***Jñānābha*, Vol. 50(2) (2020), 146-152**
(Dedicated to Honor Dr. R. C. Singh Chandel on His 75th Birth Anniversary Celebrations)

ESTIMATED SOLUTIONS OF GENERALIZED AND MULTIDIMENSIONAL CHURCHILL'S DIFFUSION PROBLEMS

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(*Received : July* 02, 2020 *; Revised: September* 17, 2020)
DOI: https://doi.org/10.58250/jnanabha.2020.50218

Abstract

In this paper to define a generalized Churchill's diffusion problem, we first extend the Churchill's diffusion problem. Then, we derive some of estimated and computational formulae of its solution. Further, we present a multidimensional Churchill's diffusion problem consisting of multidimensional Euler space derivatives and Caputo time fractional derivative. Then, on imposing certain boundary values, we obtain its solution and derive its many estimated formulae.
2010 Mathematics Subject Classifications: 26A33, 46A45, 35K58, 33E12.
Keywords and phrases: Multidimensional Euler space derivatives, Caputo time fractional derivative, Laplace transformation, a generalized Churchill's diffusion problem, a multidimensional Churchill's diffusion problem, estimation formulae.

1 Introduction

Very recently, Pathan and Kumar [17] proved the multivariable Cauchy residue theorem with the help of the Euler derivatives. On the other hand, Apostol [1] analyzed and discussed the theory of homogeneous functions in respect of Euler derivatives as if $(x_1, \dots, x_n) \in \mathbb{R}_+^n$, and $f(x_1, \dots, x_n)$ is a homogeneous function of degree k, then

(1.1) $\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} f(x_1, \dots, x_n) = k f(x_1, \dots, x_n)$,

while in 1928, for the economists, the favorite homogeneous function in the weighted geometric mean with domain $\mathbb{R}^+$, ($\mathbb{R}^+$ being the set of positive real numbers) has presented by the Cobb-Douglas function [6] as

(1.2) $f(x_1, ..., x_n) = x_1^{\alpha_1}.x_2^{\alpha_2}...x_n^{\alpha_n}$, where each $\alpha_i > 0, i = 1, \dots, n$.

It is homogeneous of degree $\alpha = \alpha_1 + ... + \alpha_n$.

In the theory of fractional calculus (see Diethelm [7, p. 148, Eqn. (7.12)]) following fractional differential equation, consisting of a system of equations, has been studied in the form

(1.3) ${}_t^C D_{0^+}^{\alpha} y(x) = \Lambda y(x) + q(x)$,

where, $0 < \alpha < 1$, an $N \times N$ matrix Λ, a given function $q : [0, h] \to \mathbb{C}^N, h > 0$, and the unknown solution $y : [0, h] \to \mathbb{C}^N$. $\mathbb{C}^N = \mathbb{C} \times \dots \times \mathbb{C}$ (N times).

For any suitable vector $u \in \mathbb{C}^N$,the solution of (1.3) is found as

(1.4) $y(x) = uE_{\alpha}(\lambda x^{\alpha})$,

where, $\lambda \in \mathbb{C}$, an eigenvalue of the matrix Λ, and the $E_{\alpha}(z)$, a Mittag - Leffler function, is defined by the series [15, p.80, Eqn. (2.1.1)] as

$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \Re(\alpha) > 0, z \in \mathbb{C}$.

In 2012, Kumar [9] considered following Churchill's diffusion problem [5]

(1.5) $x\dfrac{\partial U}{\partial x} + \dfrac{\partial U}{\partial t} + U = xF(t), x, t \in \mathbb{R}^+$,

with the boundary conditions $U(x, 0) = 0 = U(0, t)$ and they analyzed its solution

(1.6) $U(x, t) = x \int_0^t e^{-2v} F(t - v) dv, x, t \in \mathbb{R}^+$.

Again by [9], the solution (1.6) is converted into a general hypergeometric series solution to obtain various results for known special functions.

On the other hand, the theory and application of fractional differential equations in the diverse field, for example, in the dynamics of sphere immersed in an incompressible viscous fluid, oscillatory process with fractional damping, a study of the tension - deformation relationship of viscoelastic materials, and anomalous diffusion problems have been described in the literature by various researchers ([7], [8], [11], [14], [15], [16], [18] and others) consisting of the general fractional differential equations along with some operators or functions in the form of matrices. By the interacting multispecies, a system of equations in the matrix form also studied by Chandel and Kumar [4] in the ecosystem. Motivated by this work, in this paper we generalize the Churchill's diffusion equation in one and multidimensional space and then obtain their estimated solutions and computational results.

2 A Generalized Churchill type problem and its computational results

In this section on extension of the problem (1.5), we discuss a generalized Churchill type problem and then by its solution, we evaluate various results to compute the problem as

(2.1) $x\dfrac{\partial U(x,t)}{\partial x} + {}_t^C D_{0^+}^{\alpha} U(x,t) + U(x,t) = x \log_e F(t) = x\ln(F(t)), t > 0, x > 0, F(0) \geq 1,$

$0 < \alpha < 1$, with following boundary conditions

(2.2) $U(x, 0^+) = 0, \lim\limits_{x\to 0} U(x,t) = 0.$

In (1.3) and (2.1), the *Caputo derivative,* ${}_t^C D_{0^+}^{\alpha} f(t)$, of the function $f(t)$, whenever, $f^{(m)} \in L_1[a,b]$, $m-1 < \alpha \leq m$, $\forall m \in \mathbb{N}$, is defined by [7, p. 49]

(2.3) $({}_t^C D_{0^+}^{\alpha} f)(t) = (I_a^{m-\alpha} f^{(m)})(t),$

where, $f^{(m)}(t) = D_t^m f(t), \{D_t^m \equiv \frac{d^m}{dt^m} = \frac{d}{dt}(\frac{d^{m-1}}{dt^{m-1}})\}$.

In (2.3), the $(I_a^{m-q} f)(t), f \in L_1[a,b]$ is the Riemann - Liouville fractional integral [18], given by

(2.4) $(I_a^{m-q} f)(t) = \begin{cases} \frac{1}{\Gamma(m-q)} \int_a^t (t-\tau)^{m-q-1} f(\tau) d\tau, \ \forall t \in [a,b], m-1 < q < m, m \in \mathbb{N}, \\ f(t), q = m. \end{cases}$

Now, for the Laplace transform of a sufficiently well behaved function $v(t)$, denoted as $L\{V(t); s, s > 0\} = v(s)$, and again defined by $v(s) = \int_0^\infty e^{-st} V(t) dt$, then, the Laplace transformation of the Caputo operator (2.3), is presented by [7, p.134]

(2.5) $L\{({}_t^C D_{0^+}^{\alpha} V)(t); s\} = s^{\alpha} v(s) - \sum_{k=0}^{m-1} s^{\alpha-1-k} V^{(k)}(0^+) \ \forall m-1 < \alpha \leq m.$

Now, we solve the problem {(2.1) - (2.2)} in following manner:

***Theorem* 2.1** *Under the conditions* (2.2) *given as*

$U(x, 0^+) = 0, \lim\limits_{x\to 0} U(x,t) = 0,$

then, the solution of the generalized Churchill's diffusion equation (2.1) *exists and find in the form*

(2.6) $U(x,t) = x \int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-2(t-\tau)^{\alpha}) \ln(F(\tau)) d\tau, 0 < \alpha < 1, t > 0, x > 0.$

Proof. By (2.5) take Laplace transformation of the both sides of (2.1) as considering

$L\{U(x,t); s, s > 0\} = u(x,s), L\{\log(F(t)); s, s > 0\} = \bar{F}(s),$ for $0 < \alpha < 1$,

then, under the given condition $U(x, 0^+) = 0$, we find a linear differential equation as

(2.7) $\dfrac{du(x,s)}{dx} + \left\{\dfrac{s^{\alpha}+1}{x}\right\} u(x,s) = \bar{F}(s),$

with the initial condition $\lim_{x\to 0} u(x,s) = 0$.

The solution of Eqn. (2.7) is obtained as

(2.8) $u(x,s) = x^{-s^{\alpha}-1} \int x^{s^{\alpha}+1} \bar{F}(s) dx = x\left\{\frac{1}{s^{\alpha}+2}\right\} \bar{F}(s), x > 0.$

Taking inverse Laplace transform of both the sides of the Eqn. (2.8), and employing following formula due to Kilbas, Srivastava and Trujillo [8, p. 50, Eqn. (1.10.9)]

(2.9) $t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^{\alpha}) = L^{-1}\left\{\dfrac{1}{s^{\alpha}-\lambda}\right\}, \Re(s) > 0, \lambda \in \mathbb{C}, |\lambda s^{-\alpha}| < 1, t > 0,$

and thus applying the convolution theorem of Laplace transformation in (2.8) and making an appeal to (2.9) and (2.10), we derive the solution (2.6).

In the Eqn. (2.9), the $E_{\alpha,\beta}(z)$, a generalized Mittag - Leffler function, is defined by the series [15, p. 80, Eqn. (2.1.2)] as

(2.10) $E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}, \Re(\alpha) > 0, \Re(\beta) > 0, z, \alpha, \beta \in \mathbb{C}.$

***Example* 2.1** *Consider* $F(t) = exp\Big[\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}\Big]$, $\forall t > 0$, *in the problem* $\{(2.1)$ - $(2.2)\}$, *then, make an appeal to the* ***Theorem* 2.1**, *following result exists*

(2.11) $U(x,t) = xt^{\alpha} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n \frac{(-2t^{\alpha})^k}{\Gamma(\alpha k+\alpha+n+1)} t^n, 0 < \alpha < 1, t > 0, x > 0.$

Again then, the result (2.11) *gives various known results.*

Solution. In the solution (2.6) of the ***Theorem* 2.1**, choose $F(t) = \exp\Big[\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}\Big]$, we get the result (2.11).

Now in the result (2.11), choosing $a_n = \frac{\Gamma(\alpha k+\alpha+n+k+1)}{\Gamma(k+1)}$ $\forall k = 0, 1, 2, \ldots$, we obtain a known generating function

(2.12) $U(x,t) = xt^{\alpha} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{\alpha k+k+\alpha+n}{k} t^n, 0 < \alpha < 1, t > 0, x > 0.$

By an appeal to the formula $\sum_{k=0}^{\infty} \binom{\alpha k+k+\beta}{k} = \frac{(1+\zeta)^{\beta+1}}{1-\alpha\zeta}, \zeta(z) = z(1 + \zeta(z))^{\alpha+1}, \zeta(0) = 0$, due to Srivastava and Manocha [19, p. 234, Eqn. (47, for $x = 0$)], (see also, Carlitz [2]) in the result (2.12), we derive

(2.13) $U(x,t) = xt^{\alpha}\Big\{\frac{(1 + \zeta(t^*))^{\alpha+1}}{1 - \alpha\zeta(t^*)}\Big\}\Big[\frac{1}{1 - \{(1 + \zeta(t^*))t\}}\Big]$, provided that for $|(1 + \zeta(t^*))t| < 1$.

Here, $0 < \alpha < 1, t > 0, x > 0, \zeta(t^*) = t^*(1 + \zeta(t^*))^{\alpha+1}, t^* = (-2t^{\alpha})$.

Further by (2.6), we establish

$$U(x,t) = x \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(\alpha+n+1)} + x \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n+1)} \sum_{k=1}^{\infty} \frac{(-2)^k}{\Gamma(\alpha k+\alpha)} \int_0^t (t - \tau)^{\alpha+\alpha k-1} \tau^n d\tau.$$

Hence by (2.4), we get

(2.14) $U(x,t) = x \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(\alpha+n+1)} + x \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n+1)} H_n(\alpha; t).$

Here, $H_n(\alpha; t) = \sum_{k=1}^{\infty} (-2)^k I_0^{\alpha+\alpha k}(\tau^n)(t), 0 < \alpha < 1, t > 0, x > 0.$

It is noted that some other related generating functions are obtained on application of the results concerning Laguerre polynomials found in the literature by Chandel [3].

***Corollary* 2.1** *Set* $\alpha = 1, F(t) = e^{G(t)}$ $\forall t \geq 0$, *in Eqn.* (2.1) *along with the conditions* (2.2), *then, by the* ***Theorem* 2.1,** *the solution* (2.6) *becomes the solution of the Churchill's diffusion problem* (1.5) *and is found in the form*

(2.15) $U(x,t) = xe^{-2t} \int_0^t e^{2\tau} G(\tau) d\tau, t > 0, x > 0.$

***Theorem* 2.2** *If the solution* $U(x,t)$ *of the problem* $\{(2.1)$ - $(2.2)\}$, *given in* (2.6), *has separated in variables as*

(2.16) $U(x,t) = \varphi(x)\psi(t), x > 0, t > 0,$

then, there exists two curves in positive axis's

(2.17) $\varphi(x) = \eta x, \psi(t) = \dfrac{1}{\eta} \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-2(t - \tau)^{\alpha}) \ln(F(\tau)) d\tau,$

whenever, $x > 0, t > 0$; *and* $0 < \alpha < 1$, η *is constant and* $\eta \neq 0$.

Proof. In Eqn. (2.6), consider $U(x,t) = \varphi(x)\psi(t) \neq 0$, for $0 < \alpha < 1, x > 0, \eta \neq 0$, then, it may also be written by

(2.18) $\dfrac{\varphi(x)}{x} = \dfrac{1}{\psi(t)} \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-2(t - \tau)^{\alpha}) \ln(F(\tau)) d\tau = \eta.$

By the Eqn. (2.14), the parametric equations of the solution $U(x,t)$ *when*, $0 < \alpha < 1, t > 0, x > 0$,for $\eta \neq 0$, are found as

(2.19) $\varphi(x) = \eta x, \psi(t) = \dfrac{1}{\eta} \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha}(-2(t - \tau)^{\alpha}) \ln(F(\tau)) d\tau.$

Again, $\varphi(x)$ and $\psi(t)$are non-zero, when $\eta \neq 0$, and $\forall$ $t, x > 0$.

Hence, the ***Theorem* 2.2** is followed.

Now, we determine various estimation formulae of the solution of the generalized Churchill type diffusion problem for computational work:

***Theorem* 2.3** *If* $0 < \alpha < 1, \eta \neq 0$, *and* $|\ln(F(t))| \leq M, M > 0$,*then by the* ***Theorem* 2.2**, *for* $t > 0$ *there exists an estimation formula*

(2.20) $|U(x,t)| \leq Mxt^{\alpha} |E_{\alpha,\alpha+1}(-2t^{\alpha})|.$

Proof. Take $G(t) = \ln(F(t))$,then by ***Theorem* 2.2**, for $t > 0$ to get that

$\psi(t) = \frac{1}{\eta}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})G(\tau)d\tau$, and write it in the form

$$\psi(t) = \int_0^t \left\{\tfrac{1}{|\eta|}(t-\tau)^{\alpha-1}\right\}^{\frac{1}{2}}\left\{E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\right\}^{\frac{1}{2}}\left\{\tfrac{1}{|\eta|}(t-\tau)^{\alpha-1}\right\}^{\frac{1}{2}}\left\{E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\right\}^{\frac{1}{2}}G(\tau)d\tau.$$

Now, use Schwarz inequality for the integrals to find that

$$(2.21)\quad |\psi(t)| \leq \frac{1}{|\eta|}\Big[t^{\alpha}E_{\alpha,\alpha+1}(-2t^{\alpha})\int_0^t \tfrac{1}{\eta}(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})|G(\tau)|^2 d\tau\Big]^{\frac{1}{2}}.$$

An appeal to the ***Theorem* 2.3,** we have $|G(\tau)| \leq M$, then, in the integral apply the definition (2.10), again, use the ***Theorem* 2.2,** with $|U(x,t)| = |\varphi(x)||\psi(t)|$, finally, we obtain the estimation formula (2.20).

Theorem* 2.4** *If Laplace transformation of the function* $\{G(t)\}^2$ *is equal to* $P(s)$, $s > 0$, *that is* $L\{\{G(t)\}^2 : s\} = P(s)$, $s > 0$, *then by* ***Theorem* 2.3, there exists another inequality*

$$(2.22)\quad |U(x,t)| \leq x\Big[L^{-1}\Big\{\frac{P(s)}{s^{\alpha}+2} : t\Big\}\Big]^{\frac{1}{2}}\Big[L^{-1}\Big\{\frac{1}{s}\left(\frac{1}{s^{\alpha}+2}\right) : t\Big\}\Big]^{\frac{1}{2}}.$$

Proof. Since $|G(t)|^2 = \{G(t)\}^2$, so that by Eqn. (20) of ***Theorem* 2.1**, we write

$$(2.23)\quad |\psi(t)| \leq \frac{1}{|\eta|}\Big[\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})d\tau\Big]^{\frac{1}{2}}\Big[\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\{G(\tau)\}^2 d\tau\Big]^{\frac{1}{2}}.$$

Now, in Eqn. (2.19), use the formula due to Mathai and Haubold [15, p. 86, Eqn. (2.2.21)], with $L\{\{G(t)\}^2 : s\} = P(s)$, and $L\{1 : s\} = \frac{1}{s}$, $s > 0$, we get result (20).

Finally, the Eqn. (2.19), by an appeal to the ***Theorem* 2.2**, gives the result (2.18).

Theorem* 2.5** *If* $\log F(t) \in H_{\mu}[a,b]$, *where the Hölder space* $H_{\mu}[a,b] := \{\ln F(t) : [a,b] \in \mathbb{R}; \exists K > 0\ \forall (t,\tau) \in [a,b]; |\ln F(t) - \ln F(\tau)| \leq K|t-\tau|^{\mu}$ *for some* $\mu \in [0,1]\}$, *and* $0 < \alpha < 1$. *Then, by* ***Theorem* 2.2, there exists an inequality*

$$(2.24)\quad U(x,t) \leq \ln F(0)\{xt^{\alpha}E_{\alpha,\alpha+1}(-2t^{\alpha})\} + Kx\Gamma(\mu+1)O(t^{\alpha+\mu}E_{\alpha,\alpha+\mu+1}(-2t^{\alpha})).$$

Proof. Consider the Eqn. (2.1) and suppose that $F(0) \geq 1$, then by our assumption and the ***Theorem* 2.2**, we write

$$(2.25)\quad \psi(t) = \frac{1}{\eta}\ln F(0)\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})d\tau + \tfrac{1}{\eta}\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\{lnF(\tau) - \ln F(0)\}d\tau.$$

Since in ***Theorem* 2.5**, $\ln F(x) \in H_{\mu}[a,b]$, then by an appeal to Diethelm [7, p. 15], we get $\{lnF(\tau) - \ln F(0)\} = \ln\{\frac{F(t)}{F(0)}\} \leq Kt^{\mu}, \mu \in [0,1], K$ is a constant.

Now, making an appeal to the Eqns. (26) and (27), and the conditions of Hölder space, we obtain the inequality

$$(2.26)\quad \psi(t) \leq \frac{1}{\eta}logF(0)\int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})d\tau + \tfrac{K}{\eta}\int_0^t (t-\tau)^{\alpha-1}\tau^{\mu}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})d\tau.$$

Finally, an appeal to the Eqn. (2.25) and the ***Theorem* 2.2**, gives the result (2.22).

3 Churchill's Multidimensional diffusion problem and its solution

Here, in our investigation, we introduce a general equation consisting of multidimensional Euler space derivatives, Caputo time fractional derivative and the functions in trace of a given matrix as

$$(3.1)\quad x_1\frac{\partial U(x_1,\ldots,x_r,t)}{\partial x_1} + \ldots + x_r\frac{\partial U(x_1,\ldots,x_r,t)}{\partial x_r} + {}_t^C D_{0^+}^{\alpha}U(x_1,\ldots,x_r,t) + \textstyle\sum_{i=1}^{r}\beta_{ii}(U)$$
$$= \textstyle\prod_{i\neq j,i=1}^{r} V(x_i,t)\ln\prod_{j=1}^{r}(F_j(t))^{x_j}.$$

The Eqn. (3.1) can also be written as

$$(3.2)\quad \textstyle\sum_{i=1}^{r} x_i\frac{\partial U(x_1,\ldots,x_r,t)}{\partial x_i} + {}_t^C D_{0^+}^{\alpha}U(x_1,\ldots,x_r,t) + \sum_{i=1}^{r}\beta_{ii}(U) = \sum_{j=1}^{r} x_j\ln F_j(t)\prod_{i\neq j,i=1}^{r} V(x_i,t).$$

In the Eqns. (3.1) and (3.2), it is provided that $0 < \alpha < 1, V(x_i,t) \neq 0, x_i > a_i, t > 0$, the function $\log F_i : \mathbb{R}^+ \to \mathbb{R}$, such that $F_i(t) \geq 1; \beta_{ii}(U) = U(x_1,\ldots,x_r,t)\ \forall i = 1,2,3,\ldots,r$ and $\sum_{i=1}^{r}\beta_{ii}(U)$ is the Trace of square matrix $\begin{bmatrix} U & & \\ & \ddots & \\ & & U \end{bmatrix}_{r\times r}$. An empty product, when it occurs, is taken as one.

Recently, multidimensional time diffusion problems are solved as separating in space and time variables by the theory of various authors ([10], [12], [13], [14]). To solve the equation (3.1) or (3.2) by separation in variables, we impose following initial and boundary conditions:

***Theorem* 3.1** *If consider that* $U(x_1,\ldots,x_r,t)=\prod_{i=1}^{r}V(x_i,t)$, $(x_i,t)\neq 0\ \forall i=1,2,\ldots,r$ *, in the equation* (3.1) *or* (3.2)*, then, for the conditions* $U(0,x_2,\ldots,x_r,t)=\ldots=U(x_1,\ldots,x_{r-1},0,t)=0; U(x_1,\ldots,x_r,0^+)=0$*, there exists a set of problems*

$$(3.3)\quad \frac{\partial V(x_j,t)}{\partial x_j}+{}_t^C D_{0^+}^{\alpha}V(x_j,t)+V(x_j,t)=x_j\ln(F_j(t)),\ \forall j=1,2,3,\ldots.,r;$$

with the conditions given as $0<\alpha<1$, $V(0,t)=0=V(x_j,0^+)$, $\forall j=1,2,\ldots,r.$

Proof. Set $U(x_1,\ldots,x_r,t)=\prod_{i=1}^{r}V(x_i,t)$ in Eqn. (3.1) or (3.2), for $0<\alpha<1$, and then, on operating by the Caputo fractional time derivative, we find

$$(3.4)\quad \sum_{j=1}^{r}\prod_{i\neq j,i=1}^{r}V(x_i,t)x_j\frac{\partial V(x_j,t)}{\partial x_j}+\sum_{j=1}^{r}\prod_{i\neq j,i=1}^{r}V(x_i,t){}_t^C D_{0^+}^{\alpha}V(x_j,t)$$
$$+\sum_{j=1}^{r}\prod_{i\neq j,i=1}^{r}V(x_i,t)V(x_j,t)=\sum_{j=1}^{r}\prod_{i\neq j,i=1}^{r}V(x_i,t)x_j\ln(F_j(t)).$$

Equating both the sides of Eqn. (3.4), we obtain the set of equations (3.3).

Again, in the relation $U(x_1,\ldots,x_r,t)=\prod_{i=1}^{r}V(x_i,t)$, where, $V(x_i,t)\neq 0\ \forall i=1,2,\ldots,r$; set $U(0,x_2,\ldots,x_r,t)=\ldots=U(x_1,\ldots,x_{r-1},0,t)=0$; we get $V(x_i,t)\Big|_{x_i=0}=0\ \forall x_i=0, i=1,2,\ldots,r.$

By Lagrange's interpolation formula, we write

$$(3.5)\quad U(x_1,\ldots,x_r,t)\big|_{t=\tau_i,\tau_i\to 0^+}=\sum_{i=1}^{r}\left\{\frac{\prod_{j=1,j\neq i}^{r}(t-\tau_j)V(x_j,t)}{\prod_{j=1,j\neq i}^{r}(\tau_i-\tau_j)}\right\}V(x_i,t)\Big|_{t=\tau_i,\tau_i\to 0^+},$$
$$\forall i=1,2,3,\ldots,r.$$

Since, in our assumption $U(x_1,\ldots,x_r,0^+)=0$, and $V(x_i,t)\neq 0\ \forall i=1,2,\ldots,r$, thus by (3.5), we find

$$V(x_i,0^+)=0\ \forall i=1,2,\ldots,r.$$

Hence, the ***Theorem* 3.1** is proved.

***Theorem* 3.2** *The equation* (3.1) *or* (3.2) *under the given conditions* $U(0,x_2,\ldots,x_r,t)=\ldots=U(x_1,\ldots,x_{r-1},0,t)=0$; $U(x_1,\ldots,x_r,0^+)=0$, *has the solution*

$$(3.6)\quad U(x_1,\ldots,x_r,t)=\prod_{i=1}^{r}V(x_i,t)=\prod_{i=1}^{r}x_i\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\ln(F_i(\tau))d\tau,$$
$$\forall 0<\alpha<1, t>0, x_i>0, i=1,\ldots,r.$$

Proof. An appeal to the ***Theorem* 3.1** in the equation (3.1) or (3.2), we find r-equations

$$\frac{\partial V(x_j,t)}{\partial x_j}+{}_t^C D_{0^+}^{\alpha}V(x_j,t)+V(x_j,t)=x_j\ln(F_j(t)),\ \forall j=1,2,3,\ldots.,r;$$

with the conditions given by $0<\alpha<1$, $V(0,t)=0=V(x_j,0^+)$, $\forall j=1,2,\ldots,r$. Again, apply the ***Theorem* 2.1**, we find

$$V(x_j,t)=x_j\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\ln(F_j(\tau))d\tau,\ \forall j=1,2,\ldots,r.$$

Finally, use the concept and the theory of the ***Theorem* 3.1** we evaluate the solution (3.6).

***Theorem* 3.3** *For all* $0<\alpha<1, t>0, x_i>0, i=1,\ldots,r$*, if all conditions of the* ***Theorems* 3.1** *and* **3.2** *are satisfied, then, for any* $\eta'\neq 0$*, there exists* $U(x_1,\ldots,x_r,t)=\Phi(x_1,\ldots,x_r)\Psi(t)$ *such that*

$$(3.7)\quad \Phi(x_1,\ldots,x_r)=\eta' x_1\ldots x_r \text{ and } \Psi(t)=\frac{1}{\eta'}\prod_{i=1}^{r}\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\ln(F_i(\tau))d\tau.$$

Proof. An appeal to the ***Theorems* 3.1** and **3.2**, we find that

$$(3.8)\quad U(x_1,\ldots,x_r,t)=\Phi(x_1,\ldots,x_r)\Psi(t)=x_1\ldots x_r\prod_{i=1}^{r}\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\ln(F_i(\tau))d\tau.$$

By the relation (3.8), we bifurcate in separate variables as

$$(3.9)\quad \frac{\Phi(x_1,\ldots,x_r)}{x_1\ldots x_r}=\frac{\prod_{i=1}^{r}\int_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-2(t-\tau)^{\alpha})\ln(F_i(\tau))d\tau}{\Psi(t)}=\eta',\eta'\neq 0.$$

The relations (3.9) easily give the formulae (3.7).

***Theorem* 3.4** *For all* $0 < \alpha < 1, t > 0, x_i > 0, i = 1, \ldots, r$, *if all conditions of the* ***Theorems* 3.1** *and* **3.2** *are satisfied, then, for any* $\eta' \neq 0$, *and for* $|\ln(F_i(t))| \leq M_i', i = 1, 2, \ldots, r$, *then, by the* ***Theorems* 3.3,** *there exists an estimation formula via multidimensional Churchill's diffusion problem*

(3.10) $|U(x_1, \ldots, x_r, t)| \leq M_1' x_1 \ldots M_r' x_r \{t^\alpha |E_{\alpha,\alpha+1}(-2t^\alpha)|\}^r$.

Proof. Make an appeal to the ***Theorems* 3.1**, **3.2** and **3.3**, we find

(3.11) $|U(x_1, \ldots, x_r, t)| = |x_1 \ldots x_r| \prod_{i=1}^r |\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-2(t-\tau)^\alpha)| \ln(F_i(\tau))| d\tau|$.

Now, in (3.11) use the techniques of the ***Theorem* 2.3** , and for $|\ln(F_i(t))| \leq M_i', i = 1, 2, \ldots, r$, we obtain the estimation formula

(3.12) $|U(x_1, \ldots, x_r, t)| \leq |x_1 \ldots x_r| \prod_{i=1}^r M_i' \{t^\alpha |E_{\alpha,\alpha+1}(-2t^\alpha)|\}$.

The inequality (3.12) gives us the estimation formula (3.10).

***Example* 3.1** *Let in the* ***Theorem* 3.2** $F_i(t) = exp[y_i t^{\rho_i}], y_i \in \mathbb{R}^+, \rho_i \in \mathbb{N}_0 \ \forall i = 1, 2, \ldots, r$;

$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$; *then, there exists the solution of the multidimensional equation* (3.1) *or* (3.2) *with the conditions given in the* ***Theorem* 3.1**, *as*

(3.13) $U(x_1, \ldots, x_r, t) = t^\alpha \prod_{i=1}^r x_i y_i \, t^{\rho_i} \, \Gamma(\rho_i + 1) E_{\alpha,\alpha+\rho_i+1}(-2t^\alpha)$.

***Solution*.** In the ***Theorem* 3.2,** introduce $F_i(t) = \exp[y_i t^{\rho_i}], y_i \in \mathbb{R}^+, \rho_i \in \mathbb{N}_0 \ \forall i = 1, 2, \ldots, r$; where, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$; we get

$$(3.14) \quad U(x_1, \ldots, x_r, t) = \prod_{i=1}^r x_i y_i \sum_{k=0}^\infty \frac{(-2)^k}{\Gamma(\alpha k+\alpha)} \int_0^t (t-\tau)^{\alpha+\alpha k-1} \tau^{\rho_i} d\tau$$

$$= \prod_{i=1}^r x_i y_i \sum_{k=0}^\infty \frac{(-2)^k}{\Gamma(\alpha k+\alpha)} \int_0^t (t-\tau)^{\alpha+\alpha k-1} \tau^{\rho_i} d\tau.$$

By Eqn. (3.14), we easily derive the solution (3.13).

Conclusion

In 1972, Churchill studied diffusion problem and again in 2012, another form of the solution of this problem is obtained by Kumar [9] and then, converted into known and unknown hypergeometric functions. In this paper, in the ***Section* 2**, we generalize the Churchill's diffusion problem on introducing Caputo time fractional derivative and then obtain various estimation formulae and with known and unknown functions and generating relations. Again, we introduce a multidimensional time fractional diffusion problem to derive its solution on separating it in various diffusions problems. By the ***Theorem* 3.4**, at $t = x_1 \ldots x_r$, we find Cobb-Douglas [6] type functions given in (1.1).

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Jñānābha, Vol. 50(2) (2020), 153-159
(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

HYPERGEOMETRIC FORMS OF SOME MATHEMATICAL FUNCTIONS VIA DIFFERENTIAL EQUATION APPROACH

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(*Received : July* 02, 2020 *; Revised: August* 20, 2020)

Abstract

In this paper, by changing the independent and dependent variables in the suitable ordinary differential equations of first and second order, and comparing the resulting ordinary differential equations with standard ordinary differential equations of Leibnitz and Gauss, we obtain the hypergeometric forms of following functions:

$$-\frac{4}{x}\,\ell n\left(\frac{1+\sqrt{1-x}}{2}\right),\ \tan^{-1}(x),\ \ \ell n(1+x),\ \ \sin(b\sin^{-1}x),\ \ \cos(b\sin^{-1}x).$$

2010 Mathematics Subject Classifications: 33C20, 34-xx.
Keywords and phrases: Hypergeometric functions, Ordinary differential equation, Pochhammer symbol.

1 Introduction and Preliminaries

In our investigations, we shall use the following standard notations:

$\mathbb{N} :=\ \{1,2,3,\cdots\}\,;\mathbb{N}_0 := \mathbb{N}\cup\{0\}\,;\mathbb{Z}_0^- := \mathbb{Z}^-\cup\{0\} = \{0,-1,-2,-3,\cdots\}\,.$

The symbols $\mathbb{C},\mathbb{R},\mathbb{N},\mathbb{Z},\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.

Pochhammer symbol

The Pochhammer symbol (or the *shifted* factorial) $(\lambda)_\nu$ $(\lambda,\nu\in\mathbb{C})$[16, p.22 Eqn.(1), p.32 Q.N.(8) and Q.N.(9)], see also [18, p.23, Eqn.(22) and Eqn.(23)], is defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu=0;\ \lambda\in\mathbb{C}\backslash\{0\}),\\ \prod_{j=0}^{n-1}(\lambda+j) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}),\\ \frac{(-1)^k n!}{(n-k)!} & (\lambda=-n;\ \nu=k;\ n,k\in\mathbb{N}_0;\ 0\leqq k\leqq n),\\ 0 & (\lambda=-n;\ \nu=k;\ n,k\in\mathbb{N}_0;\ k>n),\\ \frac{(-1)^k}{(1-\lambda)_k} & (\nu=-k;\ k\in\mathbb{N};\ \lambda\in\mathbb{C}\backslash\mathbb{Z}),\end{cases}$$

it being assumed tacitly that the Gamma quotient exists.

Generalized hypergeometric function of one variable

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha,\beta;\gamma;z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$$(1.1)\quad {}_pF_q\left[\begin{array}{l}(\alpha_p);\\(\beta_q);\end{array} z\right] = {}_pF_q\left[\begin{array}{l}\alpha_1,\alpha_2,\ldots,\alpha_p;\\\beta_1,\beta_2,\ldots,\beta_q;\end{array} z\right] = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\ldots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\ldots(\beta_q)_n}\frac{z^n}{n!}\,,$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1,\alpha_2,\ldots,\alpha_p$ and the denominator parameters $\beta_1,\beta_2,\ldots,\beta_q$ take on complex values, provided that

$\beta_j\ \neq\ 0,-1,-2,\ldots\ ;\ \ j=1,2,\ldots,q.$

Here, none of the denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by Eqn.(1.1):

(i) converges for $|z|<\infty$, if $p\leq q$,
(ii) converges for $|z|<1$, if $p=q+1$,
(iii) diverges for all z, $z\neq 0$, if $p>q+1$,
(iv) converges absolutely for $|z|=1$, if $p=q+1$ and $\Re(\omega)>0$,