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(Dedicated to Honor Dr. R. C. Singh Chandel on His 75th Birth Anniversary Celebrations)

APPLICATION IN INITIAL VALUE PROBLEMS VIA OPERATIONAL TECHNIQUES ON A CONTOUR INTEGRAL FOR SRIVASTAVA - DAOUST FUNCTION OF TWO VARIABLES

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Abstract

In this paper, we introduce the contour integrals for two variables functions namely as Srivastava - Daoust and generalized Kampé de Fériet functions and then, by the fractional and partial derivatives operational techniques, obtain their many results and relations for various special functions useful in quantum mechanical fields. Again then, apply them to solve the fractional calculus problems involving the initial values with the Caputo fractional derivatives and Riemann - Liouville fractional integrals.
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1 Introduction

In this section, we introduce some preliminaries and formulae to be used in our investigation:

Mittag - Leffler [22], in his a series of five notes (from 1901 to 1905), defined a function

(1.1) $E_\nu(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\nu k+1)},\nu,z\in\mathbb{C},\Re(\nu)>0,$ $\mathbb{C}$ being the set of complex numbers,

and studied various properties of his named Mittag - Leffler function (1.1) (see also in, [2] and [32]) to propose the generalization of the Laplace-Abel integral in the form $\int_0^{+\infty}e^{-t}E_\nu(zt^\nu)dt$.

The properties and generalization of (1.1) are introduced as namely as generalized Mittag - Leffler function (see, Dattoli et al. [2], Humbert and Agrawal [11], Wiman [32])

(1.2) $E_{\nu,\rho}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\nu k+\rho)},z,\nu,\rho\in\mathbb{C},\Re(\nu)>0,\Re(\rho)>0.$

The functions (1.1) and (1.2) have used in various scientific and physical problems found in the literature of the authors (for example [4], [8] - [10], [12], [13] - [19], [21]). The Mittag - Leffler function (1.2) has a relation to the function (1.1) as $E_{\nu,1}(z)=E_\nu(z)$ and an integral formula of (1.2) is used in quantum mechanical problems by Borel transforms (see in Dattoli and Licciardi [3]) as

(1.3) $\int_{-\infty}^{\infty}E_{1,\beta+1}(-x^2)dx=\frac{1}{\Gamma(\beta+\frac{1}{2})},\Re(\beta)>-\frac{1}{2}.$

Recently, the contour integral has studied for Kummer confluent hypergeometric function by (see [1], [7], [24])

(1.4) $M(\alpha,\beta,y)=\dfrac{y^{1-\beta}\Gamma(\beta)}{2\pi i}\int_{-\infty}^{(0^+,1^+)}e^{yt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt,\ \alpha,\beta\in\mathbb{C},\ |\arg(y)|<\frac{\pi}{2},i=\sqrt{(-1)}.$

The function $M(\alpha,\beta,y)$ in the series form ([1], [5], [30, p.36]) is written by

$$M(\alpha,\beta,y)={}_1F_1(\alpha;\beta;y)=\frac{\Gamma(\beta)}{\Gamma(\alpha)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)}{\Gamma(\beta+k)}\frac{y^k}{k!}.$$

Here in (1.4), the relation between Gamma function and Pochhammer symbol (a generalized factorial function) is known as

$$\frac{\Gamma(\nu+k)}{\Gamma(\nu)}=(\nu)_k=\begin{cases}1,k=0;\\ \nu(\nu+1)\ldots(\nu+k-1),\forall k\in\mathbb{N};\end{cases}$$

where, $\mathbb{N}$ being the set of natural numbers.

In the present investigation, we introduce a contour integral for two variables Srivastava and Daoust function ([20], [26]-[28]) defined by

(1.5) $S^{A:B;B'}_{C:D;D'}\begin{pmatrix}z\\w\end{pmatrix}=S^{A:B;B'}_{C:D;D'}\left(\begin{matrix}[(a):\theta,\vartheta]:[(b):\psi];[(b'):\psi'];\\ [(c):\delta,\kappa]:[(d):\varphi];[(d'):\varphi'];\end{matrix}z,w\right)=\sum_{m,n=0}^{\infty}H^{A:B;B'}_{C:D;D'}(m,n)\frac{z^m}{m!}\frac{w^n}{n!},$

where,

$$H^{A\,:\,B;B'}_{C\,:\,D;D'}(m,n)=\frac{\prod_{j=1}^{A}\Gamma(a_j+\theta_j m+\vartheta_j n)}{\prod_{j=1}^{C}\Gamma(c_j+\delta_j m+\kappa_j n)}\frac{\prod_{j=1}^{B}\Gamma(b_j+\psi_j m)}{\prod_{j=1}^{D}\Gamma(d_j+\varphi_j m)}\frac{\prod_{j=1}^{B'}\Gamma(b'_j+\psi'_j n)}{\prod_{j=1}^{D'}\Gamma(d'_j+\varphi'_j n)}.$$

The series in (1.5) is convergent under the conditions given by

$$\sum_{j=1}^{C}\delta_j+\sum_{j=1}^{D}\varphi_j-\sum_{j=1}^{A}\theta_j-\sum_{j=1}^{B}\psi_j+1>0;\ \sum_{j=1}^{C}\kappa_j+\sum_{j=1}^{D'}\varphi'_j-\sum_{j=1}^{A}\vartheta_j-\sum_{j=1}^{B'}\psi'_j+1>0.$$

Again, by the formula (1.5), on setting $\theta_j=\vartheta_j=1$ with $(j=1,2,\ldots,A)$; $\psi_j=1$ with $(j=1,2,\ldots,B)$; $\psi'_j=1$ with $(j=1,2,\ldots,B')$; $\delta_j=\kappa_j=1$ with $(j=1,2,\ldots,C)$; $\varphi_j=1$ with $(j=1,2,\ldots,D)$; $\varphi'_j=1$ with $(j=1,2,\ldots,D')$; a relation between the Srivastava and Daoust function $S^{A\,:\,B;B'}_{C\,:\,D;D'}\begin{pmatrix}z\\w\end{pmatrix}$ function (1.5) and the generalized Kampé de Fériet function [31], denoted by $F^{A\,:\,B;B'}_{C\,:\,D;D'}\begin{bmatrix}z\\w\end{bmatrix}$, is found in the form

$$(1.6)\quad \frac{\prod_{j=1}^{C}\Gamma(c_j)}{\prod_{j=1}^{A}\Gamma(a_j)}\frac{\prod_{j=1}^{D}\Gamma(d_j)}{\prod_{j=1}^{B}\Gamma(b_j)}\frac{\prod_{j=1}^{D'}\Gamma(d'_j)}{\prod_{j=1}^{B'}\Gamma(b'_j)}S^{A\,:\,B;B'}_{C\,:\,D;D'}\left(\begin{matrix}[(a):1,1]:[(b):1];[(b'):1];\\ [(c):1,1]:[(d):1];[(d'):1];\end{matrix}z,w\right)$$

$$=F^{A\,:\,B;B'}_{C\,:\,D;D'}\left[\begin{matrix}(a_j)_{1,A}:(b_j)_{1,B};(b'_j)_{1,B'};\\ (c_j)_{1,C}:(d_j)_{1,D};(d'_j)_{1,D'};\end{matrix}z,w\right].$$

The generalized Kampé de Fériet function, given in right hand side of (1.6), has the relations with various one and two variables functions of Appell's and Lauricella's functions used in various fields of science and technologies (see in [6], [29], [30]). Recently in [25], Pathan and Kumar presented a representation of multi-parametric Mittag - Leffler function in terms of Srivstava and Daoust function (1.5) and used in analysis of multivariable Cauchy residue theorem. On the other hand, currently, Chandel and Kumar [1] established two contour integral representations involving Mittag - Leffler functions (i) for a two variable generalized hypergeometric function of Srivastava and Daoust function (1.5) and (ii) a sum of the Kummer's confluent hypergeometric functions (1.4). Motivated by above researches, we will introduce a new contour integral in (2.3) in the complex t- plane for Srivastava and Daoust function (1.5) and then obtain various results and relations through operational techniques. Finally, we use these results in solving of some of the initial value problems consisting of Caputo fractional derivatives and Riemann - Liouville fractional integrals.

2 The contour integral representation for Srivastava and Daoust function and related special cases

***Lemma* 2.1** *If in the complex t - plane,* $\alpha,\beta,z,y\in\mathbb{C}$, $|\arg(y)|<\frac{\pi}{2}$, *and for* $c>\Re(t)$, $\min\{\Re(t),\Re(z),\Re(\beta)\}>0$, *then, there exists a contour integral for Kummer's confluent hypergeometric function as*

$$(2.1)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{yt}t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}dt=\tfrac{y^{\beta-1}}{\Gamma(\beta)}{}_1F_1(\alpha;\beta;yz).$$

Then,

$$(2.2)\quad \frac{1}{\Gamma(\beta)}\int_0^{\infty}e^{-ty}y^{\beta-1}{}_1F_1(\alpha;\beta;yz)dy=\tfrac{1}{t^{\beta}}\left(1-\tfrac{z}{t}\right)^{-\alpha}$$

Again then,

$$\int_0^{\infty}e^{-ty}\left[\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{y\xi}\xi^{-\beta}\left(1-\tfrac{z}{\xi}\right)^{-\alpha}d\xi\right]dy=y^{-\beta}\left(1-\tfrac{z}{y}\right)^{-\alpha}\delta_{ty};$$

$$\delta_{mn}=\begin{cases}0, m\neq n,\\ 1, m=n;\end{cases}\ \textit{being the Dirac - delta function.}$$

Proof. In the left hand side of first integral in (2.2), expand ${}_1F_1(\alpha;\beta;yz)$ by the series (1.4) and thus find that

$$\sum_{k=0}^{\infty}\tfrac{(\alpha)_k z^k}{\Gamma(\beta+k)}\int_0^{\infty}e^{-ty}y^{\beta+k-1}dy=t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha},\left(\text{since, }\int_0^{\infty}e^{-ty}y^{\beta+k-1}dy=\tfrac{\Gamma(\beta+k)}{t^{\beta+k}}\right).$$

Further, apply the formula (2.1) in the left hand side of equation (2.2) to get

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\xi^{-\beta}\left(1-\tfrac{z}{\xi}\right)^{-\alpha}\left\{\int_0^{\infty}e^{-y(t-\xi)}dy\right\}d\xi=\tfrac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\tfrac{y^{-\beta}\left(1-\frac{z}{y}\right)^{-\alpha}}{y-t}dy=y^{-\beta}\left(1-\tfrac{z}{y}\right)^{-\alpha}\delta_{ty}.$$

Hence, by the statement of the ***Lemma* 2.1**, we get the equalities of (2.2). (Also, see Erdélyi et al. [5, p. 217], Srivastava and Manocha [30, p.219]).

Particularly, for $z=1$, the contour integral relation in (2.1) becomes the integral relation in (1.4).

***Theorem* 2.1** *In the complex t- plane, if $\nu,\mu \in \mathbb{R}$ such that $\nu > 0, \mu > 0$ and $(\nu+\mu) > 0$, and also $\alpha,\beta,\rho,w,z,y,\lambda \in \mathbb{C}$, $|\arg(y)| < \frac{\pi}{2}$, where, $\lambda \neq 0, \Re(\rho) > 0$, then, for $c > \Re(t)$, and $\min\{\Re(t),\Re(z),\Re(\beta)\} > 0$, there exists a contour integral $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha}dt$, and thus it has the equality for Srivastava - Daoust function as*

$$(2.3)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha}dt \quad = y^{\beta-1}S^{\,1:1;0}_{\,1:2;0}\left(\begin{matrix}[\alpha:\mu,1]:[1:1];[-:-]; \\ [\beta:\mu,1]:[\rho:\nu],[\alpha:\mu];[-:-];\end{matrix}\lambda w^{\nu}y^{\mu}, yz\right),$$

provided that $(\nu+\mu) > 0$.

Proof. In left hand side of (2.3), define the Mittag - Leffler function (1.2), to get the equality

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha}dt = \sum_{k=0}^{\infty}\frac{(\lambda w^{\nu})^k}{\Gamma(\nu k+\rho)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}t^{-\beta-\mu k}\left(1-\frac{z}{t}\right)^{-\alpha-\mu k}dt,$$

and then, in right hand side of this equality apply the formula (2.1), to find the double series

$$y^{\beta-1}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\Gamma(\alpha+\mu k+m)\Gamma(1+k)}{\Gamma(\beta+\mu k+m)\Gamma(\rho+\nu k)\Gamma(\alpha+\mu k)}\frac{(\lambda w^{\nu}y^{\mu})^k}{k!}\frac{(yz)^m}{m!}.$$

Again, with the help of convergence conditions of (1.5), this double series converges for $(\nu+\mu) > 0$. Hence, by the definition of Srivastava and Daoust function (1.5), the following formula holds

$$(2.4)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha}dt = y^{\beta-1}S^{\,1:1;0}_{\,1:2;0}\left(\begin{matrix}[\alpha:\mu,1]:[1:1];[-:-]; \\ [\beta:\mu,1]:[\rho:\nu],[\alpha:\mu];[-:-];\end{matrix}\lambda w^{\nu}y^{\mu}, yz\right),$$

provided that $(\nu+\mu) > 0$.

Hence, the equality (2.3) is followed.

***Corollary* 2.1** *For all conditions of the **Theorem 2.2** and on specialization of the parameters, with $\nu = \mu = 1$, of the formula* (2.3) *following equality holds for the Kampé de Fériet function* (1.6) *as*

$$(2.5)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{1,\rho}(\lambda w(t-z)^{-1})t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha}dt = \frac{y^{\beta-1}}{\Gamma(\beta)\Gamma(\rho)}F^{\,1:1;0}_{\,1:2;0}\left(\begin{matrix}[\alpha:1,1]:[1:1];[-:-]; \\ [\beta:1,1]:[\rho:1],[\alpha:1];[-:-];\end{matrix}\lambda wy, yz\right).$$

***Remark* 2.1** *Again, on specialization of some other parameters in the formula* (2.3) *following special cases are discussed:*

***Special case* 2.1.1** *In the formula* (2.3)*, set $\rho = 1, \alpha, w \to 0, \beta = k+1$, then, it becomes Cauchy integral formula as* [25]

$$(2.6)\quad \lim_{\alpha,w\to 0}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu,1}(\lambda w^{\nu}(t-z)^{-\mu})t^{-k-1}\left(1-\frac{z}{t}\right)^{-\alpha}dt = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}t^{-k-1}dt = \frac{y^k}{k!}.$$

***Special case* 2.1.2** *Again, set $\rho = 1, z = 1$, in the formula* (2.3)*, then, by use of the formula* (1.4)*, and for the conditions given in the **Theorem 2.2**, its limiting case for $w \to 0$, gives us the equalities*

$$(2.7)\quad \lim_{w\to 0}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu,1}(\lambda w^{\nu}(t-1)^{-\mu})t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt = \frac{z^{1-\beta}\Gamma(\beta)}{2\pi i}\int_{-\infty}^{(0^+,1^+)} e^{yt}t^{-\beta}\left(1-\frac{1}{t}\right)^{-\alpha}dt = M(\alpha,\beta,y)$$

***Special case* 2.1.3** *Further, in the Eqn.* (2.3)*, set $\nu = \mu, w = t, y = -1, z = \frac{-x}{(-t)^{\kappa-1}}, \beta = 1, |\kappa| < 1$, and consider that $f\left(\frac{x}{(-t)^{\kappa}}\right) = -\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(\alpha+\mu k)_m\frac{(\lambda)^k}{\Gamma(\mu k+\rho)}\left(\frac{x}{(-t)^{\kappa}}\right)^m$, then, it becomes the inverse Borel transformation formula of that function $f\left(\frac{x}{(-t)^{\kappa}}\right)$ as (see in* [3]*)*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-t}E_{\mu,\rho}(\lambda\left(1-\frac{x}{(-t)^{\kappa}}\right)^{-\mu})t^{-1}(1-\frac{x}{(-t)^{\kappa}})^{-\alpha}dt = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(\alpha+\mu k)_m\frac{(\lambda)^k}{\Gamma(\mu k+\rho)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{-t}}{t}\left(\frac{x}{(-t)^{\kappa}}\right)^m dt$$

$$= \frac{i}{2\pi}\int_{c-i\infty}^{c+i\infty}\frac{e^{-t}}{t}f(\frac{x}{(-t)^{\kappa}})dt, |\kappa| < 1.$$

Here,

$$(2.8)\quad f\left(\frac{x}{(-t)^{\kappa}}\right) = -\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}(\alpha+\mu k)_m\frac{(\lambda)^k}{\Gamma(\mu k+\rho)}\left(\frac{x}{(-t)^{\kappa}}\right)^m.$$

3 Operational techniques by fractional and partial derivatives on the contour integrals

In this section, we operate the contour integral defined in the ***Theorem* 2.1** by the Caputo fractional and partial derivatives and then obtain various results and relations.

The Caputo fractional derivative of the function $f(t)$, denoted by ${}_{t}^{C}D_{0^+}^{\alpha} f(t)$ where, $m-1<\alpha\le m, \forall m\in\mathbb{N}$, is defined by ([4], [12], [21])

(3.1) $({}_{t}^{C}D_{0^+}^{\alpha} f)(t) = (I^{m-\alpha} f^{(m)})(t),$

where, $f^{(m)}(t) = D_t^m f(t), \left\{D_t^m \equiv \frac{d^m}{dt^m} = \frac{d}{dt}\left(\frac{d^{m-1}}{dt^{m-1}}\right)\right\}, I^{m-\alpha}$ being the Riemann - Liouville fractional integral ([4], [12], [23])

(3.2) $(I^{m-\alpha} f)(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} f(\tau)d\tau, t>0, m-1<\alpha\le m, \\ f(t), \alpha=m, \forall m\in\mathbb{N}. \end{cases}$

The operation of Caputo derivative (3.1); $\forall m\in\mathbb{N}$, $m-1<\nu\le m$; on the Mittag - Leffler function (1.1) is found as ([4], [12])

(3.3) ${}_{w}^{C}D_{0^+}^{\nu} E_\nu(\lambda w^\nu) = \lambda E_\nu(\lambda w^\nu).$

***Theorem* 3.1** *In the complex t- plane, if $\nu,\mu\in\mathbb{R}$ such that $m-1<\nu\le m, \forall m\in\mathbb{N}, \mu>0$ and $(\nu+\mu)>0$, and $\alpha,\beta,y,w,z,\rho,\lambda\in\mathbb{C}, |\arg(y)|<\frac{\pi}{2}$, where, $\lambda\neq 0, \Re(\rho)>0$, then, for $c>\Re(t)$ and $\min\{\Re(t),\Re(z),\Re(\beta)\}>0$, there exists a contour integral* (2.3) *in the form*

(3.4) $\dfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt} E_{\nu,\rho}(\lambda w^\nu (t-z)^{-\mu}) t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha} dt = I_\lambda^{\alpha,\beta,\nu,\rho,\mu}(w,z;y),$ *(let).*

Thus, following operational formulae hold

(3.5) ${}_{w}^{C}D_{0^+}^{\nu}\left\{I_\lambda^{\alpha,\beta,\nu,1,\mu}(w,z;y)\right\} = \lambda I_\lambda^{\alpha+\mu,\beta+\mu,\nu,1,\mu}(w,z;y).$

and

(3.6) $\nu\dfrac{\partial}{\partial y}\left\{I_\lambda^{\alpha,\beta,\nu,\rho,\mu}(w,z;y)\right\} = \nu I_\lambda^{\alpha,\beta-1,\nu,\rho,\mu}(w,z;y).$

Proof. Operate (3.4) by the Caputo derivative (3.1) with respect to w to get

$$\begin{aligned} {}_{w}^{C}D_{0^+}^{\nu}\left\{I_\lambda^{\alpha,\beta,\nu,1,\mu}(w,z;y)\right\} &= {}_{w}^{C}D_{0^+}^{\nu}\left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt} E_{\nu,1}(\lambda w^\nu (t-z)^{-\mu}) t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha} dt\right\} \\ &= \left\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}\, {}_{w}^{C}D_{0^+}^{\nu}\left\{E_\nu(\lambda w^\nu (t-z)^{-\mu})\right\} t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha} dt\right\}, \end{aligned}$$

since then, in the first and last relations, applying the relations of Mittag - Leffler function, given in (1.3) and (3.3) to get the identities

$${}_{w}^{C}D_{0^+}^{\nu}\left\{I_\lambda^{\alpha,\beta,\nu,1,\mu}(w,z;y)\right\} = \left\{\frac{\lambda}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}\left\{E_{\nu,1}(\lambda w^\nu (t-z)^{-\mu})\right\} t^{-\beta-\mu}\left(1-\tfrac{z}{t}\right)^{-\alpha-\mu} dt\right\} = \lambda I_\lambda^{\alpha+\mu,\beta+\mu,\nu,1,\mu}(w,z;y).$$

In the similar manner to find

$$\nu\frac{\partial}{\partial y}\left\{I_\lambda^{\alpha,\beta,\nu,\rho,\mu}(w,z;y)\right\} = \left\{\frac{\nu}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt} E_{\nu,\rho}(\lambda w^\nu (t-z)^{-\mu}) t^{1-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha} dt\right\} = \nu I_\lambda^{\alpha,\beta-1,\nu,\rho,\mu}(w,z;y).$$

Hence, the results (3.5) and (3.6) hold good.

***Theorem* 3.2** *If in the complex t- plane, $\nu,\mu\in\mathbb{R}$ such that $m-1<\nu\le m, \forall m\in\mathbb{N}, \mu>0$ and $(\nu+\mu)>0$, and $\alpha,\beta, w,y,z,\lambda\in\mathbb{C}$, $|\arg(y)|<\frac{\pi}{2}$, where, $\lambda\neq 0$, then, for $c>\Re(t)$ and $\min\{\Re(t),\Re(z),\Re(\beta)\}>0$, there exists a contour integral*

(3.7) $\dfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt} E_\nu(\lambda w^\nu (t-z)^{-\mu}) t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha} dt = I_\lambda^{\alpha,\beta,\nu,\mu}(w,z;y).$

Again then, following operational formulae hold

(3.8) $\int_0^\infty\left[\exp(-ty + {}_{w}^{C}D_{0^+}^{\nu})\left\{I_\lambda^{\alpha,\beta,\nu,\mu}(w,z;y)\right\}\right]dy = \exp\left\{\frac{\lambda}{(t-z)^\mu}\right\} E_\nu(\lambda w^\nu (t-z)^{-\mu}) t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha}.$

and

(3.9) $\int_0^\infty\left[\exp\left(-ty + \nu\frac{\partial}{\partial y}\right)\left\{I_\lambda^{\alpha,\beta,\nu,\rho,\mu}(w,z;y)\right\}\right]dy = \exp\{\nu t\} E_\nu(\lambda w^\nu (t-z)^{-\mu}) t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha}.$

Proof. On application of the ***Theorem* 3.1**, and by the properties of Mittag - Leffler function given in (1.3) and (3.3), we obtain

(3.10) $$({}^{C}_{w}D^{\nu}_{0^+})^n\Big\{I_{\lambda}^{\alpha,\beta,\nu,\mu}(w,z;y)\Big\} = \frac{\lambda^n}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta-\mu n}\left(1-\tfrac{z}{t}\right)^{-\alpha-\mu n}dt = \lambda^n I_{\lambda}^{\alpha+\mu n,\beta+\mu n,\nu,\mu}(w,z;y).$$

and

(3.11) $$\left(\nu\frac{\partial}{\partial y}\right)^n\Big\{I_{\lambda}^{\alpha,\beta,\nu,\mu}(w,z;y)\Big\} = \frac{\nu^n}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu}(\lambda w^{\nu}(t-z)^{-\mu})t^{n-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}dt = \nu^n I_{\lambda}^{\alpha,\beta-n,\nu,\mu}(w,z;y).$$

Now, multiply by $\frac{1}{n!}$ in both of the sides of Eqns. (3.10) and (3.11) and then summing up n from $n=0$ to $n=\infty$ to get

(3.12) $$\exp({}^{C}_{w}D^{\nu}_{0^+})\Big\{I_{\lambda}^{\alpha,\beta,\nu,\mu}(w,z;y)\Big\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}\exp\Big\{\tfrac{\lambda}{(t-z)^{\mu}}\Big\}dt$$

and

(3.13) $$\exp\left(\nu\frac{\partial}{\partial y}\right)\Big\{I_{\lambda}^{\alpha,\beta,\nu,\rho,\mu}(w,z;y)\Big\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha}\exp\Big\{\nu t\Big\}dt$$

Now, multiply by e^{-ty} both the sides of (3.12) and (3.13) and then integrate to them with respect to y from $y=0$ to $y=\infty$, and with the help of the ***Lemma* 2.1**, we find the operational relations (3.8) and (3.9).

***Theorem* 3.3** *If in the complex t- plane,* $\alpha,\beta,y,z,w\in\mathbb{C}$, *and* $|\arg(y)|<\frac{\pi}{2}$, *then, for* $c>\Re(t)$ *and* $\min\Big\{\Re(w),\Re(t),\Re(\beta)\Big\}>0$*, there exists a contour integral for* $K\in\mathbb{N}^*=\Big\{2,3,4,\ldots,L\Big\}, L<\infty$,

(3.14) $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\frac{1}{K}}((zt)^{\frac{1}{K}})t^{-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt = I_{1}^{\alpha,\beta,K}(z;y,w),$$

then for $z,z_0\in\mathbb{C}$, *a function* $F(z)$ *is defined as such that* $\int_{z_0}^{z}F(z)dz<\infty$, *and thus, following equalities hold*

(3.15) $$\int_{z_0}^{z}F(z)d\Big\{I_{1}^{\alpha,\beta,K}(z;y,w)\Big\} = \sum_{r=0}^{K-1}\frac{\Big\{I_{1}^{\alpha,\beta+\frac{r}{K}-1,K}(0;y,w)\Big\}}{\Gamma(1-\frac{r}{K})}\int_{z_0}^{z}F(z)z^{-\frac{r}{K}}dz$$
$$= \sum_{r=0}^{K-1}\Big\{{}_1F_1(\alpha;\beta+\tfrac{r}{K}-1;wy)\Big\}\frac{y^{\beta+\frac{r}{K}-2}}{\Gamma(\beta+\frac{r}{K}-1)\Gamma(1-\frac{r}{K})}\int_{z_0}^{z}F(z)z^{-\frac{r}{K}}dz.$$

Proof. By the relation (3.14), we write

$$\frac{d}{dz}\Big\{I_{1}^{\alpha,\beta,K}(z;y,w)\Big\} = \frac{d}{dz}\Big\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{(y+z)t}\Big\{e^{-zt}E_{\frac{1}{K}}((zt)^{\frac{1}{K}})\Big\}t^{-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\}$$
$$= \Big\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{(y+z)t}\Big\{e^{-zt}E_{\frac{1}{K}}((zt)^{\frac{1}{K}})\Big\}t^{1-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\}$$
$$+\Big\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{(y+z)t}\frac{d}{d\phi}\Big\{e^{-\phi}E_{\frac{1}{K}}((\phi)^{\frac{1}{K}})\Big\}t^{1-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\}, \Big\{\text{ on setting } \phi=zt\Big\},$$

and since then, applying the result by Mathai and Haubold [21, p. 84], we have a relation

$$\frac{d}{d\phi}\Big\{e^{-\phi}E_{\frac{1}{K}}((\phi)^{\frac{1}{K}})\Big\} = e^{-\phi}\sum_{r=1}^{K-1}\frac{\phi^{-\frac{r}{K}}}{\Gamma(1-\frac{r}{K})}$$

and thus to find

$$\frac{d}{dz}\Big\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\frac{1}{K}}((zt)^{\frac{1}{K}})t^{-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\} = \Big\{\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\frac{1}{K}}((zt)^{\frac{1}{K}})t^{1-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\}$$
$$+\sum_{r=1}^{K-1}\frac{z^{-\frac{r}{K}}}{\Gamma(1-\frac{r}{K})}\Big\{\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}t^{1-\frac{r}{K}-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\} = \sum_{r=0}^{K-1}\frac{z^{-\frac{r}{K}}}{\Gamma(1-\frac{r}{K})}\Big\{\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}t^{1-\frac{r}{K}-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\}$$

It becomes

(3.16) $$\int_{z_0}^{z}F(z)d\Big\{\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\frac{1}{K}}((zt)^{\frac{1}{K}})t^{-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\} = \sum_{r=0}^{K-1}\int_{z_0}^{z}F(z)\frac{z^{-\frac{r}{K}}}{\Gamma(1-\frac{r}{K})}dz\Big\{\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}t^{1-\frac{r}{K}-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt\Big\}.$$

Again, by (3.14), we again write

$$\lim_{z\to 0}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\frac{1}{K}}((zt)^{\frac{1}{K}})t^{-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt = \lim_{z\to 0}I_{1}^{\alpha,\beta,K}(z;y,w)$$

and since by the definition (1.1) , there exists a result $\lim_{z\to 0}E_{\frac{1}{K}}((zt)^{\frac{1}{K}})=1$, and hence by ***Lemma* 2.1**, there implies that

(3.17) $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}t^{-\beta}\left(1-\tfrac{w}{t}\right)^{-\alpha}dt = I_{1}^{\alpha,\beta,K}(0;y,w) = \frac{y^{\beta-1}}{\Gamma(\beta)}{}_1F_1(\alpha;\beta;yw).$$

Finally, using the definition of (3.14) and the relation of (3.17) in the result (3.16), we obtain the equalities of (3.15).

***Theorem* 3.4** *If in the* ***Theorem* 3.2** *, set* $\nu = 1$, $w = x^2$, $x, \mu \in \mathbb{R}$, $\mu > 0$ *and* $\alpha, \beta, \rho, z, y \in \mathbb{C}$, $|\arg(y)| < \frac{\pi}{2}$, $\lambda = -1$, *replace* ρ *by* $\rho + 1$, *then, for* $c > \Re(t)$ *and* $\min\{\Re(t), \Re(z), \Re(\beta)\} > 0$, *there exists a contour integral*

$$(3.18)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt} E_{1,\rho+1}(-x^2(t-z)^{-\mu}) t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha} dt = I_{-1}^{\alpha,\beta,1,\rho,\mu}(x^2, z; y).$$

Again then, following formula holds

$$(3.19)\quad \int_{-\infty}^{\infty} I_{-1}^{\alpha,\beta,1,\rho,\mu}(x^2, z; y)dx = \frac{1}{\Gamma(\rho+\frac{1}{2})}\frac{y^{\beta-\frac{\mu}{2}-1}}{\Gamma(\beta-\frac{\mu}{2})}\,{}_1F_1(\alpha-\tfrac{\mu}{2}; \beta-\tfrac{\mu}{2}; yz),$$

provided that $\rho > -\frac{1}{2}$ *and* $\beta - \frac{\mu}{2} \neq 0, -1, -2, \ldots, |yz| < \infty$.

Proof. Integrate both sides of (3.18) with respect to x from $x = -\infty$ to $x = \infty$, to get

$$\int_{-\infty}^{\infty} I_{-1}^{\alpha,\beta,1,\rho,\mu}(x^2, z; y)dx = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\int_{c-i\infty}^{c+i\infty} e^{yt} E_{1,\rho+1}(-x^2(t-z)^{-\mu}) t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha} dtdx$$

Now, in the right hand side of the double integral of above equality, change the order of integration and set $x^2(t-z)^{-\mu} = y^2$, to find that

$$\int_{-\infty}^{\infty} I_{-1}^{\alpha,\beta,1,\rho,\mu}(x^2, z; y)dx = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt} t^{-\beta+\frac{\mu}{2}}\left(1-\frac{z}{t}\right)^{-\alpha+\frac{\mu}{2}} \int_{-\infty}^{\infty} E_{1,\rho+1}(-y^2)dydt,$$

where, in right hand side, in the inner integral use the Borel transformation formula (1.3), we get

$$\int_{-\infty}^{\infty} I_{-1}^{\alpha,\beta,1,\rho,\mu}(x^2, z; y)dx = \frac{1}{2\pi i}\frac{1}{\Gamma(\rho+\frac{1}{2})}\int_{c-i\infty}^{c+i\infty} e^{yt} t^{-\beta+\frac{\mu}{2}}\left(1-\frac{z}{t}\right)^{-\alpha+\frac{\mu}{2}} dt,$$

which on defining by the formula given in ***Lemma* 2.1**, we obtain

$$(3.20)\quad \int_{-\infty}^{\infty} I_{-1}^{\alpha,\beta,1,\rho,\mu}(x^2, z; y)dx = \frac{1}{\Gamma(\rho+\frac{1}{2})}\frac{y^{\beta-\frac{\mu}{2}-1}}{\Gamma(\beta-\frac{\mu}{2})}\,{}_1F_1(\alpha-\tfrac{\mu}{2}; \beta-\tfrac{\mu}{2}; yz),$$

which is valid for $\rho + \frac{1}{2} > 0$ and $\beta - \frac{\mu}{2} \neq 0, -1, -2, \ldots$ and $|yz| < \infty$.

Hence, by the result (3.20), the ***Theorem* 3.4** has followed.

4 Application with numerical examples

In this section, we apply our above results in following problems:

***Problem* 4.1** *If* $I_{\lambda}^{\alpha,\beta,\nu,\rho,\mu}(w, z; y) = ({}_{y}^{C}D_{0^+}^{\eta} F)(y)$, *where the* $I_{\lambda}^{\alpha,\beta,\nu,\rho,\mu}(w, z; y)$ *is given in (3.4) and* ${}_{t}^{C}D_{0^+}^{\eta}$ *being the Caputo fractional derivative* (3.1)*, for* $m - 1 < \eta \leq m, \forall m \in \mathbb{N}$*, and with the initial conditions* $\left.\frac{d^{m-1}}{dy^{m-1}}F(y)\right|_{y=0^+} = 0\ \forall m = 1, 2, 3, \ldots.$

Then, following solution exists

$$(4.1)\quad F(y) = I_{\lambda}^{\alpha,\beta+\eta,\nu,\rho,\mu}(w, z; y) = y^{\beta+\eta-1} S^{1:1;0}_{1:2;0}\left(\begin{matrix}[\alpha:\mu,1]:[1:1];[-:-];\\ [\beta+\eta:\mu,1]:[\rho:\nu],[\alpha:\mu];[-:-];\end{matrix}\lambda w^{\nu} y^{\mu}, yz\right),$$

provided that $(\nu + \mu) > 0$.

***Solution*.** On applying (3.4), the equation, $I_{\lambda}^{\alpha,\beta,\nu,\rho,\mu}(w, z; y) = ({}_{y}^{C}D_{0^+}^{\eta} F)(y)$, of the ***Problem* 4.1**, is written as

$$(4.2)\quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt} E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu}) t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha} dt = ({}_{y}^{C}D_{0^+}^{\eta} F)(y)$$

Now, multiply by $e^{-yt}, t > 0$, in both of the sides of (4.2) , and integrate it with respect to y, (from $y = 0$ to $y = \infty$) to get as

$$(4.3)\quad \int_{0}^{\infty}\frac{1}{2\pi i}e^{-yt}\int_{c-i\infty}^{c+i\infty} e^{yt} E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu}) t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha} dtdy = \int_{0}^{\infty} e^{-yt}\left({}_{y}^{C}D_{0^+}^{\eta} F\right)(y)dy,$$

Then, in right hand side of (4.3), apply the result by [12] as

$$\int_{0}^{\infty} e^{-yt}\left({}_{y}^{C}D_{0^+}^{\eta} F\right)(y)dy = t^{\eta} f(t) - \sum_{k=0}^{m-1} t^{\eta-1-k} F^{(k)}(0^+)\forall m - 1 < \eta \leq m,$$

where, $f(t) = \int_{0}^{\infty} e^{-yt}F(y)dy$,and inversely, $F(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt} f(t)dt$ and thus use the ***Lemma* 2.1**, with the initial conditions given in the ***Problem* 4.1**, we find

$$(4.4)\quad f(t) = E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu}) t^{-\beta-\eta}\left(1-\frac{z}{t}\right)^{-\alpha}$$

Finally, on taking contour integration (see, inverse Laplace transformation before (4.4)) of both sides of (4.4) and using by (3.4) to obtain equality in first and second results of (4.1) as

$$(4.5)\quad F(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt} f(t)dt = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt} E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu}) t^{-\beta-\eta}\left(1-\frac{z}{t}\right)^{-\alpha} dt = I_{\lambda}^{\alpha,\beta+\eta,\nu,\rho,\mu}(w, z; y).$$

In right hand side of (4.5), use the ***Theorem* 2.2**, we find equality in first and third results of (4.1).

***Problem* 4.2** *If $I_{\lambda}^{\alpha,\beta,\nu,\rho,\mu}(w,z;y) = (I^{\eta}F)(y)$, where the $I_{\lambda}^{\alpha,\beta,\nu,\rho,\mu}(w,z;y)$ is given in (3.4) and I^{η} being the Riemann - Liouville fractional integral of order η defined in (3.2) for $\eta > 0$, then*

(4.6) $F(y) = I_{\lambda}^{\alpha,\beta-\eta,\nu,\rho,\mu}(w,z;y)$.

***Solution*.** On using formula (3.4), the equation in ***Problem* 4.2** is written by

(4.7) $$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha} dt = (I^{\eta}F)(y).$$

Now, in both sides of (4.7), multiply e^{-yt}, $t > 0$, and then integrate them with respect to y from $y = 0$ to $y = \infty$, to get that

$$\int_0^{\infty} e^{-yt}\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\tfrac{z}{t}\right)^{-\alpha} dtdy = \int_0^{\infty} e^{-yt}\left\{\tfrac{1}{\Gamma(\eta)}\int_0^{y}(y-x)^{\eta-1}F(x)dx\right\}dy$$

Since, by Laplace convolution theorem, there exists a relation

$$\frac{1}{\Gamma(\eta)}\int_0^{y}(y-x)^{\eta-1}F(x)dx = \tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}t^{-\eta}f(t)dt,$$

and hence right hand side becomes

$$= \int_0^{\infty} e^{-yt}\left\{\tfrac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}t^{-\eta}f(t)dt\right\}dy.$$

Then, in both of the sides, apply the ***Lemma* 2.1**, to obtain

$$E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha} = t^{-\eta}f(t)$$

or to find

$$f(t) = E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-(\beta-\eta)}\left(1-\frac{z}{t}\right)^{-\alpha},$$

again, use the formula given in (4.4), we get

$$F(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{yt}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-(\beta-\eta)}\left(1-\tfrac{z}{t}\right)^{-\alpha} dt = I_{\lambda}^{\alpha,\beta-\eta,\nu,\rho,\mu}(w,z;y).$$

Finally, we obtain the result (4.6).

***Problem* 4.3** *If $I_{\lambda}^{\alpha,\beta,\nu,\rho,\mu}(w,z;y) = \frac{d}{dy}F(y) + \left({}_{y}^{C}D_{0^+}^{\eta}F\right)(y)$, where the $I_{\lambda}^{\alpha,\beta,\nu,\rho,\mu}(w,z;y)$ is given in* (3.4) *and ${}_{t}^{C}D_{0^+}^{\eta}$ being the Caputo fractional derivative* (3.1)*, for $m-1 < \eta \le m, \forall m \in \mathbb{N}$, and with the initial conditions , $\frac{d^{m-1}}{dy^{m-1}}F(y)\Big|_{y\,=\,0^+} = 0 \forall m = 1, 2, 3, \ldots$.*

Then

(4.8) $F(y) = \sum_{k=0}^{\infty}(-1)^k I_{\lambda}^{\alpha,\beta+\eta+(\eta-1)k,\nu,\rho,\mu}(w,z;y)$.

***Solution*.** In the similar manner of the ***Problem* 4.1**, for $m-1 < \eta \le m, \forall m \in \mathbb{N}$, by ***Problem* 4.3**, we have

(4.9) $$(t+t^{\eta})f(t) = E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta}\left(1-\frac{z}{t}\right)^{-\alpha}.$$

Then, by Eqn. (4.9), it implies that

$$f(t) = (1+t^{1-\eta})^{-1}E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta-\eta}\left(1-\frac{z}{t}\right)^{-\alpha},$$

and then we have

$$f(t) = \sum_{k=0}^{\infty}(-1)^k E_{\nu,\rho}(\lambda w^{\nu}(t-z)^{-\mu})t^{-\beta-\eta-(\eta-1)k}\left(1-\tfrac{z}{t}\right)^{-\alpha}$$

and then inversing it by formula (4.5), finally, we get the solution (4.8) as

(4.10) $F(y) = \sum_{k=0}^{\infty}(-1)^k I_{\lambda}^{\alpha,\beta+\eta+(\eta-1)k,\nu,\rho,\mu}(w,z;y)$.

Concluding remarks

This research work centralizes about a contour integral defined in the ***Theorem* 2.1** for the Srivastava and Daoust function of two variables (1.5). Specially, it gives the formula for the generalized Kampé de Fériet function (1.6) in the ***Corollary* 2.1.** Other applicable special cases are also discussed in the ***Remark* 2.1**. In the ***Section* 3**, by partial and Caputo fractional derivative operation techniques, various results and relations are obtained, in which ***Theorem* 3.2** may generate many generating functions and relations for various multiple special functions through Lie group theoretic techniques. By the Borel transforms [3], (see in the ***Theorem* 3.4**) obtained results may use in quantum mechanical problems. Next, we use and discuss our results in some numerical problems consisting of Caputo fractional derivative and Riemann Liouville integrals. These fractional operators are helpful in construction and solving of many diffusion and wave problems (see for example [8] - [10], [12], [13], [19], [21], [23] and others). It is also remarked that the Laplace operator techniques play an important role in solving the given problems consisting of contour integral in (3.4) and has a relation with two variables Srivastava - Daoust function (1.5) in Eqn. (2.3) and another relation with Kampé de Fériet function in (2.5). Also other results and relations by different contour integral formulae have carried out in the ***Sections* 2** and **3**. Hence, the contour integral in (2.3) becomes very much useful in doing of further researches in the area of fractional calculus along with the fractional diffusion and wave problems occurring in the modern science and technology.

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