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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75^{th} Birth Anniversary Celebrations)

A QUADRUPLE FIXED POINT THEOREM FOR A MULTIMAP IN A HAUSDORFF FUZZY METRIC SPACE

By

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Abstract

Rao and Rao[16] obtained a triple fixed point theorem for a multimap in Hausdorff fuzzy metric space. Extending this idea we generalize the concept of triple fixed point, we define quadruple fixed point. In this paper we have established a result regarding it in Hausdorff fuzzy metric space.
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1 Introduction and Preliminaries

Zadeh [23] introduced the concept of fuzzy sets in 1965. Since then, it was developed extensively by many authors. Fuzzy metric spaces have been defined by several researchers in several ways (e.g.[6,7]). The concepts of fuzzy metric space introduced by Kramosil and Michlek [12] have been modified by George and Veeramani [7] and also induced a Hausdro topology on such fuzzy metric space. The contraction principle in the setting of fuzzy metric spaces introduced in [7] was later proved by Grabiec[9]. Some interesting references for fixed point theorems in fuzzy metric spaces are given in [3,4,5,21].

Nadler [14] initiated the study of fixed points for multivalued contraction mappings using the Hausdor metric. In 2004, Rodrguez-Lpez and Romaguera [17] introduced Hausdors fuzzy metric on the set of the nonempty compact subsets of a given fuzzy metric space. Later some fixed point theorems for multivalued maps in fuzzy metric spaces (e.g., [1,11,20,22]) were proved by several authors. Many authors studied the existence of fixed points for various multivalued contractive mappings under dierent conditions, refer to [12-14] and the references therein.

In 2006 coupled fixed point in partially ordered metric spaces was introduced by Gnana Bhaskar and Lakshmikantham [8] and some problems of the uniqueness of a coupled fixed point was discussed and the results were applied to periodic boundary value problems. In 2011, Samet and Vetro [18] proved the coupled fixed point theorems for a multivalued mapping. Berinde and Borcut [2] also introduced the concept of triple fixed points and obtained a triple fixed point theorem for a single valued map in partially ordered metric spaces.

In this paper, we obtain a quadruple fixed point theorem for a multimap in a Hausdor fuzzy metric space and using it, we obtain a common quadruple fixed point for a multi- and single valued maps.

For this we need the following.

***Definition* 1.1** [19] *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-norm if it satisfies the following conditions:*

1. $*$ *is associative and commutative,*
2. $*$ *is continuous,*
3. $a * 1 = a$ *for all* $a \epsilon [0,1]$,
4. $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *for each* $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norm are $a * b = ab$ *and* $a * b = \min\{a, b\}$.

***Definition* 1.2** [7] *A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (nonempty) set, $*$ is a continuous t-norm, and is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and each t and $s > 0$,*
1. $M(x, y, t) > 0$,
2. $M(x, y, t) = 1$ *if and only if* $x = y$,
3. $M(x, y, t) = M(y, x, t)$,
4. $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
5. $M(x, y,) : (0, \infty) \to [0, 1]$ *is continuous.*

Let $(X, M, *)$ be a fuzzy metric space. For > 0, the open ball $B(x, r, t)$ with centre $x \in X$ and radius $0 < r < 1$ is defined by $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$.

A subset $A \subset X$ is called open if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let τ denote the family of all open subsets of X. Then τ is called the topology on X induced by the fuzzy metric M. This topology is Hausdorff and first countable. A subset A of X is said to be F-bounded if there exist $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

***Lemma* 1.1** [9] *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, t)$ is nondecreasing with respect to t for all x, y in X.*

***Definition* 1.3** [17] *Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if*

(1.1) $\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t)$,

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$, that is, whenever

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = M(x, y, t) = 1,$$

(1.2) $\lim_{n\to\infty} (x, y, t_n) = M(x, y, t)$.

***Lemma* 1.2** [17] *Let $(X, M, *)$ be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0, \infty)$. Also let us take the condition:*

(1.3) $\lim_{t\to\infty} M(x, y, t) = 1, \ \forall\, x, y \in X$.

***Lemma* 1.3** [13] *Let $\{y_n\}$ be a sequence in fuzzy metric space $(X, M, *)$ satisfying condition (3). If there exists a positive number $k < 1$ such that*

(1.4) $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t), \quad t > 0, n = 1, 2....$

***Definition* 1.4** [17] *Let B be a nonempty subset of a fuzzy metric space $(X, M, *)$. For $a \in X$ and $t > 0$, define $M(a, B, t) = \sup\left\{\frac{a,b,t}{b} \in B\right\}$.*

In this paper let $K(X)$ denotes the class of all non empty compact subsets of X.

***Lemma* 1.4** [17] *Let $(X, M, *)$ be a fuzzy metric space. Then for each $a \in X$, $B \in K(X)$ and $t > 0$, there exists $b \in B$ such that $M(a, B, t) = M(a, b, t)$.*

***Definition* 1.5** [17] *Let $(X, M, *)$ be a fuzzy metric space. For each $A, B \in K(X)$ and $t > 0$, set*

(1.5) $H_M(A, B, t) = \min\left\{\inf_{x\in A} M(x, B, t),\ \inf_{y\in B} M(A, y, t)\right\}$.

The 3-tuple $\left(K(X), H_M, *\right)$ is called a Hausdorff fuzzy metric space.

***Lemma* 1.5** [10] *Let X be a nonempty set and $g : X \to X$ be a mapping. Then there exists a subset $E \subseteq X$ such that $g(E) = g(X)$ and $g : X \to X$ is one one.*

***Definition* 1.6** *Let X be a nonempty set, $T : X{\times}X{\times}X{\times}X \to 2^X$ (collection of all nonempty subset of X) and $f : X \to X$.*
(i) The point $(s, x, y, z) \in X \times X \times X \times X$ is called a quadruple fixed point of T if

(1.6) $s \in T(s, x, y, z)$,

$y \in T(y, z, s, x)$,

$z \in T(z, s, x, y)$.

(ii) The point $(s, x, y, z) \in X \times X \times X \times X$ is called a quadruple coincident point of T and f if

(1.7) $f_s \in T(s, x, y, z)$,

$f_x \in T(x, y, z, s)$,

$f_y \in T(y, z, s, x)$,

$f_z \in T(z, s, x, y)$.

(iii) The point $(s, x, y, z) \in X \times X \times X \times X$ is called a quadruple common fixed point of T and f if

(1.8) $s = f_s \in T(s, x, y, z)$,

$x = f_x \in T(x, y, z, s)$,

$y = f_y \in T(y, z, s, x)$,

$z = f_z \in T(z, s, x, y)$.

***Definition* 1.7** *Let $T : X \times X \times X \times X \to 2^X$ be a multivalued map and f be a self map on X. The Hybrid pair $\{T, f\}$ is called w-compatible if $f(T(s, x, y, z)) \subseteq T(fs, fx, fy, fz)$ whenever (s, x, y, z) is quadruple coincident point of T and f.*

2 Main Result

Let us prove a slightly different result from Lemma 1.3 which we will use to prove our main result.

***Lemma* 2.1** *Let $\{s_n\}, \{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequences in fuzzy metric space $(X, M, *)$ satisfying condition (1.3). If there exists a positive number $k < 1$ such that*

$$(2.1) \quad \min\left\{M(s_n, s_{n+1}, kt)(x_n, x_{n+1}, kt)(y_n, y_{n+1}, kt)(z_n, z_{n+1}, kt)\right\}$$
$$\geq \min\left\{M(s_{n-1}, s_n, t)(x_{n-1}, x_n, t)(y_{n-1}, y_n, t)(z_{n-1}, z_n, t)\right\},$$

for all $t > 0$, $n = 1, 2, ...,$ then $\{s_n\}, \{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences in X.

Proof. We have

$$(2.2) \quad \min\left\{M(s_n, s_{n+1}, kt), M(x_n, x_{n+1}, kt), M(y_n, y_{n+1}, kt), M(z_n, z_{n+1}, kt)\right\}$$
$$\geq \min\left\{M(s_{n-1}, s_n, t), M(x_{n-1}, x_n, t), M(y_{n-1}, y_n, t), M(z_{n-1}, z_n, t)\right\}$$
$$\geq \min\left\{M\left(s_{n-2}, s_{n-1}, \frac{t}{k^2}\right), M\left(x_{n-2}, x_{n-1}, \frac{t}{k^2}\right), M\left(y_{n-2}, y_{n-1}, \frac{t}{k^2}\right),\right.$$
$$\left.M\left(z_{n-2}, z_{n-1}, \frac{t}{k^2}\right)\right\}$$
$$\vdots$$
$$\geq \min\left\{M\left(s_0, s_1, \frac{t}{k^n}\right), M\left(x_0, x_1, \frac{t}{k^n}\right), M\left(y_0, y_1, \frac{t}{k^n}\right), M\left(z_0, z_1, \frac{t}{k^n}\right)\right\}$$

.

Hence,

$$(2.3) \quad M(s_n, s_{n+1}, t) \geq \min\left\{M\left(s_0, s_1, \frac{t}{k^n}\right), M\left(x_0, x_1, \frac{t}{k^n}\right), M\left(y_0, y_1, \frac{t}{k^n}\right), M\left(z_0, z_1, \frac{t}{k^n}\right)\right\}.$$

Now, for any positive integer p,

$$(2.4) \quad M\left(s_n, s_{n+p}, t\right) \geq M\left(s_n, s_{n+1}, \frac{t}{p}\right) * M\left(s_{n+1}, s_{n+2}, \frac{t}{p}\right) * ... * M\left(s_{n+p-1}, s_{n+p}, \frac{t}{p}\right)$$
$$\geq \min\left\{M\left(s_0, s_1, \frac{t}{pk^n}\right), M\left(x_0, x_1, \frac{t}{pk^n}\right), M\left(y_0, y_1, \frac{t}{pk^n}\right), M\left(z_0, z_1, \frac{t}{pk^n}\right)\right\}$$
$$* \min\left\{M\left(s_0, s_1, \frac{t}{pk^{n+1}}\right)\left(x_0, x_1, \frac{t}{pk^{n+1}}\right)\left(y_0, y_1, \frac{t}{pk^{n+1}}\right)\left(z_0, z_1, \frac{t}{pk^{n+1}}\right)\right\}$$
$$*...* \geq \min\left\{M\left(s_0, s_1, \frac{t}{pk^{n+p-1}}\right), M\left(x_0, x_1, \frac{t}{pk^{n+p-1}}\right), M\left(y_0, y_1, tpk^{n+p-1}\right), M(z_0, z_1, \frac{t}{pk^{n+p-1}})\right\}.$$

Letting $n \to \infty$ and using condition (3), we have

$$\lim_{n\to\infty} M\Big(x_n, x_{n+p}, t\Big) \geq 1 * 1 * ... * 1 = 1.$$

Hence,

$$\lim_{n\to\infty} M\Big(x_n, x_{n+p}, t\Big) = 1. \tag{2.5}$$

Thus $\{s_n\}$ is a Cauchy sequence in X. Similarly, we can show that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences in X.

Now, let us prove our first main result.

***Theorem* 2.1** *Let $(X, M, *)$ be a complete fuzzy metric space satisfying condition (1.3) and $F : X \times X \times X \times X \to K(X)$ be a set valued mapping satisfying*

$$H_M\Big(F(s, x, y, z), F(h, u, v, w), kt\Big) \geq \min\Big\{M(s, h, t), M(x, u, t)M(y, v, t), M(z, w, t)\Big\} \tag{2.6}$$

for each $s, x, y, z, h, u, v, w \in X$, $t > 0$, where $0 < k < 1$.

Then F has a quadruple fixed point.

Proof. Let $s_0, x_0, y_0, z_0 \in X$.

Choose $s_1 \in F(s_0, x_0, y_0, z_0), x_1 \in F(x_0, y_0, z_0, s_0), y_1 \in (y_0, z_0, s_0, x_0), z_1 \in F(z_0, s_0, x_0, y_0)$.

Since F is compact valued, by Lemma 1.4, there exists $s_2 \in F(s_1, x_1, y_1, z_1)$ such that

$$\begin{aligned} M(s_1, s_2, kt) &= \sup_{x\in F(s_1,x_1,y_1,z_1)} M(s_1, s, kt) \\ &\geq H_M\Big(F(s_0, x_0, y_0, z_0), F(s_1, x_1, y_1, z_1), kt\Big) \\ &\geq \min\Big\{M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t)\Big\}. \end{aligned} \tag{2.7}$$

Since F is compact valued, by Lemma 1.4, there exists $x_2 \in F(x_1, y_1, z_1, s_1)$ such that

$$\begin{aligned} M(x_1, x_2, kt) &= \sup_{x\in F(x_1,y_1,z_1,s_1)} M(x_1, x, kt) \\ &\geq H_M(F(x_0, y_0, z_0, s_0), F(x_1, y_1, z_1, s_1), kt) \\ &\geq \min\{M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t), M(s_0, s_1, t)\} \\ &\geq \min\{M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t)\}. \end{aligned} \tag{2.8}$$

Since F is compact valued, by Lemma 1.4, there exists $y_2 \in F(y_1, z_1, s_1, x_1)$ such that

$$\begin{aligned} M(y_1, y_2, kt) &= \sup_{x\in F(y_1,z_1,s_1,x_1)} M(y_1, y, kt) \\ &\geq H_M(F(y_0, z_0, s_0, x_0), F(y_1, z_1, s_1, x_1), kt) \\ &\geq \min\{M(y_0, y_1, t), M(z_0, z_1, t), M(s_0, s_1, t), M(x_0, x_1, t)\}. \\ &\geq \min\{M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t)\}. \end{aligned} \tag{2.9}$$

Since F is compact valued, by Lemma 1.4, there exists $z_2 \in F(z_1, s_1, x_1, y_1)$ such that

$$\begin{aligned} M(z_1, z_2, kt) &= \sup_{x\in F(z_1,s_1,x_1,y_1)} M(z_1, z, kt) \\ &\geq H_M(F(z_0, s_0, x_0, y_0), F(z_1, s_1, x_1, y_1), kt) \\ &\geq \min\{M(z_0, z_1, t), M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t)\} \\ &\geq \min\{M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t)\}. \end{aligned} \tag{2.10}$$

Thus,

$$\begin{aligned} &\min\{M(s_1, s_2, kt), M(x_1, x_2, kt), M(y_1, y_2, kt), M(z_1, z_2, kt)\} \\ &\quad\geq \min\{M(s_0, s_1, t), M(x_0, x_1, t), M(y_0, y_1, t), M(z_0, z_1, t)\}. \end{aligned} \tag{2.11}$$

Continuing in this way we can find the sequences $\{s_n\}, \{x_n\}, \{y_n\}$ and $\{z_n\}$ in X such that

$$s_{n+1} \in F(s_n, x_n, y_n, z_n), x_{n+1} \in F(x_n, y_n, z_n, s_n), y_{n+1} \in F(y_n, z_n, s_n, x_n),$$
$$z_{n+1} \in F(z_n, s_n, x_n, y_n)$$

Such that

(2.12) $\min\{M(s_n, s_{n+1}, kt), M(x_n, x_{n+1}, kt), M(y_n, y_{n+1}, kt), M(z_n, z_{n+1}, kt)\}$
$\geq \min\{M(s_{n-1}, s_n, t), M(x_{n-1}, x_n, t), M(y_{n-1}, y_n, t), M(z_{n-1}, z_n, t)\}$.

Hence, by Lemma 2.1, $\{s_n\}, \{x_n\}, \{y_n\}$ and $\{z_n\}$ are Cauchy sequences in X.

Since X is complete, there exists $s, x, y, z \in X$ such that $\lim_{n\to\infty}\{s_n\} = s, \lim_{n\to\infty}\{x_n\} = x, \lim_{n\to\infty}\{y_n\} = y, \lim_{n\to\infty}\{z_n\} = z$.

Consider

(2.13) $H_M(F(s_n, x_n, y_n, z_n), F(s, x, y, z), kt)$
$\geq \min\{M(s_n, s, t), M(x_n, x, t), M(y_n, y, t), M(z_n, z, t)\}$.

Let $n \to \infty$, we get

(2.14) $\lim_{n\to\infty} H_M(F(s_n, x_n, y_n, z_n), F(s, x, y, z), kt) = 1$ so that
$\lim_{n\to\infty} H_M(F(s_n, x_n, y_n, z_n), F(s, x, y, z), t) = 1$.

Similarly we can show that

(2.15) $\lim_{n\to\infty} H_M(F(x_n, y_n, z_n, s_n), F(x, y, z, s), t) = 1$,
$\lim_{n\to\infty} H_M(F(y_n, z_n, s_n, x_n), F(y, z, s, x), t) = 1$,
$\lim_{n\to\infty} H_M(F(z_n, s_n, x_n, y_n), F(z, s, x, y), t) = 1$.

Since
$s_{n+1} \in F(s_n, x_n, y_n, z_n), x_{n+1} \in F(x_n, y_n, z_n, s_n), y_{n+1} \in F(y_n, z_n, s_n, x_n)$,
$z_{n+1} \in F(z_n, s_n, x_n, y_n)$,

From (2.12) and (2.13), we have

(2.16) $\lim_{n\to\infty} \sup_{a\in F(s,x,y,z)} M(s_{n+1}, a, t) = 1$,
$\lim_{n\to\infty} \sup_{b\in F(x,y,z,s)} M(x_{n+1}, b, t) = 1$,
$\lim_{n\to\infty} \sup_{c\in F(y,z,s,x)} M(y_{n+1}, c, t) = 1$,
$\lim_{n\to\infty} \sup_{d\in F(z,s,x,y)} M(z_{n+1}, d, t) = 1$.

Hence there exist sequences $l_n \in F(s, x, y, z), p_n \in F(x, y, z, s), q_n \in F(y, z, s, x)$ and $r_n \in F(z, s, x, y)$ such that

(2.17) $\lim_{n\to\infty} M(s_{n+1}, l_n, t) = 1$,
$\lim_{n\to\infty} M(x_{n+1}, p_n, t) = 1$,
$\lim_{n\to\infty} M(y_{n+1}, q_n, t) = 1$,
$\lim_{n\to\infty} M(z_{n+1}, r_n, t) = 1$.

for each $t > 0$.

Now for each $n \in N$, we have

(2.18) $M(l_n, s, t) \geq M(l_n, s_{n+1}, t/2) * M(s_{n+1}, s, t/2)$.

Letting $n \to \infty$, we obtain

(2.19) $\lim_{n\to\infty} M(l_n, s, t) = 1$ so that $\lim_{n\to\infty} l_n = s$.

Similarly, we can show that

(2.20) $\lim_{n\to\infty} p_n = x, \quad \lim_{n\to\infty} q_n = y, \quad \lim_{n\to\infty} r_n = z$.

Since $F(s, x, y, z), F(x, y, z, s), F(y, z, s, x)$ and $F(z, s, x, y)$ are compact, we have
$s \in F(s, x, y, z), x \in F(x, y, z, s), y \in F(y, z, s, x)$ and $z \in F(z, s, x, y)$.

Thus (s, x, y, z) is a quadruple fixed point of F.

***Theorem* 2.2** *Let $(X, M, *)$ be a complete fuzzy metric space satisfying condition (1.3) and $F : X \times X \times X \times X \to K(X)$ and $gX \to X$ be a mappings satisfying*

$$(2.21)\quad H_M\Big(F(s,x,y,z), F(h,u,v,w), kt\Big) \geq \min\Big\{M(gs,gh,t), M(gx,gu,t), M(gy,gv,t), M(gz,gw,t)\Big\},$$

for all $s, x, y, z, h, u, v, w \in X$, $t > 0$ and $0 < k < 1$. further assume that $F(X \times X \times X \times X) \subseteq g(x)$, then F and g have a quadrupled coincidence point. Moreover, F and g have a quadrupled common fixed point if one of the following conditions holds.

(a) The pair F, g is called w-compatible and there exists $\mu, \alpha, \beta, \gamma \in X$ such that $\lim_{n\to\infty} g^n s = \mu \lim_{n\to\infty} g^n x = \alpha, \lim_{n\to\infty} g^n y = \beta, \lim_{n\to\infty} g^n z = \gamma$, whenever (s, x, y, z) is a quadrupled coincidence point of F and g and g is continuous at $\mu, \alpha, \beta, \gamma$.

(b) There exist $\mu, \alpha, \beta, \gamma \in X$ such that $\lim_{n\to\infty} g^n\mu = s, \lim_{n\to\infty} g^n\alpha = x, \lim_{n\to\infty} g^n\beta = y, \lim_{n\to\infty} g^n\gamma = z$, whenever (s, x, y, z) is a quadrupled coincidence point of F and g and g is continuous at s, x, y, z.

Proof. By ***Lemma* 1.5**, there exists $E \subseteq X$ such that $g : E \to X$ is one to one and $g(E) = g(X)$.

Now, define $G : g(E) \times g(E) \times g(E) \times g(E) \to K(X)$ by $G(gs, gx, gy, gz) = F(x, y, z)$ for all $gs, gx, gy, gz \in g(E)$. Since g is one-one one E, G is well defined.

Now,

$$(2.22)\quad H_M\Big(G(gs,gx,gy,gz), G(gh,gu,gv,gw), kt\Big) = H_M\Big(F(s,x,y,z), F(h,u,v,w), kt\Big) \geq \min\Big\{M(gs,gh,t), M(gx,gu,t), M(gy,gv,t), M(gz,gw,t)\Big\}.$$

Hence G satisfies (2.6) and all the conditions of ***Theorem* 2.1**.

By ***Theorem* 2.1**, G has a quadruple fixed point $(h, u, v, w) \in g(E) \times g(E) \times g(E) \times g(E)$. Thus,

$$(2.23)\quad \begin{aligned} &h \in (h,u,v,w),\\ &u \in (u,v,w,h),\\ &v \in (v,w,h,u),\\ &w \in (w,h,u,v). \end{aligned}$$

Since $F(X \times X \times X \times X) \subseteq g(x)$, there exist $h, u, v, w \in X \times X \times X \times X$ such that $gh_1 = h, gu_1 = u, gv_1 = v$ and $gw_1 = w$. so from (2.23) we have

$$\begin{aligned} gh_1 &\in G(gh_1, gu_1, gv_1, gw_1) = F(h_1, u_1, v_1, w_1)\\ gu_1 &\in G(gu_1, gv_1, gw_1, gh_1) = F(u_1, v_1, w_1, h_1)\\ gv_1 &\in G(gv_1, gw_1, gh_1, gu_1) = F(v_1, w_1, h_1, u_1)\\ gw_1 &\in G(gv_1, gw_1, gh_1, gu_1) = F(w_1, h_1, u_1, v_1). \end{aligned}$$

This implies that $h_1, u_1, v_1, w_1 \in X \times X \times X \times X$ is a quadruple fixed point of F and g.

Now following as in [15] except from the inequalities satisfied by M we can show that F and g have a quadruple fixed point.

3 Conclusion

Thus, our paper establishes the results regarding the quadruple fixed point in Hausdorff fuzzy metric space.

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