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A CLASS OF TWO VARIABLES SEQUENCE OF FUNCTIONS SATISFYING ABEL'S INTEGRAL EQUATION AND THE PHASE SHIFTS

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Abstract

In this paper, we introduce a class of two variables sequence of functions as satisfying Abel's integral equation in which unknown function is the potential function and again consider that the Riemann - Liouville fractional integral of this class of functions equals to the slope of that potential function and then discuss some of its oscillatory properties and use them to evaluate the phase shifts in terms of arcsine of the series consisting of the Srivastava and Daoust's triple hypergeometric function.

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1 Introduction

In our investigation, we introduce a class of two variables functions in the form

$$f(r;\lambda,q)=\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}a_{p,n}(\lambda,q)\frac{r^{p+2}}{p+2!},\lambda\neq 0,q>0. \tag{1.1}$$

The class (1.1) consists of a general sequences $a_{p,n}(\lambda,q),\forall p\geq 0,n\geq 0,\lambda\neq 0,q>0$.

Set $a_{p,n}(\lambda,q)=\frac{\prod_{j=1}^{m}\Gamma(A_j+p+n)\prod_{j=1}^{\mu}\Gamma(C_j+p)\prod_{j=1}^{\nu}\Gamma(D_j+n)}{\prod_{j=1}^{m'}\Gamma(B_j+p+n)\prod_{j=1}^{\mu'}\Gamma(C'_j+p)\prod_{j=1}^{\nu'}\Gamma(D'_j+n)}\frac{\Gamma(p+3)\lambda^p}{\Gamma(p+1)}\frac{q^n}{n!}$ in Eqn. (1.1), then

$$f(r;\lambda,q)=r^2F^{m:\mu;\nu}_{m':\mu';\nu'}\left[\begin{matrix}(A)_{1,m}:(C)_{1,\mu};(D)_{1,\nu}\\(B)_{1,m'}:(C')_{1,\mu'};(D')_{1,\nu'}\end{matrix}\middle|\begin{matrix}\lambda r\\q\end{matrix}\right] \tag{1.2}$$

provided that for the convergence,

(i) $m+\mu<m'+\mu'+1,m+\nu<m'+\nu'+1;|\lambda r|<\infty,|q|<\infty,or$

(ii) $m+\mu=m'+\mu'+1,m+\nu=m'+\nu'+1;|\lambda r|<\infty,|q|<\infty$, and

(iii)

$$\begin{cases}|\lambda r|^{\frac{1}{(m-m')}}+|q|^{\frac{1}{(m-m')}}<1,if m>m';\\ \max\{|\lambda r|,|q|\}<1,if m\leq m'.\end{cases} \tag{1.3}$$

Here in Eqn. (1.2) - (1.3), the function $F^{m:\mu;\nu}_{m':\mu';\nu'}[\lambda r,q]$ is a generalized Kampé de Fériet function, a generalization of the Appell and Lauricella functions (see, Appell and Kampé de Fériet [2], Srivastava and Panda [22], Srivastava and Manocha [21]). Therefore, by the Eqns. (1.1), (1.2) and (1.3), several one and two variables hypergeometric functions may be found and studied.

The phase shifts are useful in various physical problems, for example the photo - ionization cross - sections and a reliable calculation of the impact ionization cross - section of an atom requires an accurate determination of the continuum wave functions in the incident and in the exit channels. In the theory of collisions, phase shifts determine the scattering cross - sections (Pain [12], Ikot et al. [6], Mahajan and Varma [10], Raghuwanshi and Sharma [14]). Many of workers have calculated and computed the phase shifts of scattering of electrons on considering the solvable potential functions for example, the exponential functions (Teitz [24], Bhattacharjie and Sudarshan [3], Ikot et al. [6]), binomial functions (Agrawal and Kumar [1], Kumar, Chandel and Agrawal [8]) and other special functions as Mittag - Leffler functions (Kumar and Singh [9]), Coulomb potentials (Pain [12]).

For further extensions and developments of this theory of collision and phase shifts, we introduce some more parameters in the potential function found by Abel's integral equation and consider an angular momentum, where, ε_{nL} is the binding energy of the excited electron in nL orbital, L the quantum number of $s-$ wave, then, for radius vector r and $K^2 = 2\varepsilon_{nL}$, $0 < \alpha < 1, \lambda \neq 0, q > 0, 0 \leq a \leq r \leq b$, the asymptotic solution of the s - wave Schrödinger equation

$$\frac{d^2}{dr^2}U_L(r) + [K^2 - V^\alpha(r;0,\lambda,q) - \frac{L(L+1)}{r^2}]U_L(r) = 0 \tag{1.4}$$

has been represented as (see Pain [12])

$$U_L(Kr) = C_L\sqrt{\frac{\pi K r}{2}}[\cos\{\eta_L(K)\}J_{L+\frac{1}{2}}(Kr) - sin\{\eta_L(K)\}J_{L-\frac{1}{2}}(Kr)]. \tag{1.5}$$

In Eqns. (1.4) and (1.5), C_L is a constant, $V^\alpha(r;0,\lambda,q)$ the potential function, with $0 < \alpha < 1, \lambda \neq 0, q > 0, 0 \leq r < \infty$, and $\eta_L(K)$ the phase shift for the quantum number L, $J_\nu(z), \nu > -1$ the classical Bessel function (see Rainville [15]).

Again, by the Eqn. (1.5), the phase shift difference formula has been given in the form (see Tietz [24])

$$\eta_L(K) - \eta_{L+1}(K) = arcsin[\frac{\pi}{2K}\int_0^\infty r\frac{d}{dr}V^\alpha(r;0,\lambda,q)J_{L+\frac{1}{2}}(Kr)J_{L+\frac{3}{2}}(Kr)dr]. \tag{1.6}$$

To make application of the fractional calculus techniques ([5], [16]) in our present work, we claim following definitions:

The Riemann - Liouville fractional differential operator is defined by (see Diethelm [5, p. 27])

$$D_a^\alpha f = D^m J_a^{m-\alpha} f, m-1 < \alpha \leq m, \forall \alpha \in \mathbb{R}, D^m f(x) = \frac{d^m}{dx^m}f(x), \tag{1.7}$$

where $m \in \mathbb{N}$, f be a function such that analytic on the interval $[a, b]$.

Here in Eqn. (1.7), the Riemann - Liouville integral $J_a^\alpha f$ is given by

$$J_a^\alpha f = \frac{1}{\Gamma(\alpha)}\int_a^x (x-\xi)^{\alpha-1} f(\xi)d\xi, a \leq x \leq b. \tag{1.8}$$

We also make an application of multivariable representation of the Kampé de Fériet function (1.2) - (1.3) and evaluate the phase shifts in terms of arcsine of the series consisting of the Srivastava and Daoust's triple hypergeometric function.

The multivariable representation of the Kampé de Fériet function (1.2) - (1.3) is the multivariable Srivastava and Daoust's hypergeometric function ([11], [17] - [19], [20], [21]) defined by

$$S^{A:B^{(')};\ldots;B^{(k)}}_{C:D^{(')};\ldots;D^{(k)}}\begin{pmatrix}[(a):\theta',\ldots,\theta^{(k)}]:((b'):\varphi');\ldots;((b^{(k)}):\varphi^{(k)}) & |z_1,\ldots,z_k\\ [(c):\psi',\ldots,\psi^{(k)}]:((d'):\delta');\ldots;((d^{(k)}):\delta^{(k)}) & \end{pmatrix}$$

$$= \sum_{m_1,\ldots,m_k=0}^{\infty} \frac{\prod_{j=1}^{A}\Gamma(a_j+m_1\theta_j'+\ldots+m_k\theta_j^{(k)})\prod_{j=1}^{B^{(')}}\Gamma(b_j'+m_1\varphi_j')\prod_{j=1}^{B^{(k)}}\Gamma(b_j^{(k)}+m_k\varphi_j^{(k)})}{\prod_{j=1}^{C}\Gamma(c_j+m_1\psi_j'+\ldots+m_k\psi_j^{(k)})\prod_{j=1}^{D^{(')}}\Gamma(d_j'+m_1\delta_j')\prod_{j=1}^{D^{(k)}}\Gamma(d_j^{(k)}+m_k\delta_j^{(k)})} \times \frac{z_1^{m_1}}{m_1!}\cdots\frac{z_k^{m_k}}{m_k!}$$

provided that for the convergence conditions,

$$1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\varphi_j^{(i)}\geq 0, |z_i|<\infty\forall i=1,\ldots,k. \tag{1.9}$$

In the next section 2, we consider the function (1.1) as a known function of an Abel's integral equation (2.1) and then, utilize its solution in obtaining of the phase shifts formula in terms of arcsine of the series consisting of the Srivastava and Daoust's triple hypergeometric function (1.9) (for $k=3$).

2 The function satisfying Abel's integral equation and the potential functions

To proceed our work, we suppose that the function defined in Eqn. (1.1) satisfies the Abel's integral equation given by

$$f(r-a;\lambda,q)=\Gamma(1-\alpha)\int_a^r (r-\xi)^{-\alpha}V^{\alpha}(\xi;a,\lambda,q)d\xi,$$
$$\text{where, } 0\leq a\leq r\leq b, 0<\alpha<1, \lambda\neq 0, q>0. \tag{2.1}$$

Then, the solution of (2.1) gives us a function

$$V^{\alpha}(r;a,\lambda,q)=\frac{1}{\Gamma(\alpha)}\frac{d}{dr}\int_a^r (r-\xi)^{\alpha-1}f(\xi-a;\lambda,q)d\xi, \forall 0\leq a\leq r\leq b, 0<\alpha<1. \tag{2.2}$$

Again then due to Eqns. (1.1) and (2.2), we find

$$V^{\alpha}(r;a,\lambda,q)=\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}a_{p,n}(\lambda,q)\frac{(r-a)^{p+\alpha+1}}{\Gamma(p+\alpha+2)}, \forall 0<\alpha<1, \lambda\neq 0, q>0, 0\leq a\leq r\leq b. \tag{2.3}$$

Particularly, put $a_{p,n}(\lambda,q)=(-1)^n\lambda^p\frac{\Gamma(p+\alpha+2)}{\Gamma(p+1)}\frac{(n+1)^p}{q^{n+1}}$ $\forall p\geq 0, n\geq 0,$ $\lambda\neq 0$ in (2.3), and on changing the order of summation to find that

$$V^{\alpha}(r;a,\lambda,q)=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}(-1)^n\lambda^p(r-a)^{\alpha+1}\frac{(n+1)^p}{q^{n+1}}\frac{(r-a)^p}{\Gamma(p+1)}$$
$$=(r-a)^{\alpha+1}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{q^n}e^{n\lambda(r-a)}=(r-a)^{\alpha+1}\{\frac{e^{\lambda(r-a)}}{q+e^{\lambda(r-a)}}\}. \tag{2.4}$$

Again, in Eqn. (2.3), set $a_{p,n}(\lambda,q)=\frac{\Gamma(p+\alpha+2)}{\Gamma(p+1)}\lambda^p q^{-n}$ to get that

$$V^{\alpha}(r;a,\lambda,q)=\frac{q(r-a)^{\alpha+1}e^{\lambda(r-a)}}{q-1}, \forall 0<\alpha<1, \lambda\neq 0, q>1, 0\leq a\leq r\leq b. \tag{2.5}$$

Hence, the function in Eqn. (2.2) may be a general potential function.

Remark 2.1. *It is remarkable that the Eqns. (1.1) and (2.3) give us the relation*

$$\lim_{\alpha\to 1} V^{\alpha}(r;a,\lambda,q) = f(r-a;\lambda,q) \tag{2.6}$$

Now from (2.2), we find

$$\frac{d}{dr}V^{\alpha}(r;a,\lambda,q) = \frac{d^2}{dr^2}W^{\alpha}(r;a,\lambda,q),$$

where, $W^{\alpha}(r;a,\lambda,q)$

$$= \frac{1}{\Gamma(\alpha)}\int_a^r (r-\xi)^{\alpha-1} f(\xi-a;\lambda,q)d\xi, 0<\alpha<1, \lambda\neq 0, q>0, 0\leq a\leq r\leq b. \tag{2.7}$$

Here, the function $W^{\alpha}(r;a,\lambda,q)$ *is defined as the Riemann - Liouville fractional derivative, given in the Eqns. (1.7) - (1.8), of the function* $f(.)$.

Again, suppose that the slope of the function $V^{\alpha}(r;a,\lambda,q)$ *is negatively proportional to the* $W^{\alpha}(r;a,\lambda,q)$ *that is*

$$\frac{d}{dr}V^{\alpha}(r;a,\lambda,q) \propto -W^{\alpha}(r;a,\lambda,q). \tag{2.8}$$

Then, on application of the Eqns. (2.7) and (2.8), we find

$$\frac{d^2}{dr^2}W^{\alpha}(r;a,\lambda,q) = -\mu^2 W^{\alpha}(r;a,\lambda,q), \mu\neq 0, \tag{2.9}$$

μ^2 *is the proportionality constant.*

Again, by the Eqn. (2.7) the function $W^{\alpha}(r;a,\lambda,q)$ *satisfies the initial condition*

$$W^{\alpha}(a;a,\lambda,q) = 0. \tag{2.10}$$

Now, we suppose that $W^{\alpha}(b;a,\lambda,q)=0, b>a$, *then, the problem (2.9) along with the condition (2.10) becomes oscillatory in the interval [a,b].*

Theorem 2.1. *If* $W^{\alpha}(r;a,\lambda,q) = \frac{1}{\Gamma(\alpha)}\int_a^r (r-\xi)^{\alpha-1} f(\xi-a;\lambda,q)d\xi, 0<\alpha<1, \lambda\neq 0, q> 0, 0\leq a\leq r\leq b$ *and* $\frac{d}{dr}V^{\alpha}(r;a,\lambda,q) = -\mu^2 W^{\alpha}(r;a,\lambda,q)$, *where,* $V^{\alpha}(r;a,\lambda,q)$ *is found due to the Abel's integral equation (2.1), then, for the conditions* $W^{\alpha}(a;a,\lambda,q) = 0 = W^{\alpha}(b;a,\lambda,q)$, *a and b are integral values,* $b>a$, *the eigen solutions of Eqn. (2.9) are given by* $W_n^{\alpha}(r;a,\lambda,q) = \sqrt{\frac{2}{(b-a)}} sinn\pi r, b>a$, *for the corresponding eigen values* $\mu_n = n\pi \forall n = 1,2,\dots$ *on* $[a,b]$ *and the Green function*

$G(r,\xi) = \frac{2}{(b-a)}\sum_{n=1}^{\infty}\frac{sinn\pi r sinn\pi\xi}{n^2\pi^2}, b>a$. *Also, the Eqn. (2.9) has infinite number of solutions*

$$W^{\alpha}(r;a,\lambda,q) = \gamma sinn\pi r, \ \gamma \text{ is any constant } \forall n = 1,2,3,\dots. \tag{2.11}$$

Proof. Take Riemann - Liouville fractional derivative by Eqns. (1.7) and (1.8) of the function (1.1) and then make an appeal to the definition (2.7) to find

$$W^{\alpha}(r;a,\lambda,q) = \frac{1}{\Gamma(\alpha)}\int_a^r (r-\xi)^{\alpha-1} f(\xi-a;\lambda,q)d\xi, 0<\alpha<1, \lambda\neq 0, q>0, 0\leq a\leq r\leq b. \tag{2.12}$$

Now, make an appeal to the Eqn. (2.2) and then the Eqns. (2.7) - (2.10) and with the help of (2.12) and the theory of integral equations we prove the theme of the Theorem 2.1. □

3 Application to find out the phase shifts

In this section, we make an appeal to the Eqn. (2.9) to get that linear in r in the form $\int_0^r \frac{\frac{d^2}{dr^2}W^\alpha(r;0,\lambda,q)}{W^\alpha(r;0,\lambda,q)}dr = -\mu^2 r$, whenever $W^\alpha(r;0,\lambda,q)$ is never zero, now to make further developments in this theory here, to introduce $\mu^2 = K^2 - L(L+1)$ and difference of the integrals is taken as $\int_0^r \frac{\frac{d^2}{dr^2}W_L^\alpha(r;0,\lambda,q)}{W_L^\alpha(r;0,\lambda,q)}dr - \int_0^r \{V^\alpha(r;0,\lambda,q) + (1-r^2)\frac{L(L+1)}{r^2}\}dr = \{L(L+1) - K^2\}r$, then prove following theorem:

Theorem 3.1. *If $W_L^\alpha(r;0,\lambda,q) \neq 0, \forall\ 0 < \alpha < 1, \lambda \neq 0, q > 0, 0 \leq r < \infty$ and the integral is given by*

$$\int_0^r [\frac{\frac{d^2}{dr^2}W_L^\alpha(r;0,\lambda,q) - \{V^\alpha(r;0,\lambda,q) + (1-r^2)\frac{L(L+1)}{r^2}\}W_L^\alpha(r;0,\lambda,q)}{W_L^\alpha(r;0,\lambda,q)}]dr = \{L(L+1)-K^2\}r. \tag{3.1}$$

Then, there exists a real number $\forall K > 0,\ 0 < \alpha < 1, \lambda \neq 0, q > 0, 0 \leq r < \infty$, and it is equivalent to the phase shifts by the formula

$$\{\eta_L(K) - \eta_{L+1}(K)\} = arcsin[\frac{\pi(K)^{2L+1}}{(2)^{2L+3}\Gamma(L+\frac{3}{2})\Gamma(L+\frac{5}{2})}$$
$$\times \sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_{p,n}(\lambda,q)}{\Gamma(p+\alpha+1)}\int_0^\infty r^{p+2L+\alpha+3}{}_1F_2[{}^{L+2;}_{L+\frac{5}{2}, 2L+3;} \quad -K^2r^2]dr]. \tag{3.2}$$

Proof. Differentiate both sides of the Eqn. (3.1) with respect to r to find the differential equation equivalent to the Eqn. (1.4) as

$$\frac{d^2}{dr^2}W_L^\alpha(r;0,\lambda,q) + \{K^2 - V^\alpha(r;0,\lambda,q) - \frac{L(L+1)}{r^2}\}W_L^\alpha(r;0,\lambda,q) = 0. \tag{3.3}$$

Consider the formulae (1.4) - (1.6) in the Eqn. (3.3) to get $\sin\{\eta_L(K) - \eta_{L+1}(K)\} = [\frac{\pi}{2K}\int_0^\infty r\frac{d}{dr}V^\alpha(r;0,\lambda,q)J_{L+\frac{1}{2}}(Kr)J_{L+\frac{3}{2}}(Kr)dr]$, $0 < \alpha < 1$, and then on using the Eqn. (2.3), to introduce here
$\frac{d}{dr}V^\alpha(r;0,\lambda,q) = \sum_{p=0}^\infty\sum_{n=0}^\infty a_{p,n}(\lambda,q)\frac{(r)^{p+\alpha}}{\Gamma(p+\alpha+1)}, \forall\ 0 < \alpha < 1, \lambda \neq 0, q > 0, 0 \leq r < \infty$,
to find that

$$\sin\{\eta_L(K) - \eta_{L+1}(K)\} = [\frac{\pi}{2K}\sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_{p,n}(\lambda,q)}{\Gamma(p+\alpha+1)}\int_0^\infty r^{p+\alpha+1}J_{L+\frac{1}{2}}(Kr)J_{L+\frac{3}{2}}(Kr)dr]. \tag{3.4}$$

Now use the formula [15, p. 121]
$J_n(z)J_m(z) = \frac{(\frac{z}{2})^{n+m}}{\Gamma(n+1)\Gamma(m+1)}{}_2F_3[{}^{\frac{1}{2}(n+m+1), \frac{1}{2}(n+m+2);}_{n+1, m+1, n+m+1;} \quad -z^2]$ in right hand side of (3.4) to get
$\sin\{\eta_L(K) - \eta_{L+1}(K)\} = \frac{\pi(K)^{2L+1}}{(2)^{2L+3}\Gamma(L+\frac{3}{2})\Gamma(L+\frac{5}{2})}$

$$\times \sum_{p=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_{p,n}(\lambda,q)}{\Gamma(p+\alpha+1)}\int_0^\infty r^{p+2L+\alpha+3}{}_1F_2[{}^{L+2;}_{L+\frac{5}{2}, 2L+3;} \quad -K^2r^2]dr. \tag{3.5}$$

The Eqn. (3.5) immediately gives the result (3.2). □

Remark 3.1. *In the result (3.2), in right side of the integrand, the generalized hypergeometric function* ${}_1F_2[.]$ *has in the denominator more parameters than its numerator, so that it is an entire function (see Titchmarsh [25, p. 285, Ex. 6]) and thus due to Weierstrass factorization theorem (see Titchmarsh [25, p. 247]) one of its factor may be in the form* $\exp[-K^{\alpha+2}r^{\alpha+2}]\cos Kr\ H(K)$ *and then by Csordas and Varga [4], Pathan and Kumar [13], the integral has Pólya class, and hence, due to this integral in the right hand side has some real zeros for* $K > 0, r \in (0, \infty)$ *and thus the phase shift in Eqn. (3.2) has some real zeros (see also Kumar [7]). Hence, certainly all phase shifts coincide at a real point.*

4 An application of Srivastava and Daoust's triple hypergeometric function to compute the phase shifts

In this section, we make an application of the Theorem 3.1 and obtain the phase shifts in terms of the arcsine of the series containing the Srivastava and Daoust function (1.9).

Theorem 4.1. *For the sequence* $a_{p,n}(\lambda, q) = (-1)^{n+p}\lambda^p \frac{(n+1)^p}{q^{n+1}}, \lambda \neq 0, q > 0, \forall p = 0, 1, 2, \ldots; n = 1, 2, 3, \ldots,$ *there exists the phase shift difference formula*

$$\{\eta_L(K) - \eta_{L+1}(K)\} = arcsin[\tfrac{\sqrt{\pi}}{2}\tfrac{(K)^{2L+1}}{(\beta)^{2L+\alpha+4}}\sum_{n=1}^{\infty}\tfrac{(-1)^{n-1}}{q^n}$$
$$\times S^{1:1;-;1}_{-:1;-;2}\left(\begin{matrix}[2L+\alpha+4:1,1,2]:(1:1);(-:-);(L+2:1)\\ [-:-,-,-]:(1+\alpha:1);(-:-);(L+\frac{5}{2}:1),(2L+3:1)\end{matrix}\Big|\tfrac{-n\lambda}{\beta}, 1, \tfrac{-K^2}{\beta^2}\right)]$$

provided that

$$0 \neq \lambda < \frac{\gamma_1\beta}{n}, \forall n = 1, 2, 3, \ldots; 0 < K < \sqrt{\gamma_2}\beta; \beta > 0, \gamma_1 < \infty, \gamma_2 < \infty,\ q > 0, 0 < \alpha < 1. \tag{4.1}$$

Proof. Consider the sequence $a_{p,n}(\lambda, q) = (-1)^{n+p}\lambda^p \frac{(n+1)^p}{q^{n+1}}, \lambda \neq 0, q > 0, \forall p = 0, 1, 2, \ldots; n = 1, 2, 3, \ldots$ in Eqn. (3.2), then, for $0 < \alpha < 1$, and on changing the order of summation to get that in the form

$$\sin\{\eta_L(K) - \eta_{L+1}(K)\} = \frac{\pi(K)^{2L+1}}{(2)^{2L+3}\Gamma(L+\frac{3}{2})\Gamma(L+\frac{5}{2})}$$
$$\times\sum_{n=1}^{\infty}\sum_{p=0}^{\infty}\frac{(-1)^{n-1}(-\lambda n)^p}{q^n\Gamma(p+\alpha+1)}\int_0^{\infty} r^{p+2L+\alpha+3}{}_1F_2\left[\begin{matrix}L+2;\\ L+\frac{5}{2}, 2L+3;\end{matrix} -K^2r^2\right]dr. \tag{4.2}$$

Now, from the Eqn. (4.2), for $\beta > 0$, we may write

$$\sin\{\eta_L(K) - \eta_{L+1}(K)\} = \frac{1}{\Gamma(1+\alpha)}\frac{\pi(K)^{2L+1}(2)^{-2L-3}}{\Gamma(L+\frac{3}{2})\Gamma(L+\frac{5}{2})}\sum_{n=1}^{\infty}(-1)^{n-1}(\frac{1}{q})^n$$
$$\times\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\frac{(1)_p(-\lambda n)^p}{(1+\alpha)_p p!}\frac{\beta^m}{m!}\int_0^{\infty}e^{-\beta r}r^{2L+3+p+m+\alpha}{}_1F_2\left[\begin{matrix}L+2;\\ L+\frac{5}{2}, 2L+3;\end{matrix} -K^2r^2\right]dr. \tag{4.3}$$

Then, on simplifying the Eqn. (4.3) and on using the definition (1.9) to find the result

$$\sin\{\eta_L(K) - \eta_{L+1}(K)\} = \frac{\sqrt{\pi}}{2}\frac{(K)^{2L+1}}{(\beta)^{2L+\alpha+4}}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{q^n}$$
$$\times S^{1:1;-;1}_{-:1;-;2}\left(\begin{matrix}[2L+\alpha+4:1,1,2]:(1:1);(-:-);(L+2:1)\\ [-:-,-,-]:(1+\alpha:1);(-:-);(L+\frac{5}{2}:1),(2L+3:1)\end{matrix}\Big|\frac{-n\lambda}{\beta}, 1, \frac{-K^2}{\beta^2}\right), \tag{4.4}$$

provided that $0 \neq \lambda < \frac{\gamma_1 \beta}{n}, \forall n = 1, 2, 3, \ldots; 0 < K < \sqrt{\gamma_2}\beta; \beta > 0, \gamma_1 < \infty, \gamma_2 < \infty,\ q > 0, 0 < \alpha < 1$.

The result (4.4) immediately gives us the formula (4.1). □

Concluding Remarks

The result (4.1) on specializing of the parameters, under the given convergence conditions, may help for computation of, a physical quantity, the phase shifts as using MATLAB to analyse the photo - ionization cross - sections and a reliable calculation of the impact ionization cross - section of an atom requires an accurate determination of the continuum wave functions in the incident and in the exit channels. In the integral operator (3.1), as claiming the theory of Suffridge [23] on the univalent functions, the starlikeness, convexity and other geometric properties of holomorphic maps in one and higher dimensions may be studied.

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