Using the theory of ordinary differential equation and changing of independent and dependent variables in suitable differential equation, we can derive the following hypergeometric forms.

***Theorem* 2.1** *If* $|x| < 1$, *then following hypergeometric forms hold true:*

(2.1) $-\frac{4}{x}\,\ell n\left(\frac{1+\sqrt{1-x}}{2}\right) = {}_3F_2\left[\begin{array}{c} 1, 1, \frac{3}{2}; \\ 2, 2; \end{array} x\right]$,

(2.2) $\tan^{-1}(x) = x\,{}_2F_1\left[\begin{array}{c} 1, \frac{1}{2}; \\ \frac{3}{2}; \end{array} -x^2\right]$,

(2.3) $\ell n(1+x) = x\,{}_2F_1\left[\begin{array}{c} 1, 1; \\ 2; \end{array} -x\right]$,

(2.4) $\sin(b\sin^{-1}x) = bx\,{}_2F_1\left[\begin{array}{c} \frac{1+b}{2}, \frac{1-b}{2}; \\ \frac{3}{2}; \end{array} x^2\right]$,

(2.5) $\cos(b\sin^{-1}x) = {}_2F_1\left[\begin{array}{c} \frac{b}{2}, -\frac{b}{2}; \\ \frac{1}{2}; \end{array} x^2\right]$.

Proof of hypergeometric form of (2.1)

Consider,

(2.1,I) $y \equiv y(x) = -\frac{4}{x}\,\ell n\left(\frac{1+\sqrt{1-x}}{2}\right)$.

For $(0/0)$ indeterminate form, applying L'Hospital's rule in right hand side of the (2.1,I), the value of y at $x=0$, will be 1. That is

(2.1,II) $y(0) = 1$,

(2.1,III) $xy = -4\,\ell n\left(\frac{1+\sqrt{1-x}}{2}\right)$.

On differentiating result (2.1,III) w.r.t. x and applying product rule, after simplification we get

$$\frac{dy}{dx} + \frac{y}{x} = \frac{2}{x\{1+\sqrt{1-x}\}\sqrt{1-x}}$$
$$= \frac{2\{1-\sqrt{1-x}\}}{x\{1+\sqrt{1-x}\}\{1-\sqrt{1-x}\}\sqrt{1-x}}.$$

Therefore

(2.1,IV) $\frac{dy}{dx} + \frac{y}{x} = \frac{2}{x^2}\{(1-x)^{-\frac{1}{2}} - 1\}$.

The differential equation (2.1,IV) is written in the standard form of Leibnitz linear differential equation ($\frac{dy}{dx}+Py=Q$), therefore integrating factor for differential equation (2.1,IV) will be

(2.1,V) $I.F.=exp\left\{\int\frac{1}{x}dx\right\}=\exp\{\ell n(x)\}=x$.

The general solution of differential equation (2.1,IV) will be

(2.1,VI) $y=\dfrac{1}{x}\int x\left[\frac{2}{x^2}\left\{(1-x)^{-\frac{1}{2}}-1\right\}\right]dx+C,$

where C is the constant of integration.

Therefore

$$y=\frac{1}{x}\int\frac{2}{x}\left\{{}_1F_0\left[\begin{array}{c}\frac{1}{2};\\ -;\end{array}x\right]-1\right\}dx+C=\textstyle\sum_{r=0}^{\infty}\frac{\left(\frac{3}{2}\right)_r\,(1)_r\,(1)_r\,x^r}{(2)_r\,(2)_r\,r!}+C$$

(2.1,VII) $y(x)=1+\sum_{r=1}^{\infty}\frac{\left(\frac{3}{2}\right)_r\,(1)_r\,(1)_r\,x^r}{(2)_r\,(2)_r\,r!}+C,$

(2.1,VIII) $y(x)={}_3F_2\left[\begin{array}{c}1,\ 1,\ \frac{3}{2};\\ 2,\ 2;\end{array}x\right]+C.$

When $x=0$ in the equation(2.1,VII), we get $C=0$.

Therefore particular solution of the differential equation (2.1,IV) will be

(2.1,IX) $y={}_3F_2\left[\begin{array}{c}1,\ 1,\ \frac{3}{2};\\ 2,\ 2;\end{array}x\right],$

therefore

(2.1,X) $-\dfrac{4}{x}\,\ell n\left(\dfrac{1+\sqrt{1-x}}{2}\right)={}_3F_2\left[\begin{array}{c}1,\ 1,\ \frac{3}{2};\\ 2,\ 2;\end{array}x\right],$

which is satisfied by $x=0$.

Proof of Hypergeometric Form of (2.2)

Consider,

(2.2,I) $y\equiv y(x)=\tan^{-1}(x),$

(2.2,II) $y(0)=0.$

Differentiate the equation (2.2,I) w.r.t. x, we get

(2.2,III) $(1+x^2)\dfrac{dy}{dx}=1.$

Again differentiate the equation (2.2,III) w.r.t. x and use product rule, after simplification we have

(2.2,IV) $(1+x^2)\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}=0$.

Put $x^2=-t$ or $x=i\sqrt{t}$, then use values of equations (1.4) and (1.5) in above differential equation (2.2,IV), after simplification we get

(2.2,V) $t(1-t)\dfrac{d^2y}{dt^2}+\left\{\dfrac{1}{2}-\dfrac{3t}{2}\right\}\dfrac{dy}{dt}=0$.

Now substitute $y=i\sqrt{t}z$, where z is the function of t and put the values of equations (1.6) and (1.7) in above differential equation (2.2,V), after simplification we obtain

(2.2,VI) $t(1-t)\dfrac{d^2z}{dt^2}+\left\{\dfrac{3}{2}-\dfrac{5t}{2}\right\}\dfrac{dz}{dt}-\dfrac{1}{2}z=0$.

Now compare the coefficients of above differential equation (2.2,VI) with Gauss' standard differential equation (1.16), we get

$$\gamma=\frac{3}{2},\ \alpha+\beta+1=\frac{5}{2},\ \alpha\beta=\frac{1}{2}\ .$$

Now solve the above algebraic equations simultaneously, we get

$$\alpha=1,\ \beta=\frac{1}{2}\ .$$

Therefore one of the series solution of above differential equation (2.2,VI) will be

$$z={}_2F_1\left[\begin{array}{c}1,\ \frac{1}{2};\\ \frac{3}{2};\end{array}t\right],$$

$$y=i\sqrt{t}\,{}_2F_1\left[\begin{array}{c}1,\ \frac{1}{2};\\ \frac{3}{2};\end{array}t\right],$$

$$\tan^{-1}(x)=x\,{}_2F_1\left[\begin{array}{c}1,\ \frac{1}{2};\\ \frac{3}{2};\end{array}-x^2\right],$$

which is satisfied by $x=0$.

Proof of Hypergeometric Form of (2.3)

Consider,

(2.3,I) $y \equiv y(x) = \ell n(1+x)$,

(2.3,II) $y(0) = 0$.

Differentiate the equation (2.3,I) w.r.t. x, we get

(2.3,III) $(1+x)\dfrac{dy}{dx} = 1$.

Again differentiate the equation (2.3,III) w.r.t. x and use product rule, after simplification we have

(2.3,IV) $(1+x)\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 0$.

Put $x = -t$, then use values of equations (1.9) and (1.10) in above differential equation (2.3,IV), after simplification we get

(2.3,V) $(1-t)\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt} = 0$.

Now substitute $y = -tz$, where z is the function of t and put the values of equations (1.11) and (1.12) in above differential equation (2.3,V), after simplification we obtain

(2.3,VI) $t(1-t)\dfrac{d^2z}{dt^2} + \{2-3t\}\dfrac{dz}{dt} - z = 0$.

Now compare the coefficients of above differential equation (2.3,VI) with Gauss' standard differential equation (1.16), we get

$\gamma = 2,\ \alpha + \beta + 1 = 3,\ \alpha\beta = 1.$

Now solve the above algebraic equations simultaneously, we get

$\alpha = 1,\ \beta = 1$.

Therefore one of the series solution of above differential equation (2.3,VI) will be

$$z = {}_2F_1\left[\begin{array}{c} 1,\ 1; \\ 2; \end{array}\ t\right],$$

$$y = -t\,{}_2F_1\left[\begin{array}{c} 1,\ 1; \\ 2; \end{array}\ t\right],$$

$$\ell n(1+x) = x\,{}_2F_1\left[\begin{array}{c} 1,\ 1; \\ 2; \end{array}\ -x\right],$$

which is satisfied by $x = 0$.

Proof of Hypergeometric Form of (2.4)

Consider,

(2.4,I) $y \equiv y(x) = \sin(b\sin^{-1}x)$,

(2.4,II) $y(0) = 0$.

Differentiate the equation (2.4,I) w.r.t. x, we get

(2.4,III) $\sqrt{1-x^2}\,\dfrac{dy}{dx} = b\cos(b\sin^{-1}x)$.

Again differentiate the equation (2.4,III) w.r.t. x and use product rule, after simplification we have

(2.4,IV) $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + b^2y = 0$.

Put $x^2 = t$ or $x = \sqrt{t}$, then use values of equations (1.14) and (1.15) in above differential equation (2.4,IV), after simplification we get

(2.4,V) $t(1-t)\dfrac{d^2y}{dt^2} + \left\{\dfrac{1}{2} - t\right\}\dfrac{dy}{dt} + \dfrac{b^2}{4}y = 0$.

Now compare the coefficients of above differential equation (2.4,V) with Gauss' standard differential equation (1.16), we get

$$\gamma = \frac{1}{2},\ \alpha + \beta + 1 = 1,\ \alpha\beta = -\frac{b^2}{4}.$$

Now solve the above algebraic equations simultaneously, we get

$$\alpha = \frac{b}{2},\ \beta = -\frac{b}{2}\ .$$

Therefore one of the suitable power series solution of above differential equation (2.4,V) will be

$$y = t^{\frac{1}{2}}\ {}_2F_1\left[\begin{array}{c} \frac{1+b}{2},\ \frac{1-b}{2};\\ \frac{3}{2}; \end{array}\ t\right],$$

$$\sin(b\sin^{-1}x) = x\ {}_2F_1\left[\begin{array}{c} \frac{1+b}{2},\ \frac{1-b}{2};\\ \frac{3}{2}; \end{array}\ x^2\right].$$

More general solution will be

$$\sin(b\sin^{-1}x) = Ax\ {}_2F_1\left[\begin{array}{c} \frac{1+b}{2},\ \frac{1-b}{2};\\ \frac{3}{2}; \end{array}\ x^2\right],$$

$$\text{or}\quad \frac{\sin(b\sin^{-1}x)}{Ax} = {}_2F_1\left[\begin{array}{c} \frac{1+b}{2},\ \frac{1-b}{2};\\ \frac{3}{2}; \end{array}\ x^2\right],$$

(2.4,VI) $\frac{b}{A}\left(\frac{\sin^{-1}x}{x}\right) - \frac{b^3}{3!A}\left(\frac{\sin^{-1}x}{x}\right)(\sin^{-1}x)^2 + \frac{b^5}{5!A}\left(\frac{\sin^{-1}x}{x}\right)(\sin^{-1}x)^4 - \cdots = {}_2F_1\left[\begin{array}{c} \frac{1+b}{2},\ \frac{1-b}{2};\\ \frac{3}{2}; \end{array}\ x^2\right].$

Now taking $\lim_{x\to 0}$ in the equation (2.4,VI), we get $A = b$.

Therefore more general solution will be

$$\sin(b\sin^{-1}x) = bx\ {}_2F_1\left[\begin{array}{c} \frac{1+b}{2},\ \frac{1-b}{2};\\ \frac{3}{2}; \end{array}\ x^2\right],$$

which is satisfied by $x = 0$ or $b = 0$ or both $b, x = 0$.

Proof of Hypergeometric Form of (2.5)

Consider,

(2.5,I) $y \equiv y(x) = \cos(b\sin^{-1}x)$,

(2.5,II) $y(0) = 1$.

Differentiate the equation (2.5,I) w.r.t. x, we get

(2.5,III) $\sqrt{1-x^2}\,\frac{dy}{dx} = -b\sin(b\sin^{-1}x)$.

Again differentiate the equation (2.5,III) w.r.t. x and use product rule, after simplification we have

(2.5,IV) $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + b^2y = 0$.

Put $x^2 = t$ or $x = \sqrt{t}$, then use values of equations (1.14) and (1.15) in above differential equation (2.5,IV), after simplification we obtain

(2.5,V) $t(1-t)\frac{d^2y}{dt^2} + \left\{\frac{1}{2} - t\right\}\frac{dy}{dt} + \frac{b^2}{4}y = 0$.

Now compare the coefficients of above differential equation (2.5,V) with Gauss' standard differential equation (1.16), we get

$$\gamma = \frac{1}{2},\ \alpha + \beta + 1 = 1,\ \alpha\beta = -\frac{b^2}{4}.$$

Now solve the above algebraic equations simultaneously, we get

$$\alpha = \frac{b}{2},\ \beta = -\frac{b}{2}\ .$$

Therefore one of the series solution of above differential equation (2.5,V) will be

$$y = {}_2F_1\left[\begin{array}{c} \frac{b}{2},\ -\frac{b}{2};\\ \frac{1}{2}; \end{array}\ t\right],$$

$$\cos(b\sin^{-1}x) = {}_2F_1\left[\begin{array}{c} \frac{b}{2},\ -\frac{b}{2};\\ \frac{1}{2}; \end{array}\ x^2\right],$$

which is satisfied by $x = 0$ or $b = 0$ or both $b, x = 0$.

3 Conclusion

In our present investigation, we derived the hypergeometric forms of some functions by using differential equation approach. Moreover, the results derived in this paper are expected to have useful applications in wide range of problems of Mathematics, Statistics and Physical sciences. Similarly, we can derive the hypergeometric forms of other functions in an analogous manner.

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References

[1] M. Abramowitz and I.A. Stegun ; *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*,Reprint of the 1972 Edition, Dover Publications, Inc., New York, 1992.

[2] G.E. Andrews, R. Askey and R. Roy ; *Special Functions*, Cambridge University Press, Cambridge, UK, 1999.

[3] L.C. Andrews; *Special Functions for Engineers and Applied Mathematicians*, Macmillan Publishing Company, New York, 1985.

[4] L.C. Andrews; *Special Functions of Mathematics for Engineers*, Reprint of the 1992 Second Edition, SPIE Optical Engineering Press, Bellingham, W.A, Oxford University Press, Oxford, 1998.

[5] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi; *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York /Toronto / London, 1953.

[6] I.S. Gradshteyn and I.M. Ryzhik ; *Table of Integrals, Series and Products*, 5th Ed. Academic Press, New York, 1994.

[7] W. Magnus, F. Oberhettinger and R.P. Soni ; *Some Formulas and Theorems for the Special Functions of Mathematical Physics*, Third Enlarged Edition, Springer-Verlag, New York, 1966.

[8] A.P. Prudnikov, Yu. A. Brychkov and O.I. Marichev; *Integrals and Series*, Vol.3: More special functions, Nauka, Moscow, 1986 (in Russian); (Translated from the Russian by G. G. Gould), Gordon and Breach Science Publishers, New York, Philadelphia. London, Paris, Montreux, Tokyo, Melbourne, 1990.

[9] M.I. Qureshi and M.S. Baboo; Power series and hypergeometric representations with positive integral powers of arctangent function, *South Asian Journal of Mathematics*, **8**(1) (2018), 11-17.

[10] M.I. Qureshi and M.S. Baboo; Power series and hypergeometric representations associated with positive integral powers of logarithmic function, *South Asian Journal of Mathematics*, **8**(3) (2018), 144-150.

[11] M.I. Qureshi, I. H. Khan and M.P. Chaudhary; Hypergeometric forms of well known partial fraction expansions of some meromorphic functions, *Global Journal of Science Frontier Research*, **11**(6) (2011), 45-52.

[12] M.I. Qureshi, S.H. Malik and T.R. Shah; Hypergeometric representations of some mathematical functions via Maclaurin series, *Jñānābha*, **50**(1) (2020), 179-188.

[13] M.I. Qureshi, S.H. Malik and T.R. Shah; Hypergeometric forms of some functions involving arcsine(x) using differential equation approach, *South East Asian J. of Mathematics and Mathematical Sciences*, **16**(2) (2020), (To appear).

[14] M. I. Qureshi, S. Porwal, D. Ahamad and K. A. Quraishi; Successive differentiations of tangent, cotangent, secant, cosecant functions and related hyperbolic functions (A hypergeometric approach), *International Journal of Mathematics Trends and Technology* , **65**(7) (2019), 352-367.

[15] E. D. Rainville; *Intermediate Course in Differential Equations*, John Wiley and Sons, New York, 1943.

[16] E. D. Rainville; *Special Functions*, The Macmillan Co. Inc., New York 1960, Reprinted by Chelsea publ. Co., Bronx, New York, 1971.

[17] H. M. Srivastava and P. W. Karlsson; *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley, New York/ Chichester/ Brisbane/ Toronto, 1985.

[18] H. M. Srivastava and H. L. Manocha; *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester Brisbane, Toronto, 1984.

[19] G. Szegö; *Orthogonal Polynomials*, Amer. Math. Soc., Colloquium publ., New York, 1939.