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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75th Birth Anniversary Celebrations)

COMMON FIXED POINT THEOREMS IN BICOMPLEX VALUED b-METRIC SPACES FOR RATIONAL CONTRACTIONS

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Abstract

In this paper we prove some common fixed point theorems in a complete bicomplex valued b-metric spaces for rational contractions.

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1 Introduction, Definitions and Notations.

Segre's [41] paper, published in 1892 made a pioneering attempt in the development of special algebras. He conceptualized commutative generalization of complex numbers as bicomplex numbers, tricomplex numbers, etc. as elements of an infinite set of algebras. Unfortunately this significant work of Segre failed to earn the attention of the mathematicians for almost a century. However, recently a renewed interest in this subject contributes a lot in the different fields of mathematical sciences and other branches of science and technology.

Price [36] developed the bicomplex algebra and function theory. In this field an impressive body of work has been developed by different researchers during the last few years. One can see some of the attempts in (cf.[3]-[6], [15], [16], [24]-[33], [39], [40], [42], [43]). Azam et al. [1] introduced a concept of complex valued metric space and established a common fixed point theorem for a pair of self contracting mappings. Rouzkard & Imdad [37] generalized the result obtained by Azam et al. [1] and proved another common fixed point theorem satisfying some rational inequality in complex valued metric space. The Banach contraction principle (cf. [12]) is a very popular and effective tool to solve the existence problems in many branches of mathematical analysis and it is an active area of research since 1922. The famous Banach theorem (cf. [12]) states that "Let (X, d) be a metric space and T be a mapping of X into itself satisfying $d(Tx, Ty) \leq kd(x, y), \forall x, y \in X$, where k is a constant in $(0, 1)$. Then T has a unique fixed point $x^* \in X$". In this connection Choudhury et al. ([13]&[14]) proved some fixed point results in partially ordered complex valued metric spaces for rational type expressions. Datta & Ali [7] proved common fixed point theorems for four mappings in complex valued metric space. Also one can see the attempts in (cf. [2], [8], [9], [44], [46], 47).

The concept of complex-valued b-metric spaces introduced by Rao et al.[38] proved a common fixed point theorem in complex valued b-metric spaces. Mukheimer [34] proved some common fixed point theorems in complex-valued b-metric spaces. Also Dubey et al.[18] proved some common fixed point theorems for contractive mappings in complex-valued b-metric spaces and Singh et al.[45] common fixed point theorems in complex-valued b-metric spaces. In this connection Mitra[35] proved a common fixed point theorem in complex valued b-metric spaces and Kumar et al.[19] proved common fixed point theorem in complex valued b-metric space for rational contraction. We write the set of real, complex and bicomplex numbers respectively as $\mathbb{C}_0, \mathbb{C}_1$ and $\mathbb{C}_2$. In this paper we are going to prove some common fixed point theorem in bicomplex valued b-metric space for rational contraction.

Let $z_1, z_2 \in \mathbb{C}_1$ be any two complex numbers, then the partial order relation $\precsim$ on $\mathbb{C}_1$ is defined as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$,

i.e., $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(1) $Re(z_1) = Re(z_2),\ Im(z_1) = Im(z_2)$,

(2) $Re(z_1) < Re(z_2),\ Im(z_1) = Im(z_2)$,

(3) $Re(z_1) = Re(z_2),\ Im(z_1) < Im(z_2)$ and

(4) $Re(z_1) < Re(z_2),\ Im(z_1) < Im(z_2)$.

In particular, we can say $z_1 \lnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e. one of (2), (3) and (4) is satisfied and $z_1 \prec z_2$ if only (4) is satisfied. We can easily check the following fundamental properties of partial order relation $\precsim$ on $\mathbb{C}_1$:

1. If $0 \precsim z_1 \lnsim z_2$, then $|z_1| < |z_2|$,
2. If $z_1 \precsim z_2, z_2 \prec z_3$then $z_1 \prec z_3$ and
3. If$z_1 \precsim z_2$ and $\lambda > 0$ is a real number then $\lambda z_1 \precsim \lambda z_2$.

1.1 Complex valued metric space.

Azam et al.[1] defined the complex valued metric spaces as

***Definition* 1.1** *Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{C}_1$ satisfies the following conditions:*

1. $0 \precsim d(x, y)$ *for all* $x, y \in X$,
2. $d(x, y) = 0$ *if and only if* $x = y$,
3. $d(x, y) = d(y, x)$ *for all* $x, y \in X$ *and*
4. $d(x, y) \precsim d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called the complex valued metric space.

***Definition* 1.2** *Let X be a nonempty set and let $s \geq 1$. Suppose the mapping $d : X \times X \to \mathbb{C}_1$ satisfies the following conditions:*

1. $0 \precsim d(x, y)$ *for all* $x, y \in X$,
2. $d(x, y) = 0$ *if and only if* $x = y$,
3. $d(x, y) = d(y, x)$ *for all* $x, y \in X$ *and*
4. $d(x, y) \precsim s\left[d(x, z) + d(z, y)\right]$ *for all* $x, y, z \in X$.

Then d is called a complex valued b-metric on X and (X, d) is called the complex valued b-metric space.

1.2 Bicomplex Number.

Segre [41] defined the bicomplex number as:

$$\xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2,$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}_0$ and the independent units i_1, i_2 are such that $i_1^2 = i_2^2 = -1$ and $i_1 i_2 = i_2 i_1$. We denote $i_1 i_2 = j$, which is known as the hyperbolic unit and such that $j^2 = 1$, $i_1 j = j i_1 = -i_2$, $i_2 j = j i_2 = -i_1$. Also $\mathbb{C}_2$ is defined as:

$$\mathbb{C}_2 = \{\xi : \xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2, a_1, a_2, a_3, a_4 \in \mathbb{C}_0\}$$

i.e.,

$$\mathbb{C}_2 = \{\xi : \xi = z_1 + i_2 z_2, z_1, z_2 \in \mathbb{C}_1\},$$

where $z_1 = a_1 + a_2 i_1 \in \mathbb{C}_1$ and $z_2 = a_3 + a_4 i_1 \in \mathbb{C}_1$.

If $\xi = z_1 + i_2 z_2$ and $\eta = w_1 + i_2 w_2$ be any two bicomplex numbers then the sum is $\xi \pm \eta = (z_1 + i_2 z_2) \pm (w_1 + i_2 w_2) = (z_1 \pm w_1) + i_2 (z_2 \pm w_2)$ and the product is $\xi.\eta = (z_1 + i_2 z_2) . (w_1 + i_2 w_2) = (z_1 w_1 - z_2 w_2) + i_2 (z_1 w_2 + z_2 w_1)$.

1.2.1 Idempotent representation of bicomplex number.

There are four idempotent elements in $\mathbb{C}_2$, they are $0, 1, e_1 = \frac{1+i_1 i_2}{2}$, and $e_2 = \frac{1-i_1 i_2}{2}$ out of which e_1 and e_2 are nontrivial such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. Every bicomplex number $z_1 + i_2 z_2$ can uniquely be expressed as the combination of e_1 and e_2, namely

$$\xi = z_1 + i_2 z_2 = (z_1 - i_1 z_2) e_1 + (z_1 + i_1 z_2) e_2.$$

This representation of ξ is known as the idempotent representation of bicomplex number and the complex coefficients $\xi_1 = (z_1 - i_1 z_2)$ and $\xi_2 = (z_1 + i_1 z_2)$ are known as idempotent components of the bicomplex number ξ.

1.2.2 Non-Singular and Singular elements.

An element $\xi = z_1 + i_2 z_2 \in \mathbb{C}_2$ is said to be invertible if there exists another element η in $\mathbb{C}_2$ such that $\xi\eta = 1$ and η is said to be the inverse (multiplicative) of ξ. Consequently ξ is said to be the inverse (multiplicative) of η. An element which has an inverse in $\mathbb{C}_2$ is said to be the nonsingular element of $\mathbb{C}_2$ and an element which does not have an inverse in $\mathbb{C}_2$ is said to be the singular element of $\mathbb{C}_2$.

An element $\xi = z_1 + i_2 z_2 \in \mathbb{C}_2$ is nonsingular if and only if $\left|z_1^2 + z_2^2\right| \neq 0$ and singular if and only if $\left|z_1^2 + z_2^2\right| = 0$ and the inverse of ξ is defined as

$$\xi^{-1} = \eta = \frac{z_1 - i_2 z_2}{z_1^2 + z_2^2}.$$

Zero is the only one element in $\mathbb{R}$ which does not have multiplicative inverse and in $\mathbb{C}, 0 = 0 + i0$ is the only one element which does not have multiplicative inverse. We denote the set of singular elements of $\mathbb{R}$ and $\mathbb{C}$ by O_0 and O_1 respectively. But there are more than one element in $\mathbb{C}_2$ which do not have multiplicative inverse; we denote this set by O_2 and clearly $O_0 = O_1 \subset O_2$.

1.2.3 Norm of a bicomplex number.

The norm $\|\cdot\|$ of $\mathbb{C}_2$ is a positive real valued function and $\|\cdot\| : \mathbb{C}_2 \to \mathbb{R}^+$ is defined by

$$\begin{aligned}\|\xi\| &= \|z_1 + i_2 z_2\| = \left\{|z_1|^2 + |z_2|^2\right\}^{\frac{1}{2}} \\ &= \left[\frac{|(z_1 - i_1 z_2)|^2 + |(z_1 + i_1 z_2)|^2}{2}\right]^{\frac{1}{2}} = \left(a_1^2 + a_2^2 + a_3^2 + a_4^2\right)^{\frac{1}{2}},\end{aligned}$$

where $\xi = a_1 + a_2 i_1 + a_3 i_2 + a_4 i_1 i_2 = z_1 + i_2 z_2 \in \mathbb{C}_2$.

The linear space $\mathbb{C}_2$ with respect to defined norm is a norm linear space, also $\mathbb{C}_2$ is complete; therefore $\mathbb{C}_2$ is the Banach space. If $\xi, \eta \in \mathbb{C}_2$ then $\|\xi\eta\| \leq \sqrt{2}\|\xi\|\|\eta\|$ holds instead of $\|\xi\eta\| \leq \|\xi\|\|\eta\|$, therefore $\mathbb{C}_2$ is not the Banach algebra.

Now we define the partial order relation $\precsim_{i_2}$ on $\mathbb{C}_2$ as follows:

Let $\mathbb{C}_2$ be the set of bicomplex numbers and $\xi = z_1 + i_2 z_2, \eta = w_1 + i_2 w_2 \in \mathbb{C}_2$ then $\xi \precsim_{i_2} \eta$ if and only if $z_1 \precsim w_1$. and $z_2 \precsim w_2$ i.e., $\xi \precsim_{i_2} \eta$ if one of the following conditions is satisfied:

(1) $z_1 = w_1,\ z_2 = w_2$,
(2) $z_1 \prec w_1,\ z_2 = w_2$,
(3) $z_1 = w_1,\ z_2 \prec w_2$ and
(4) $z_1 \prec w_1,\ z_2 \prec w_2$.

In particular we can write $\xi \lnsim_{i_2} \eta$ if $\xi \precsim_{i_2} \eta$ and $\xi \neq \eta$ i.e. one of (2), (3) and (4) is satisfied and we will write $\xi \prec_{i_2} \eta$ if only (4) is satisfied.

For any two bicomplex numbers $\xi, \eta \in \mathbb{C}_2$ we can verify the followings:

(i) $\xi \precsim_{i_2} \eta \to \|\xi\| \leq \|\eta\|$,
(ii) $\|\xi + \eta\| \leq \|\xi\| + \|\eta\|$,
(iii) $\|a\xi\| = a\|\xi\|$ if a is a non negative real number,
(iv) $\|\xi\eta\| \leq \sqrt{2}\|\xi\|\|\eta\|$ and the equality holds only when at least one of ξ and η is equal to zero,
(v) $\left\|\xi^{-1}\right\| = \|\xi\|^{-1}$ if ξ is a nonsingular bicomplex number with $0 \prec \xi$,
(vi) $\left\|\frac{\xi}{\eta}\right\| = \frac{\|\xi\|}{\|\eta\|}$, if η is a nonsingular bicomplex number.

1.3 Bicomplex valued metric space.

Choi et al.[17] defined the bicomplex valued metric space as follows:

***Definition* 1.3** *Let X be a nonempty set. Suppose the mapping $d : X \times X \to \mathbb{C}_2$ satisfies the following conditions:*

1. $0 \precsim_{i_2} d(x, y)$ *for all* $x, y \in X$,
2. $d(x, y) = 0$ *if and only if* $x = y$,
3. $d(x, y) = d(y, x)$ *for all* $x, y \in X$ *and*
4. $d(x, y) \precsim_{i_2} d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then d is called a bicomplex valued metric on X and (X, d) is called the bicomplex valued metric space.

***Definition* 1.4** *Let X be a nonempty set and let $s \geq 1$. Suppose the mapping $d : X \times X \to \mathbb{C}_2$ satisfies the following conditions:*

1. $0 \precsim_{i_2} d(x, y)$ *for all* $x, y \in X$,

2. $d(x,y)=0$ *if and only if* $x=y$,
3. $d(x,y)=d(y,x)$ *for all* $x,y\in X$ *and*
4. $d(x,y)\precsim_{i_2} s\left[d(x,z)+d(z,y)\right]$ *for all* $x,y,z\in X$.

Then d *is called a bicomplex valued b-metric on* X *and* (X,d) *is called the bicomplex valued b-metric space.*

***Example* 1.1** *Let* $X=[0,1]$, *and consider the mapping* $d: X\times X\to\mathbb{C}_2$ *as defined by* $d(x,y)=(1+i_1+i_2+i_1i_2)|x-y|^2$.

Then for all $x,y,z\in X$,

$$\begin{aligned} d(x,y) &= (1+i_1+i_2+i_1i_2)|x-y|^2 \\ &= (1+i_1+i_2+i_1i_2)|x-z+z-y|^2 \\ &= (1+i_1+i_2+i_1i_2)\left(|x-z|^2+|z-y|^2+2|x-z||z-y|\right) \\ &\precsim_{i_2} (1+i_1+i_2+i_1i_2)\left(|x-z|^2+|z-y|^2+|x-z|^2+|z-y|^2\right) \\ &\precsim_{i_2} 2\left[d(x,z)+d(z,y)\right] \end{aligned}$$

therefore (X,d) is a bicomplex valued b-metric space as $s=2$.

***Definition* 1.5** (i). *Let* $A\subseteq X$ *and* $a\in A$ *is said to be an interior point of* A *if there exists a* $0\prec_{i_2} r\in\mathbb{C}_2$ *such that*

$$B(a,r)=\{x\in X: d(a,x)\prec_{i_2} r\}\subseteq A$$

and the subset $A\subseteq X$ *is said to be an open set if each point of* A *is an interior point of* A.
(ii). *A point* $a\in X$ *is said to be a limit point of* A *if for all* $0\prec_{i_2} r\in\mathbb{C}_2$ *such that*

$$B(a,r)\cap\{A-\{a\}\}\neq\phi$$

and the subset $A\subseteq X$ *is said to be a closed set if all the limit points of* A *belong to* A.
(iii). *The family*

$$F=\{B(a,r): a\in X, 0\prec_{i_2} r\in\mathbb{C}_2\}$$

is a sub-basis for a Hausdorff topology τ *on* X.

***Definition* 1.6** *For a bicomplex valued metric space* (X,d)

(i). *A sequence* $\{x_n\}$ *in* X *is said to be a convergent sequence and converges to a point* x *if for any* $0\prec_{i_2} r\in\mathbb{C}_2$ *there is a natural number* $n_0\in\mathbb{N}$ *such that* $d(x_n,x)\prec_{i_2} r$, *for all* $n>n_0$ *and we write* $\lim_{n\to\infty} x_n=x$ *or* $x_n\to x$ *as* $n\to\infty$.

(ii). *A sequence* $\{x_n\}$ *in* X *is said to be a Cauchy sequence in* (X,d) *if for any* $0\prec_{i_2} r\in\mathbb{C}_2$ *there is a natural number* $n_0\in\mathbb{N}$ *such that* $d(x_n,x_{n+m})\prec_{i_2} r$, *for all* $m,n\in\mathbb{N}$ *and* $n>n_0$.

(iii). *If every cauchy sequence in* X *is convergent in* X *then* (X,d) *is said to be a complete bicomplex valued metric space.*

***Definition* 1.7** *Let* (X,d) *be a bicomplex valued metric space and* $S,T: X\to X$ *be two self-mappings then* S *and* T *are said to be compatible if* $\lim_{n\to\infty} d(STx, TSx_n)=0$, *whenever* $\{x_n\}$ *is a sequence in* X *such that* $\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Tx_n=u$ *for some* $u\in X$.

***Definition* 1.8** *Let* $S,T: X\to X$ *be two self-mappings then* S *and* T *are said to be weakly compatible if* $STx=TSx$ *whenever* $Sx=Tx$ *for all* $x\in X$.

***Definition* 1.9** *Let* $S,T: X\to X$ *be two self-mappings then,* S *and* T *are said to be commuting if* $TSx=STx$ *for all* $x\in X$.

***Definition* 1.10** *Let* (X,d) *be a bicomplex valued metric space and* $S,T: X\to X$ *be two Self-mappings then* S *and* T *are said to be weakly commuting if* $d(STx,TSx)\precsim_{i_2} d(Sx,Tx)$ *for all* $x\in X$.

***Definition* 1.11** *Let* (X,d) *be a cone metric space then the self-mapping* $T: X\to X$ *is said to be almost Jaggi contraction if it satisfies the following condition:*

$$d(Tx,Ty)\leq\alpha\frac{d(x,Tx)\,d(y,Ty)}{d(x,y)}+\beta d(x,y)+L\min\{d(x,Ty),d(y,Tx)\} \tag{1.1}$$

for all $x,y\in X$,*where* $L\geq 0$ *and* α, β *are non-negative real numbers with* $\alpha+\beta<1$.

***Definition* 1.12** *Let (X, d) be a cone metric space then the self-mapping $T : X \to X$ is said to be Jaggi contraction if it satisfies the following condition:*

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx)\, d(y, Ty)}{d(x, y)} + \beta d(x, y)$$

for all $x, y \in X$, where $L \geq 0$ and α and β are non-negative real numbers with $\alpha + \beta < 1$.

***Definition* 1.13** *Let (X, d) be a complete complex valued $b-$metric space then the self-mapping $T : X \to X$ is said to be Jaggi contraction if it satisfies the following condition:*

$$d(Tx, Ty) \precsim \alpha \frac{d(x, Tx)\, d(y, Ty)}{d(x, y)} + \beta d(x, y)$$

for all $x, y \in X$, where α and β are non-negative real numbers with $s(\alpha + \beta) < 1$.

***Definition* 1.14** *Let (X, d) be a cone metric space then the self-mapping $T : X \to X$ is said to be Dass-Gupta contraction if it satisfies the following condition:*

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) + L \min\{d(x, Tx), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$, where $L \geq 0$ and α, β are non-negative real numbers with $\alpha + \beta < 1$.

***Definition* 1.15** *Let (X, d) be a complete complex valued $b-$metric space with coefficient $s \geq 1$, then the self-mapping $T : X \to X$ is said to be Dass-Gupta contraction if it satisfies the following condition:*

$$d(Tx, Ty) \precsim \alpha \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) + L \min\{d(x, Tx), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$, where $L \geq 0$ and α, β are non-negative real numbers with $s(\alpha + \beta) < 1$.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

***Lemma* 2.1** *[20] Let (X, d) be a bicomplex valued metric space and a sequence $\{x_n\}$ in X is said to be convergent to a point x if and only if $\lim_{n\to\infty} \|d(x_n, x)\| = 0$.*

***Lemma* 2.2** *[20] Let (X, d) be a bicomplex valued metric space and a sequence $\{x_n\}$ in X is said to be a Cauchy sequence in X if and only if $\lim_{n\to\infty} \|d(x_n, x_{n+m})\| = 0$.*

3 Main Theorems.

In this section we prove some fixed point theorems on bicomplex valued b-metric space for rational contraction.

***Theorem* 3.1** *Let (X, d) be a complete bicomplex valued b-metric space with the coefficient $s \geq 1$. Let the self-mapping $T : X \to X$ be almost Jaggi contraction satisfying the condition*

$$d(Tx, Ty) \precsim_{i_2} \alpha \frac{d(x, Tx)\, d(y, Ty)}{d(x, y)} + \beta d(x, y) + L \min\{d(x, Ty), d(y, Tx)\}. \tag{3.1}$$

for all $x, y \in X$ and $d(x, y)$ is nonsingular where $L \geq 0$ and α, β are non-negative real numbers with $s\left(\sqrt{2}\alpha + \beta\right) < 1$. Then T has a unique fixed point in X.

Proof. Let $\{x_n\}$ be a sequence in X such that

$$x_n = Tx_{n-1}, \text{ for all } n = 1, 2, ...$$

where x_0 is an arbitrary fixed point in X.
Therefore by using (3.1) we obtain that

$$\begin{aligned} d(x_n, x_{n+1}) =& d(Tx_{n-1}, Tx_n) \\ &\precsim_{i_2} \alpha \frac{d(x_{n-1}, Tx_{n-1})\, d(x_n, Tx_n)}{d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n) + \\ &\quad L \min\{d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\} \\ &\precsim_{i_2} \alpha \frac{d(x_{n-1}, x_n)\, d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n) + \end{aligned}$$

$$L\min\{d(x_{n-1},x_{n+1}),d(x_n,x_n)\}$$
$$\precsim_{i_2}\alpha\frac{d(x_{n-1},x_n)\,d(x_n,x_{n+1})}{d(x_{n-1},x_n)}+\beta d(x_{n-1},x_n)$$

Hence
$$\|d(x_n,x_{n+1})\|\le\sqrt{2}\alpha\frac{\|d(x_{n-1},x_n)\|\,\|d(x_n,x_{n+1})\|}{\|d(x_{n-1},x_n)\|}+\beta\|d(x_{n-1},x_n)\|$$
$$\le\sqrt{2}\alpha\|d(x_n,x_{n+1})\|+\beta\|d(x_{n-1},x_n)\|,$$

i.e.,$\|d(x_n,x_{n+1})\|\le\frac{\beta}{1-\sqrt{2}\alpha}\|d(x_{n-1},x_n)\|,$

i.e.,$\|d(x_n,x_{n+1})\|\le h\|d(x_{n-1},x_n)\|,$

where $h=\frac{\beta}{1-\sqrt{2}\alpha}$ and $0\le h<1$, as $s\left(\sqrt{2}\alpha+\beta\right)<1$ and $s\ge1$. Therefore for all $n=1,2,3,...$
$$\|d(x_n,x_{n+1})\|\le h\|d(x_{n-1},x_n)\|\le h^2\|d(x_{n-2},x_{n-1})\|\le...\le h^n\|d(x_0,x_1)\|.$$

Thus

(3.2) $\|d(x_{n+1},x_{n+2})\|\le h^{n+1}\|d(x_0,x_1)\|.$

Since $s\left(\sqrt{2}\alpha+\beta\right)<1$ and $s\ge1\to sh=\frac{s\beta}{1-\sqrt{2}\alpha}<1$.

Then for any two positive integers m,n with $m>n$ we get that
$$d(x_n,x_m)\precsim_{i_2}s[d(x_n,x_{n+1})+d(x_{n+1},x_m)].$$

Therefore,
$$\begin{aligned}&\|d(x_n,x_m)\|\\&\le s\|d(x_n,x_{n+1})\|+s\|d(x_{n+1},x_m)\|\\&\le s\|d(x_n,x_{n+1})\|+s^2\|d(x_{n+1},x_{n+2})\|+s^2\|d(x_{n+2},x_m)\|\\&\le s\|d(x_n,x_{n+1})\|+s^2\|d(x_{n+1},x_{n+2})\|+s^3\|d(x_{n+2},x_{n+3})\|+s^3\|d(x_{n+3},x_m)\|\\&\quad ... \qquad ... \qquad ...\\&\quad ... \qquad ... \qquad ...\end{aligned}$$

i.e.,$\|d(x_n,x_m)\|\le s\|d(x_n,x_{n+1})\|+s^2\|d(x_{n+1},x_{n+2})\|+s^3\|d(x_{n+2},x_{n+3})\|+...+s^{m-n-1}\|d(x_{m-1},x_m)\|$
$$\le\left\{\begin{array}{c}\|d(x_n,x_{n+1})\|+s^2\|d(x_{n+1},x_{n+2})\|+\\ s^3\|d(x_{n+2},x_{n+3})\|+...+s^{m-n}\|d(x_{m-1},x_m)\|\end{array}\right\},\text{ as } s\ge1.$$

Therefore by using(3.2) we get that
$$\|d(x_n,x_m)\|\le sh^n\|d(x_0,x_1)\|+s^2h^{n+1}\|d(x_0,x_1)\|$$
$$+s^3h^{n+2}\|d(x_0,x_1)\|+...+s^{m-n}h^{m-1}\|d(x_0,x_1)\|$$

i.e., $\|d(x_n,x_m)\|\sum_{i=1}^{m-n}s^ih^{i+n-1}\|d(x_0,x_1)\|,$

i.e.,$\|d(x_n,x_m)\|\sum_{i=1}^{m-n}s^{i+n-1}h^{i+n-1}\|d(x_0,x_1)\|,$ as $s\ge1$,

i.e.,$\|d(x_n,x_m)\|\sum_{j=n}^{m-1}s^jh^j\|d(x_0,x_1)\|,$

i.e.,$\|d(x_n,x_m)\|\sum_{j=n}^{\infty}(sh)^j\|d(x_0,x_1)\|,$

i.e.,$\|d(x_n,x_m)\|\le\frac{(sh)^n}{1-sh}\|d(x_0,x_1)\|.$

Since $\frac{(sh)^n}{1-sh}\to0$ as $n\to\infty$, therefore for any $\varepsilon>0$ there exists a positive integer n_0 such that $\|d(x_n,x_m)\|<\varepsilon$, for all $m,n>n_0$. Hence $\{x_n\}$ is Cauchy in X. Since X is a complete bicomplex valued b-metric space, then there exists $u\in X$ such that $\lim_{n\to\infty}x_n=u$.

Now we show that $u=Tu$, if not then there exists $0\prec_{i_2}\xi\in\mathbb{C}_2$ such that $d(u,Tu)=\xi$.

Therefore,
$$\begin{aligned}\xi&=d(u,Tu)\\&\precsim_{i_2}sd(u,x_{n+1})+sd(x_{n+1},Tu)\\&\precsim_{i_2}sd(u,x_{n+1})+sd(Tx_n,Tu)\\&\precsim_{i_2}sd(u,x_{n+1})+s\alpha\frac{d(x_n,Tx_n)\,d(u,Tu)}{d(x_n,u)}+s\beta d(x_n,u)+sL\min\{d(x_n,Tu),d(u,Tx_n)\}\\&\precsim_{i_2}sd(u,x_{n+1})+s\alpha\frac{d(x_n,x_{n+1})\,\xi}{d(x_n,u)}+s\beta d(x_n,u)+sL\min\{d(x_n,Tu),d(u,x_{n+1})\}.\end{aligned}$$

Therefore,

$$\|\xi\| \leq s \|d(u, x_{n+1})\| + s\sqrt{2}\alpha \frac{\|d(x_n, x_{n+1})\| \|\xi\|}{\|d(x_n, u)\|} + s\beta \|d(x_n, u)\| + s \|L \min\{d(x_n, Tu), d(u, x_{n+1})\}\|.$$

Since $\lim_{n\to\infty} x_n = u$, taking limit both sides as $n \to \infty$ we get that $\|\xi\| \leq 0$, which is a contradiction, hence $\|\xi\| = 0 \to \|d(u, Tu)\| = 0 \to u = Tu$. Therefore u is a fixed point of T.

Now we show that T has a unique fixed point. If possible let $u^* \in X$ be another fixed point of T.

Then

$$\begin{aligned} d(u, u^*) &= d(Tu, Tu^*) \\ &\precsim_{i_2} \alpha \frac{d(u, Tu)\, d(u^*, Tu^*)}{d(u, u^*)} + \beta d(u, u^*) + L \min\{d(u, Tu^*), d(u^*, Tu)\} \\ &\precsim_{i_2} \alpha \frac{d(u, u)\, d(u^*, u^*)}{d(u, u^*)} + \beta d(u, u^*) + L \min\{d(u, u^*), d(u^*, u)\} \\ &\precsim_{i_2} (\beta + L)\, d(u, u^*) \end{aligned}$$

i.e., $\|d(u, u^*)\| \leq (\beta + L)\|d(u, u^*)\|$

i.e., $\|d(u, u^*)\| = 0$

i.e., $u = u^*$

This completes the proof of the ***Theorem* 3.1.**

***Example* 3.1** *Let* $X = [0, 1]$ *and consider the mapping* $d : X \times X \to \mathbb{C}_2$ *defined by* $d(x, y) = (1 + i_1 + i_2 + i_1 i_2)\,|x - y|^2$

Then for all $x, y, z \in X$, we can easily show that

$$d(x, y) \precsim_{i_2} 2\,[d(x, z) + d(z, y)]$$

therefore (X, d) is a bicomplex valued b-metric space with $s = 2$.

Let us consider the mapping $T : X \to X$ by $Tx = \frac{x}{2}$, then

$$\begin{aligned} d(Tx, Ty) &= d\left(\frac{x}{2}, \frac{y}{2}\right) \\ &= (1 + i_1 + i_2 + i_1 i_2)\left|\frac{x}{2} - \frac{y}{2}\right|^2 \\ &= \frac{1}{4}(1 + i_1 + i_2 + i_1 i_2)\,|x - y|^2 \\ &= \frac{1}{4} d(x, y). \end{aligned}$$

if we choose $\alpha = \frac{3}{16\sqrt{2}}$ and $\beta = \frac{1}{4}$ then $s\left(\sqrt{2}\alpha + \beta\right) = 2\left(\sqrt{2}\frac{3}{16\sqrt{2}} + \frac{1}{4}\right) = \frac{7}{8} < 1$ and for all $L \geq 0$ then all conditions of the ***Theorem* 3.1** is satisfied. And clearly 0 is the unique fixed point of T.

***Corollary* 3.1** *Let* (X, d) *be a complete bicomplex valued b-metric space with the coefficient* $s \geq 1$. *Let the self-mapping* $T : X \to X$ *be Jaggi contraction satisfying the condition*

$$d(Tx, Ty) \precsim_{i_2} \alpha \frac{d(x, Tx)\, d(y, Ty)}{d(x, y)} + \beta d(x, y)$$

for all $x, y \in X$ *and* $d(x, y)$ *is nonsingular where* α, β *are non-negative real numbers with* $s\left(\sqrt{2}\alpha + \beta\right) < 1$. *Then* T *has a unique fixed point in X.*

Proof. This can be proved by taking $L = 0$ in ***Theorem* 3.1.**

***Theorem* 3.2** *Let* (X, d) *be a complete bicomplex valued b-metric space with the coefficient* $s \geq 1$. *Let the mappings* $S, T : X \to X$ *be almost Jaggi contraction satisfying the condition*

$$(3.3)\quad d(Sx, Ty) \precsim_{i_2} \alpha \frac{d(x, Sx)\, d(y, Ty)}{d(x, y)} + \beta d(x, y) + L \min\{d(x, Ty), d(y, Sx)\}$$

for all $x, y \in X$ *and* $d(x, y)$ *is nonsingular where* $L \geq 0$ *and* α, β *are non-negative real numbers with* $s\left(\sqrt{2}\alpha + \beta\right) < 1$. *Then the mappings S and T have a unique common fixed point in X.*

Proof. Let $\{x_n\}$ be any sequence in X and x_0 be an arbitrary point in X. We define

$$x_{2n+1} = Sx_{2n}, \quad x_{2n+2} = Tx_{2n+1}, \quad n = 0, 1, 2, ...$$

Therefore by using (3.3) we obtain that

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\precsim_{i_2} \alpha \frac{d(x_{2n}, Sx_{2n})\, d(x_{2n+1}, Tx_{2n+1})}{d(x_{2n}, x_{2n+1})} + \beta d(x_{2n}, x_{2n+1}) + L \min\{d(x_{2n}, Tx_{2n+1}), d(x_{2n+1}, Sx_{2n})\} \\ &\precsim_{i_2} \alpha \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+1})} + \beta d(x_{2n}, x_{2n+1}) + L \min\{d(x_{2n}, x_{2n+2}), d(x_{2n+1}, x_{2n+1})\} \\ &\precsim_{i_2} \alpha \frac{d(x_{2n}, x_{2n+1})\, d(x_{2n+1}, x_{2n+2})}{d(x_{2n}, x_{2n+1})} + \beta d(x_{2n}, x_{2n+1}). \end{aligned}$$

Therefore,

$$\begin{aligned} \|d(x_{2n+1}, x_{2n+2})\| &\le \sqrt{2}\alpha \frac{\|d(x_{2n}, x_{2n+1})\|\, \|d(x_{2n+1}, x_{2n+2})\|}{\|d(x_{2n}, x_{2n+1})\|} + \beta \|d(x_{2n}, x_{2n+1})\| \\ &\le \sqrt{2}\alpha \|d(x_{2n+1}, x_{2n+2})\| + \beta \|d(x_{2n}, x_{2n+1})\| \end{aligned}$$

(3.4) i.e., $\|d(x_{2n+1}, x_{2n+2})\| \le \dfrac{\beta}{1 - \sqrt{2}\alpha} \|d(x_{2n}, x_{2n+1})\|$

Similarly we get that

(3.5) $\|d(x_{2n+2}, x_{2n+3})\| \le \dfrac{\beta}{1 - \sqrt{2}\alpha} \|d(x_{2n+1}, x_{2n+2})\|$

From (3.4) and (3.5) we can say that

$$\|d(x_{n+1}, x_{n+2})\| \le \frac{\beta}{1 - \sqrt{2}\alpha} \|d(x_n, x_{n+1})\|$$

Let $h = \frac{\beta}{1 - \sqrt{2}\alpha}$. Then $0 \le h < 1$, as $s\left(\sqrt{2}\alpha + \beta\right) < 1$ and $s \ge 1$. Therefore for all $n = 0, 1, 2, ...$

(3.6) $\|d(x_{n+1}, x_{n+2})\| \le h \|d(x_n, x_{n+1})\| \le h^2 \|d(x_{n-1}, x_n)\| \le ... \le h^{n+1} \|d(x_0, x_1)\|.$

Since $s\left(\sqrt{2}\alpha + \beta\right) < 1$ and $s \ge 1$, therefore $sh = \frac{s\beta}{1 - \sqrt{2}\alpha} < 1$

Then for any two positive integers m, n with $m > n$, we obtain that

$$d(x_n, x_m) \precsim_{i_2} s\left[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)\right].$$

Therefore,

$$\begin{aligned} &\|d(x_n, x_m)\| \\ \le & s\|d(x_n, x_{n+1})\| + s\|d(x_{n+1}, x_m)\| \\ \le & s\|d(x_n, x_{n+1})\| + s^2\|d(x_{n+1}, x_{n+2})\| + s^2\|d(x_{n+2}, x_m)\| \\ \le & s\|d(x_n, x_{n+1})\| + s^2\|d(x_{n+1}, x_{n+2})\| + s^3\|d(x_{n+2}, x_{n+3})\| + s^3\|d(x_{n+3}, x_m)\| \\ & ... \qquad ... \qquad ... \\ & ... \qquad ... \qquad ... \end{aligned}$$

$$\begin{aligned} i.e., \|d(x_n, x_m)\| \le & s\|d(x_n, x_{n+1})\| + s^2\|d(x_{n+1}, x_{n+2})\| \\ & + s^3\|d(x_{n+2}, x_{n+3})\| + ... + s^{m-n-1}\|d(x_{m-1}, x_m)\| \\ \le & \left\{ \begin{array}{c} \|d(x_n, x_{n+1})\| + s^2\|d(x_{n+1}, x_{n+2})\| + \\ s^3\|d(x_{n+2}, x_{n+3})\| + ... + s^{m-n}\|d(x_{m-1}, x_m)\| \end{array} \right\}, \text{as} \quad s \ge 1. \end{aligned}$$

Therefore by using(3.6) we obtain that

$$\begin{aligned} \|d(x_n, x_m)\| \le & sh^n\|d(x_0, x_1)\| + s^2h^{n+1}\|d(x_0, x_1)\| \\ & + s^3h^{n+2}\|d(x_0, x_1)\| + ... + s^{m-n}h^{m-1}\|d(x_0, x_1)\| \end{aligned}$$

i.e., $\|d(x_n, x_m)\| \sum_{i=1}^{m-n} s^i h^{i+n-1} \|d(x_0, x_1)\|$

i.e.,$\|d(x_n, x_m)\| \sum_{i=1}^{m-n} s^{i+n-1} h^{i+n-1} \|d(x_0, x_1)\|$, as $\quad s \ge 1$.

i.e.,$\|d(x_n, x_m)\| \sum_{j=n}^{m-1} s^j h^j \|d(x_0, x_1)\|$

i.e., $\|d(x_n, x_m)\| \sum_{j=n}^{\infty} (sh)^j \|d(x_0, x_1)\|$

i.e.,$\|d(x_n, x_m)\| \leq \frac{(sh)^n}{1-sh} \|d(x_0, x_1)\|$.

Since $\frac{(sh)^n}{1-sh} \to 0$ as $n \to \infty$, therefore for any $\varepsilon > 0$ there exists a positive integer n_0 such that $\|d(x_n, x_m)\| < \varepsilon$, for all $m, n > n_0$. Hence $\{x_n\}$ is a Cauchy in X. Again since X is a complete bicomplex valued b-metric space, there exists a $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Now we show that $u = Su$, if not then there exists $0 \prec_{i_2} \xi \in \mathbb{C}_2$ such that $d(u, Su) = \xi$.

Therefore,

$$\begin{aligned}
\xi &= d(u, Su)\\
&\precsim_{i_2} sd(u, x_{2n+2}) + sd(x_{2n+2}, Su)\\
&\precsim_{i_2} sd(u, x_{2n+2}) + sd(Su, Tx_{2n+1})\\
&\precsim_{i_2} sd(u, x_{2n+2}) + s\alpha\frac{d(u, Su)\, d(x_{2n+1}, Tx_{2n+1})}{d(u, x_{2n+1})} + s\beta d(u, x_{2n+1}) + sL\min\{d(u, Tx_{2n+1}), d(x_{2n+1}, Su)\}\\
&\precsim_{i_2} sd(u, x_{2n+2}) + s\alpha\frac{\xi d(x_{2n+1}, x_{2n+2})}{d(u, x_{2n+1})} + s\beta d(u, x_{2n+1}) + sL\min\{d(u, x_{2n+2}), d(x_{2n+1}, Su)\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\xi\| \leq & s\|d(u, x_{2n+2})\| + s\alpha\frac{\|\xi\|\,\|d(x_{2n+1}, x_{2n+2})\|}{\|d(u, x_{2n+1})\|} +\\
& s\beta\|d(u, x_{2n+1})\| + s\|L\min\{d(u, x_{2n+2}), d(x_{2n+1}, Su)\}\|.
\end{aligned}$$

Since $\lim_{n\to\infty} x_n = u$, taking limit both sides as $n \to \infty$ we get that $\|\xi\| \leq 0$, which is a contradiction, hence $\|\xi\| = 0 \to \|d(u, Su)\| = 0 \to u = Su$. Therefore u is a fixed point of S. Similarly we can show that $Tu = u$.

Now we show that S and T have a unique common fixed point. For this let $u^* \in X$ be another common fixed point of S and T, i.e. $Su^* = Tu^* = u^*$.

Then

$$\begin{aligned}
d(u, u^*) &= d(Tu, Tu^*)\\
&\precsim_{i_2} \alpha\frac{d(u^*, Tu^*)[1 + d(u, Tu)]}{1 + d(u, u^*)} + \beta d(u, u^*) + L\min\{d(u, Tu), d(u, Tu^*), d(u^*, Tu)\}\\
&\precsim_{i_2} \alpha\frac{d(u^*, u^*)[1 + d(u, u)]}{d(u, u^*)} + \beta d(u, u^*) + L\min\{d(u, u), d(u, u^*), d(u^*, u)\}\\
&\precsim_{i_2} \beta d(u, u^*)
\end{aligned}$$

i.e., $\|d(u, u^*)\| \leq \beta\|d(u, u^*)\|$

i.e., $\|d(u, u^*)\| = 0$

i.e., $u = u^*$

Hence the proof of the ***Theorem* 3.2**. is established.

***Theorem* 3.3** *Let (X, d) be a complete bicomplex valued $b-$metric space with coefficient $s \geq 1$. Let the self-mapping $T : X \to X$ be a Dass-Gupta contraction satisfying the condition*

$$(3.7)\quad d(Tx, Ty) \precsim_{i_2} \alpha\frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y) + L\min\{d(x, Tx), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$ and $1 + d(x, y)$ be nonsingular, where $L \geq 0$ and α, β are non-negative real numbers with $s\left(\sqrt{2}\alpha + \beta\right) < 1$. Then T has a unique fixed point in X.

Proof. Let $\{x_n\}$ be a sequence in X such that

$$x_n = Tx_{n-1}, \text{ for all } n = 1, 2, ...$$

where x_0 is an arbitrary fixed point in X.

Therefore by using (3.7) we obtain that

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n)\\
&\precsim_{i_2} \alpha\frac{d(x_n, Tx_n)[1 + d(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n)\\
&\quad + L\min\{d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1})\}
\end{aligned}$$

$$\precsim_{i_2} \alpha \frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n) +$$
$$L \min\{d(x_{n-1}, x_n), d(x_{n-1}, x_{n+1}), d(x_n, x_n)\}$$
$$\precsim_{i_2} \alpha \frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} + \beta d(x_{n-1}, x_n).$$

Therefore,

$$\|d(x_n, x_{n+1})\| \leq \sqrt{2}\alpha \frac{\|d(x_n, x_{n+1})\| \, \|1 + d(x_{n-1}, x_n)\|}{\|1 + d(x_{n-1}, x_n)\|} + \beta \|d(x_{n-1}, x_n)\|$$
$$\leq \sqrt{2}\alpha \|d(x_n, x_{n+1})\| + \beta \|d(x_{n-1}, x_n)\|,$$

i.e.,$\|d(x_n, x_{n+1})\| \leq \frac{\beta}{1-\sqrt{2}\alpha} \|d(x_{n-1}, x_n)\|$,
i.e.,$\|d(x_n, x_{n+1})\| \leq h \|d(x_{n-1}, x_n)\|$,
where $h = \frac{\beta}{1-\sqrt{2}\alpha}$, then $0 \leq h < 1$, since $s\left(\sqrt{2}\alpha + \beta\right) < 1$ and $s \geq 1$. Therefore for all $n = 1, 2, 3, ...$

$$\|d(x_n, x_{n+1})\| \leq h \|d(x_{n-1}, x_n)\| \leq h^2 \|d(x_{n-2}, x_{n-1})\| \leq ... \leq h^n \|d(x_0, x_1)\|.$$

Therefore,

$$\|d(x_{n+1}, x_{n+2})\| \leq h^{n+1} \|d(x_0, x_1)\|$$

Since $s(\alpha + \beta) < 1$ and $s \geq 1 \rightarrow sh = \frac{s\beta}{1-\sqrt{2}\alpha} < 1$
Then for any two positive integers m, n with $m > n$

$$d(x_n, x_m) \precsim_{i_2} s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)].$$

Therefore,

$$\|d(x_n, x_m)\|$$
$$\leq s \|d(x_n, x_{n+1})\| + s \|d(x_{n+1}, x_m)\|$$
$$\leq s \|d(x_n, x_{n+1})\| + s^2 \|d(x_{n+1}, x_{n+2})\| + s^2 \|d(x_{n+2}, x_m)\|$$
$$\leq s \|d(x_n, x_{n+1})\| + s^2 \|d(x_{n+1}, x_{n+2})\| + s^3 \|d(x_{n+2}, x_{n+3})\| + s^3 \|d(x_{n+3}, x_m)\|$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$\ldots \qquad \ldots \qquad \ldots$$

$$i.e., \|d(x_n, x_m)\| \leq s \|d(x_n, x_{n+1})\| + s^2 \|d(x_{n+1}, x_{n+2})\| + s^3 \|d(x_{n+2}, x_{n+3})\| + ... + s^{m-n-1} \|d(x_{m-1}, x_m)\|$$
$$\leq \left\{ \begin{array}{c} \|d(x_n, x_{n+1})\| + s^2 \|d(x_{n+1}, x_{n+2})\| + \\ s^3 \|d(x_{n+2}, x_{n+3})\| + ... + s^{m-n} \|d(x_{m-1}, x_m)\| \end{array} \right\} \text{ as } s \geq 1.$$

Therefore by using (3.2) we get that

$$\|d(x_n, x_m)\| \leq sh^n \|d(x_0, x_1)\| + s^2 h^{n+1} \|d(x_0, x_1)\| + s^3 h^{n+2} \|d(x_0, x_1)\| + ... + s^{m-n} h^{m-1} \|d(x_0, x_1)\|$$

i.e.,$\|d(x_n, x_m)\| \sum_{i=1}^{m-n} s^i h^{i+n-1} \|d(x_0, x_1)\|$
i.e.,$\|d(x_n, x_m)\| \sum_{i=1}^{m-n} s^{i+n-1} h^{i+n-1} \|d(x_0, x_1)\|$, since $s \geq 1$.
i.e., $\|d(x_n, x_m)\| \sum_{j=n}^{m-1} s^j h^j \|d(x_0, x_1)\|$
i.e., $\|d(x_n, x_m)\| \sum_{j=n}^{\infty} (sh)^j \|d(x_0, x_1)\|$
i.e., $\|d(x_n, x_m)\| \leq \frac{(sh)^n}{1-sh} \|d(x_0, x_1)\|$.

Since $\frac{(sh)^n}{1-sh} \rightarrow 0$ as $n \rightarrow \infty$, therefore for any $\varepsilon > 0$ there exists a positive integer n_0 such that $\|d(x_n, x_m)\| < \varepsilon$, for all $m, n > n_0$. Hence $\{x_n\}$ is Cauchy in X. Since X is a complete bicomplex valued b-metric space, then there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Now we show that $u = Tu$, if not then there exists $0 \prec_{i_2} \xi \in \mathbb{C}_2$ such that $d(u, Tu) = \xi$.
Therefore,

$$\xi = d(u, Tu)$$
$$\precsim_{i_2} sd(u, x_{n+1}) + sd(x_{n+1}, Tu)$$
$$\precsim_{i_2} sd(u, x_{n+1}) + sd(Tx_n, Tu)$$
$$\precsim_{i_2} sd(u, x_{n+1}) + s\alpha \frac{d(u, Tu)[1 + d(x_n, Tx_n)]}{1 + d(x_n, u)} +$$

$$s\beta d(x_n,u)+sL\min\{d(x_n,Tx_n),d(x_n,Tu),d(u,Tx_n)\}$$

$$\precsim_{i_2} sd(u,x_{n+1})+s\alpha\frac{\xi[1+d(x_n,x_{n+1})]}{d(x_n,u)}+s\beta d(x_n,u)+$$

$$sL\min\{d(x_n,x_{n+1}),d(x_n,Tu),d(u,x_{n+1})\}.$$

Thus

$$\|\xi\|\leq s\|d(u,x_{n+1})\|+s\sqrt{2}\alpha\frac{\|\xi\|\,\|1+d(x_n,x_{n+1})\|}{\|1+d(x_n,u)\|}+s\beta\|d(x_n,u)\|+$$

$$s\|L\min\{d(x_n,x_{n+1}),d(x_n,Tu),d(u,x_{n+1})\}\|.$$

Since $\lim_{n\to\infty}x_n=u$, taking limit both sides as $n\to\infty$ we get that $\|\xi\|\leq s\alpha\|\xi\|$, which is a contradiction. Hence $\|\xi\|=0\to\|d(u,Tu)\|=0\to u=Tu$. Therefore u is a fixed point of T.

Now we show that T has a unique fixed point. For this let $u^*\in X$ be another fixed point of T.

Then

$$d(u,u^*)=d(Tu,Tu^*)$$

$$\precsim_{i_2}\alpha\frac{d(u^*,Tu^*)[1+d(u,Tu)]}{1+d(u,u^*)}+\beta d(u,u^*)+$$

$$L\min\{d(u,Tu),d(u,Tu^*),d(u^*,Tu)\}$$

$$\precsim_{i_2}\alpha\frac{d(u^*,u^*)[1+d(u,u)]}{d(u,u^*)}+\beta d(u,u^*)+$$

$$L\min\{d(u,u),d(u,u^*),d(u^*,u)\}$$

$$\precsim_{i_2}\beta d(u,u^*)$$

i.e., $\|d(u,u^*)\|\leq\beta\|d(u,u^*)\|$

i.e., $\|d(u,u^*)\|=0$

i.e., $u=u^*$.

This completes the proof of the ***Theorem*** **3.3**.

Corollary **3.2** *Let (X,d) be a complete bicomplex valued $b-$metric space with coefficient $s\geq 1$. Let the self-mapping $T:X\to X$ be a Dass-Gupta rational contraction satisfying the condition*

(3.8) $$d(Tx,Ty)\precsim_{i_2}\alpha\frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)}+\beta d(x,y)$$

for all $x,y\in X$ and $1+d(x,y)$ be non degenerated, where α,β are non-negative real numbers with $s\left(\sqrt{2}\alpha+\beta\right)<1$. Then T has a unique fixed point in X.

Proof. This can be proved by taking $L=0$ in ***Theorem*** **3.3**.

Corollary **3.3** *Let (X,d) be a complete bicomplex valued $b-$metric space with coefficient $s\geq 1$. Let the self-mapping $T:X\to X$ be a Dass-Gupta rational contraction satisfying the condition*

(3.9) $$d(T^nx,T^ny)\precsim_{i_2}\alpha\frac{d(y,T^ny)[1+d(x,T^nx)]}{1+d(x,y)}+\beta d(x,y)$$

for all $x,y\in X$ and $1+d(x,y)$ be nonsingular, where α,β are non-negative real numbers with $s\left(\sqrt{2}\alpha+\beta\right)<1$. Then T has a unique fixed point in X.

By ***Corollary*** **3.2** there exists a unique point $u\in X$ such that

$$T^nu=u.$$

Therefore,

$$d(Tu,u)=d(TT^nu,T^nu)=d(T^nTu,T^nu)\precsim_{i_2}\alpha\frac{d(u,T^nu)[1+d(Tu,T^nTu)]}{1+d(Tu,u)}+\beta d(Tu,u)$$

i.e., $d(Tu,u)\precsim_{i_2}\alpha\frac{d(u,u)[1+d(Tu,T^nTu)]}{1+d(Tu,u)}+\beta d(Tu,u)$

i.e., $d(Tu,u)\precsim_{i_2}\alpha d(Tu,u)$

i.e., $\|d(Tu,u)\|\leq\alpha\|d(Tu,u)\|$

i.e., $\|d(Tu,u)\|=0$

i.e., $Tu=u$.

This completes the proof of the ***Corollary*** **3.3**.

4 Future prospect.

In the line of the works as carried out in the paper one may think of the deduction of fixed point theorems using fuzzy metric, quasi metric, partial metric, probabilistic metric, p-adic metric (where p is a prime number), cone metric, quasi semi metric, convex metric, D-metric and other different types of metrics under the flavour of bicomplex analysis. This may be regarded as an active area of research to the future workers in this branch.

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