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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75th Birth Anniversary Celebrations)

UPPER BOUND ON FOURTH HANKEL DETERMINANT FOR CERTAIN SUBCLASS OF MULTIVALENT FUNCTIONS

By

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Abstract

The present investigation is concerned with the estimation of the upper bound to the $H_4(p)$ Hankel determinant for a subclass of p-valent functions in the open unit disc $E = \{z :| z |< 1\}$. This work will motivate the researchers to work in the direction of investigation of fourth Hankel determinant for several other subclasses of univalent and multivalent functions.

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1 Introduction

Let P denote the class of analytic functions $p(z)$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

whose real parts are positive in E.

By A_p, we denote the class of functions of the form

(1.1) $$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, (p \in N = \{1, 2, 3, ...\}),$$

which are analytic in the unit disc $E = \{z :| z |< 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$.

Let S be the class $A_1 \equiv A$ consisting of functions of the form (1.1) and which are univalent in E.

Let R represent the class of functions $f \in A$, which satisfy the condition

$$Re(f'(z)) > 0.$$

The class R was introduced by MacGregor [12] and functions in this class are called bounded turning functions.

By R_1, we denote the class of functions $f \in A$, with the condition that

$$Re\left(\frac{f(z)}{z}\right) > 0.$$

R_1 is a subclass of close-to-star functions and was studied by MacGregor [13].

Further, Murugusundramurthi and Magesh [15] introduced the following class:

$$R(\alpha) = \left\{f : f \in A, Re\left\{(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right\} > 0, 0 \le \alpha \le 1, z \in E\right\}.$$

In particular, $R(1) \equiv R$ and $R(0) \equiv R_1$.

Later on, Vamshee Krishna et al. [8] introduced a subclass of p-valent functions as follows:

$$RT_p = \left\{f : f \in A_p, Re\left(\frac{f'(z)}{pz^{p-1}}\right) > 0, z \in E\right\}.$$

For $p = 1$, $RT_1 \equiv R$.

Motivated by the above defined classes, Amourah et al. [2] defined the following subclass of p-valent functions:

$$R_p(\alpha) = \left\{f : f \in A_p, Re\left\{(1-\alpha)\frac{f(z)}{z^p} + \alpha\frac{f'(z)}{pz^{p-1}}\right\} > 0, 0 \le \alpha \le 1, z \in E\right\}.$$

The following observations are obvious:

(i) $R_1(\alpha) \equiv R(\alpha)$,
(ii) $R_p(1) \equiv RT_p$,
(iii) $R_1(1) \equiv R$,
(iv) $R_1(0) \equiv R_1$.

In 1976, Noonan and Thomas [16] stated the qth Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q+1} \\ a_{n+1} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n+q+1} & \dots & \dots & a_{n+2q-2} \end{vmatrix}.$$

In the particular cases, $q = 2, n = p, a_1 = 1$ and $q = 2, n = p + 1$, the Hankel determinant simplifies respectively to $H_2(p) = |a_{p+2} - a_{p+1}^2|$ and $H_2(p+1) = |a_{p+1}a_{p+3} - a_{p+2}^2|$.

This paper is concerned with the Hankel determinant in the case $q = 3$ and $n = p$ as

$$H_3(p) = \begin{vmatrix} a_p & a_{p+1} & a_{p+2} \\ a_{p+1} & a_{p+2} & a_{p+3} \\ a_{p+2} & a_{p+3} & a_{p+4} \end{vmatrix},$$

which is known as Hankel determinant of order 3.

For $f \in A_p$ and $a_p = 1$, we have

$$H_3(p) = a_{p+2}(a_{p+1}a_{p+3} - a_{p+2}^2) - a_{p+3}(a_{p+3} - a_{p+1}a_{p+2}) + a_{p+4}(a_{p+2} - a_{p+1}^2),$$

and using the triangle inequality, it yields

$$(1.2) \quad |H_3(p)| \leq |a_{p+2}||a_{p+1}a_{p+3} - a_{p+2}^2| + |a_{p+3}||a_{p+3} - a_{p+1}a_{p+2}| + |a_{p+4}||a_{p+2} - a_{p+1}^2|.$$

For any $f \in A_p$ of the form (1.1), we can represent the fourth Hankel determinant as

$$(1.3) \quad H_{4,p}(f) = a_{p+6}H_3(p) - a_{p+5}D_1 + a_{p+4}D_2 - a_{p+3}D_3,$$

where D_1, D_2 and D_3 are determinants of order 3 given by

$$(1.4) \quad D_1 = (a_{p+2}a_{p+5} - a_{p+3}a_{p+4}) - a_{p+1}(a_{p+1}a_{p+5} - a_{p+2}a_{p+4}) + a_{p+3}(a_{p+1}a_{p+3} - a_{p+2}^2),$$

$$(1.5) \quad D_2 = (a_{p+3}a_{p+5} - a_{p+4}^2) - a_{p+1}(a_{p+2}a_{p+5} - a_{p+3}a_{p+4}) + a_{p+2}(a_{p+2}a_{p+4} - a_{p+3}^2),$$

$$(1.6) \quad D_3 = a_{p+1}(a_{p+3}a_{p+5} - a_{p+4}^2) - a_{p+2}(a_{p+2}a_{p+5} - a_{p+3}a_{p+4}) + a_{p+3}(a_{p+2}a_{p+4} - a_{p+3}^2).$$

Hankel determinant has been considered by several authors. For example, Noor [17] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions given by Eq.(1.1) with bounded boundary. Ehrenborg [5] studied the Hankel determinant of exponential polynomials and in [10], the Hankel transform of an integer sequence is defined and some of its properties have been discussed by Layman.

Second Hankel determinant for various classes has been extensively studied by various authors including Mehrok and Singh [14], Janteng et al.[7] and many others. Third Hankel determinants for various classes were studied by some of the researchers including Babalola [3], Shanmugam et al.[18], Altinkaya and Yalcin [1] and Singh and Singh [19]. Also the Hankel determinant for various subclasses of p-valent functions were studied by various authors including Krishna and Ramreddy [8] and Hayami and Owa [6].

In this paper, we seek upper bound for the functional $H_{4,p}(f)$ for the functions belonging to the class $R_p(\alpha)$. This paper will motivate the future researchers to investigate the fourth Hankel determinant for some other subclasses of univalent and multivalent functions.

2 Preliminary results

***Lemma* 2.1**[4,11] *If* $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in P$, *then for* $n, k \in N = \{1, 2, 3, ...\}$, *we have the following inequalities*:

$$|c_{n+k} - \lambda c_n c_k| \leq 2, 0 \leq \lambda \leq 1,$$

and

$$|c_n| \leq 2.$$

***Lemma* 2.2** *If* $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in P$, *then for* $n, k \in N = \{1, 2, 3, ...\}$, *we have*:

$$|c_{n+k} - \lambda c_n c_k| \leq 4\lambda - 2, \lambda \geq 1.$$

Proof. For $\lambda \geq 1$, we have

$$|c_{n+k} - \lambda c_n c_k| \leq |c_n c_k - c_{n+k}| + (\lambda - 1)|c_n c_k|.$$

Using ***Lemma* 2.1**, the above inequality yields

$$|c_{n+k} - \lambda c_n c_k| \leq 4\lambda - 2.$$

***Lemma* 2.3**[2] *If $f \in R_p(\alpha)$, then*

$$|a_{p+j}| \leq \frac{2p}{p + j\alpha}.$$

***Lemma* 2.4**[2] *If $f \in R_p(\alpha)$, then*

$$|a_{p+2} - a_{p+1}^2| \leq \frac{2p}{p + 2\alpha}.$$

***Lemma* 2.5**[9] *If $f \in R_p(\alpha)$, then*

$$|a_{p+1}a_{p+3} - a_{p+2}^2| \leq \frac{4p^2}{(p + 2\alpha)^2}.$$

***Lemma* 2.6**[2] *If $f \in R_p(\alpha)$, then*

$$|a_{p+1}a_{p+2} - a_{p+3}| \leq \begin{cases} 2 & \text{if } \alpha = 0, \\ \dfrac{2p(6\alpha^2 + 3p\alpha + p^2)^{\frac{3}{2}}}{3\sqrt{6}\alpha(p + \alpha)(p + 2\alpha)(p + 3\alpha)} & \text{if } 0 < \alpha \leq 1. \end{cases}$$

***Lemma* 2.7** *If $f \in R_p(\alpha)$, then*

$$|H_3(p)| \leq \begin{cases} 16 & \text{for } \alpha = 0, \\ \dfrac{4p^2}{p + 2\alpha}\left[\dfrac{2p}{(p + 2\alpha)^2} + \dfrac{1}{p + 4\alpha} + \dfrac{(6\alpha^2 + 3p\alpha + p^2)^{\frac{3}{2}}}{3\sqrt{6}\alpha(p + \alpha)(p + 3\alpha)^2}\right] & \text{for } 0 < \alpha \leq 1. \end{cases}$$

Proof. From ***Lemma* 2.3**, we have

(2.1) $|a_{p+2}| \leq \dfrac{2p}{p + 2\alpha},$

(2.2) $|a_{p+3}| \leq \dfrac{2p}{p + 3\alpha},$

and

(2.3) $|a_{p+4}| \leq \dfrac{2p}{p + 4\alpha}.$

Using equations (2.1),(2.2) and (2.3), ***Lemma* 2.4**, ***Lemma* 2.5** and ***Lemma* 2.6** in (1.2), the result is obvious.
For $p = 1$, ***Lemma* 2.7** yields the following result:

***Corollary* 2.1** *If $f \in R(\alpha)$, then*

$$|H_3(1)| \leq \begin{cases} 16 & \text{for } \alpha = 0, \\ \dfrac{4}{1 + 2\alpha}\left[\dfrac{2}{(1 + 2\alpha)^2} + \dfrac{1}{1 + 4\alpha} + \dfrac{(6\alpha^2 + 3\alpha + 1)^{3/2}}{3\sqrt{6}\alpha(1 + \alpha)(1 + 3\alpha)^2}\right] & \text{for } 0 < \alpha \leq 1. \end{cases}$$

For $p = 1, \alpha = 1$, ***Lemma* 2.7** gives the following result proved by Babalola [3]:

***Corollary* 2.2** *If $f \in R$, then*

$$|H_3(1)| \leq 0.7423.$$

3 Fourth Hankel determinant for the class $R_p(\alpha)$

***Theorem* 3.1** *If $f \in R_p(\alpha)$, then*

$$(3.1)\quad |H_4(p)| \leq \begin{cases} 152.0866 & \text{for } \alpha = 0, \\ \dfrac{8p^3}{(p + 2\alpha)(p + 6\alpha)}\left[\dfrac{2p}{(p + 2\alpha)^2} + \dfrac{1}{p + 4\alpha} + \dfrac{(6\alpha^2 + 3p\alpha + p^2)^{3/2}}{3\sqrt{6}\alpha(p + \alpha)(p + 3\alpha)^2}\right] & \\ + \dfrac{2p}{(p + 5\alpha)}u(p, \alpha) + \dfrac{2p}{(p + 4\alpha)}v(p, \alpha) + \dfrac{2p}{(p + 3\alpha)}w(p, \alpha) & \text{for } 0 < \alpha \leq 1, \end{cases}$$

where

$$(3.2)\quad u(p,\alpha) = 2p^2(4p-2)\left[\frac{1}{(p+\alpha)^2(p+5\alpha)} + \frac{1}{(p+3\alpha)(p+2\alpha)^2} + \frac{1}{(p+\alpha)(p+3\alpha)^2}\right] + \frac{174p^2(4p-2)+4p^2}{48(p+\alpha)(p+2\alpha)(p+4\alpha)},$$

$$(3.3)\quad v(p,\alpha) = \left[\frac{63p^2(4p-2)}{25(p+\alpha)(p+2\alpha)(p+5\alpha)} + \frac{18p^2(4p-2)}{5(p+4\alpha)(p+2\alpha)^2} + \frac{150p^2(4p-2)+4p^2}{75(p+2\alpha)(p+3\alpha)^2}\right]$$

and

$$(3.4)\quad w(p,\alpha) = 2p^2(4p-2) \times\left[\frac{1}{(p+2\alpha)^2(p+5\alpha)} + \frac{1}{(p+\alpha)(p+3\alpha)(p+5\alpha)} + \frac{2}{(p+3\alpha)^3} + \frac{1}{(p+\alpha)(p+4\alpha)^2}\right] + \frac{34p^2(4p-2)}{16(p+2\alpha)(p+3\alpha)(p+4\alpha)} + \frac{p^2}{(p+\alpha)(p+2\alpha)^2(p+3\alpha)(p+4\alpha)^2(p+5\alpha)}.$$

Proof. Using ***Lemma* 2.3** in (1.4), (1.5) and (1.6), it gives

$$(3.5)\quad D_1 = \frac{p^2c_2c_5}{(p+2\alpha)(p+5\alpha)} - \frac{p^2c_3c_4}{(p+3\alpha)(p+4\alpha)} - \frac{p^3c_1^2c_5}{(p+\alpha)^2(p+5\alpha)} + \frac{p^3c_1c_2c_4}{(p+\alpha)(p+2\alpha)(p+4\alpha)} + \frac{p^3c_1c_3^2}{(p+\alpha)(p+3\alpha)^2} - \frac{p^3c_3c_2^2}{(p+3\alpha)(p+2\alpha)^2},$$

$$(3.6)\quad D_2 = \frac{p^2c_3c_5}{(p+3\alpha)(p+5\alpha)} - \frac{p^2c_4^2}{(p+4\alpha)^2} - \frac{p^3c_1c_2c_5}{(p+\alpha)(p+2\alpha)(p+5\alpha)} + \frac{p^3c_1c_3c_4}{(p+\alpha)(p+3\alpha)(p+4\alpha)} + \frac{p^3c_4c_2^2}{(p+2\alpha)^2(p+4\alpha)} - \frac{p^3c_2c_3^2}{(p+2\alpha)(p+3\alpha)^2}$$

and

$$(3.7)\quad D_3 = \frac{p^3c_1c_3c_5}{(p+\alpha)(p+3\alpha)(p+5\alpha)} - \frac{p^3c_1c_4^2}{(p+\alpha)(p+4\alpha)^2} - \frac{p^3c_2^2c_5}{(p+2\alpha)^2(p+5\alpha)} + \frac{2p^3c_2c_3c_4}{(p+2\alpha)(p+3\alpha)(p+4\alpha)} - \frac{p^3c_3^3}{(p+3\alpha)^3}.$$

On rearranging the terms in (3.5), (3.6) and (3.7), it yields

$$(3.8)\quad D_1 = \frac{p^2c_5(c_2-pc_1^2)}{(p+\alpha)^2(p+5\alpha)} + \frac{p^2c_3(c_4-pc_2^2)}{(p+3\alpha)(p+2\alpha)^2} - \frac{p^2c_3(c_4-pc_1c_3)}{(p+\alpha)(p+3\alpha)^2} - \frac{67p^2c_4(c_3-pc_1c_2)}{48(p+\alpha)(p+2\alpha)(p+4\alpha)} + \frac{19p^2c_2(c_5-pc_1c_4)}{48(p+\alpha)(p+2\alpha)(p+4\alpha)} + \frac{p^2c_2c_5}{48(p+\alpha)(p+2\alpha)(p+4\alpha)},$$

$$(3.9)\quad D_2 = \frac{p^2c_5(c_3-pc_1c_2)}{(p+\alpha)(p+2\alpha)(p+5\alpha)} - \frac{p^2c_4(c_4-pc_2^2)}{(p+4\alpha)(p+2\alpha)^2} - \frac{p^2c_3(c_5-pc_2c_3)}{(p+2\alpha)(p+3\alpha)^2} - \frac{4p^2c_4(c_4-pc_1c_3)}{5(p+4\alpha)(p+2\alpha)^2} - \frac{13p^2c_3(c_5-pc_1c_4)}{50(p+\alpha)(p+2\alpha)(p+5\alpha)} + \frac{p^2c_3c_5}{75(p+2\alpha)(p+3\alpha)^2}$$

and

$$(3.10)\quad D_3 = \frac{p^2c_5(c_4-pc_2^2)}{(p+2\alpha)^2(p+5\alpha)} - \frac{p^2c_5(c_4-pc_1c_3)}{(p+\alpha)(p+3\alpha)(p+5\alpha)} + \frac{p^2c_3(c_6-pc_3^2)}{(p+3\alpha)^3} - \frac{p^2c_3(c_6-pc_2c_4)}{(p+3\alpha)^3} + \frac{p^2c_4(c_5-pc_1c_4)}{(p+\alpha)(p+4\alpha)^2} - \frac{17p^2c_4(c_5-pc_2c_3)}{16(p+2\alpha)(p+3\alpha)(p+4\alpha)} + \frac{p^2c_4c_5}{4(p+\alpha)(p+2\alpha)^2(p+3\alpha)(p+4\alpha)^2(p+5\alpha)}.$$

Using ***Lemma* 2.2** and applying triangle inequality in (3.8), (3.9) and (3.10), we obtain

$$(3.11)\quad |D_1| \le u(p,\alpha),$$

$$(3.12)\quad |D_2| \le v(p,\alpha)$$

and

(3.13) $|D_3| \leq w(p,\alpha)$,

where $u(p,\alpha), v(p,\alpha)$ and $w(p,\alpha)$ are defined in (3.2), (3.3) and (3.4) respectively.
Hence using ***Lemma* 2.3**, ***Lemma* 2.7** and equations (3.11), (3.12), (3.13) in equation (1.3) and applying triangle inequality, the result (3.1) is obvious.
On putting $p = 1$ in ***Theorem* 3.1**, we obtain the following result:
***Corollary* 3.1** *If $f \in R(\alpha)$, then*

$$|H_4(1)| \leq \begin{cases} 152.0866 & \text{for } \alpha = 0, \\ \dfrac{8}{(1+2\alpha)(1+6\alpha)}\left[\dfrac{2}{(1+2\alpha)^2} + \dfrac{1}{1+4\alpha} + \dfrac{(6\alpha^2+3\alpha+1)^{\frac{3}{2}}}{3\sqrt{6}\alpha(1+\alpha)(1+3\alpha)^2}\right] & \\ +\dfrac{2}{(1+5\alpha)}p(\alpha) + \dfrac{2}{(1+4\alpha)}q(\alpha) + \dfrac{2}{(1+3\alpha)}r(\alpha) & \text{for } 0 < \alpha \leq 1, \end{cases}$$

where

$$p(\alpha) = 4\left[\frac{1}{(1+\alpha)^2(1+5\alpha)} + \frac{1}{(1+3\alpha)(1+2\alpha)^2} + \frac{1}{(1+\alpha)(1+3\alpha)^2}\right] + \frac{29}{4(1+\alpha)(1+2\alpha)(1+4\alpha)},$$

$$q(\alpha) = 4\left[\frac{63}{50(1+\alpha)(1+2\alpha)(1+5\alpha)} + \frac{9}{5(1+4\alpha)(1+2\alpha)^2} + \frac{76}{75(1+2\alpha)(1+3\alpha)^2}\right]$$

and

$$r(\alpha) = 4\left[\frac{1}{(1+2\alpha)^2(1+5\alpha)} + \frac{1}{(1+\alpha)(1+3\alpha)(1+5\alpha)} + \frac{2}{(1+3\alpha)^3} + \frac{1}{(1+\alpha)(1+4\alpha)^2}\right] + \frac{68}{16(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{1}{(1+\alpha)(1+2\alpha)^2(1+3\alpha)(1+4\alpha)^2(1+5\alpha)}.$$

On putting $p = 1, \alpha = 1$ in ***Theorem* 3.1**, the following result is obvious:
***Corollary* 3.2** *If $f \in R$, then*

$$|H_{4,1}(f)| \leq 0.7973.$$

4 Conclusion.

In the present work, we estimated the bounds for the fourth Hankel determinant for a subclass of multivalent bounded turning functions. The estimation of fourth Hankel determinant for the various subclasses of analytic functions is a new concept in the field of geometric function theory. Till now much work has been done on the study of second and third Hankel determinants for various subclasses of univalent functions, so this paper will work as a milestone to the future researchers in this field.

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