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A SHORT NOTE ON EXTON'S RESULT

By

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ABSTRACT

In this paper, authors derived some generating functions (partly bilateral and partly unilateral) involving exponential and Mittag-Leffer's functions in view of Exton's result [3].

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1. Introduction and Definition. The function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad \alpha > 0 \tag{1.1}$$

was introduced by Mittag-Leffler's [5] and was investigated systimatically by several other authors (for detail, see [2, Chapter XVIII]).

The function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \qquad \alpha,\beta > 0 \tag{1.2}$$

has properties very similar to those of Mittag-Leffler's function $E_\alpha(z)$ (See Wiman [9], Agarwal [1]).

In 1971, Prabhakar [7] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ in the form

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha k + \beta)} \frac{z^k}{k!}, \qquad \alpha,\beta,\gamma > 0 \tag{1.3}$$

where $(\gamma)_k$ is the Pochhammer symbol (Rainville [8])

$(\gamma)_0 = 1, (\gamma)_k = \gamma(\gamma+1)(\gamma+2)...(\gamma+k-1)$.

The function $E^{\gamma}_{\alpha,\beta}(z)$ is most natural generalization of the exponential function *exp*(*z*), Mittag-Leffler function $E_\alpha(z)$ and Wiman's function $E_{\alpha,\beta}(z)$.

We note that

$$\left.\begin{aligned}&E^1_{\alpha,\beta}(z)=E_{\alpha,\beta}(z),E_{\alpha,1}(z)=E_\alpha(z),E_{1,\beta}(z)=\frac{1}{\Gamma\beta}\,{}_1F_1[1,\beta;z]\\&E^1_{1,1}(z)=E_{1,1}(z)=E_1(z)=e^z,E_2(z^2)=cosh\,z\end{aligned}\right\}\tag{1.4}$$

An interesting (partly bilateral and partly unilateral) generating function for $F_n^m(x)$, due to Exton [3, p.147(3)] is recalled here in the following (modified) from [see [6]):

$$exp\left(s+t-\frac{xt}{s}\right)=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}s^m t^n F_n^m(x),\tag{1.5}$$

where $F_n^m(x)={}_1F_1(-n;m+1;x)/m!n!=L_n^m(x)/(m+n)!,$ (1.6)

and $L_n^m(x)$ denotes the classical Laguerre polynomials, (see [8] and in what folllows

$$m^*=max(0,-m),(m\varepsilon Z=0,1,2,...)\tag{1.7}$$

so that all factorials in equation (1.5) have meaning.

2. Generating Reltions.

Result-1. If $p,q,l\in N$, then

$$exp\left(s^p+t^q-\left(\frac{xt}{s}\right)^t\right)=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}s^m t^n\sum_{r=0}^{[n/q]}\frac{(-x)^{lr}}{r!\left(\frac{m+lr}{p}\right)!\left(\frac{n-lr}{q}\right)!}\tag{2.1}$$

Special Cases

(i) For $p=q=l$, equation (2.1) reduces to

$$exp\left(s^p+t^p-\left(\frac{xt}{s}\right)^p\right)=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{(m/p)!(n/p)!}\,{}_1F_1\left[\begin{matrix}-n/p & ; \\ m/p+1 & ;\end{matrix}-(-x)^p\right].\tag{2.2}$$

(ii) When $p=2$ in (2.2) or $p=q=l=2$ in equation (2.1), we have

$$exp\left(s^2+t^2-\left(\frac{xt}{s}\right)^2\right)=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{(m/2)!(n/2)!}\,{}_1F_1\left[\begin{matrix}-n/2 & ; \\ m/2+1 & ;\end{matrix}x^2\right].\tag{2.3}$$

(iii) When $p=q=1$ in equation (2.1), we get

$$exp\left(s+t-(xt/s)^l\right)=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{m!n!}\,{}_1F_1\left[\begin{matrix}\prod_{j=1}^{l}\frac{-n+j-1}{l} & ; \\ & x^l \\ \prod_{j=1}^{l}\frac{(m+1)+j-1}{l}+1 & ;\end{matrix}\right].\tag{2.4}$$

For $l=1$, equation (2.4) reduces to (1.5). It can also be obtained from (2.1) by taking $p=q=l=1$.

Result-2. If $p,q,l \in N$ and $E^{\gamma}_{\alpha,\beta}$ is defined by (1.3), then

$$E^{\gamma_1}_{\alpha_1,\beta_1}\left(s^p\right)E^{\gamma_2}_{\alpha_2,\beta_2}\left(t^q\right)E^{\gamma_3}_{\alpha_3,\beta_3}\left(-xt/s\right)^l$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{\Gamma\beta_1\Gamma\beta_2\Gamma\beta_3}\sum_{r=0}^{[n/q]}\frac{\left(\gamma_1\right)_{\left(\frac{m+lr}{p}\right)}\left(\gamma_2\right)_{\left(\frac{n-lr}{q}\right)}\left(\gamma_3\right)_r\left(-x\right)^{lr}}{\left(\beta_1\right)_{\alpha_1\left(\frac{m+lr}{p}\right)}\left(\beta_2\right)_{a_2\left(\frac{n-lr}{q}\right)}\left(\beta_3\right)_{\alpha_3 r}}. \tag{2.5}$$

Special Cases.

(i) For $\gamma_1=\gamma_2=\gamma_3=1$, equation (2.5) reduces to

$$E_{\alpha_1,\beta_1}\left(s^p\right)E_{\alpha_2,\beta_2}\left(t^q\right)E_{\alpha_e,\beta_e}\left(\frac{-xt}{s}\right)^l$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{\Gamma\beta_1\Gamma\beta_2\Gamma\beta_3}\sum_{r=0}^{[n/q]}\frac{\left(-x\right)^{lr}}{\left(\beta_1\right)_{\alpha_1\left(\frac{m+lr}{p}\right)}\left(\beta_2\right)_{\alpha_2\left(\frac{n-lr}{q}\right)}\left(\beta_3\right)_{\alpha_3 r}}. \tag{2.6}$$

(ii) When $p=q=l$ in equation (2.5), we get

$$E_{\alpha_1,\beta_1}\left(s^p\right)E_{\alpha_2,\beta_2}\left(t^q\right)E_{\alpha_3,\beta_3}\left(\frac{-xt}{s}\right)^p$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{\Gamma\beta_1\Gamma\beta_2\Gamma\beta_3}\sum_{r=0}^{[n/p]}\frac{\left(-x\right)^{pr}}{\left(\beta_1\right)_{\alpha_1\left(\frac{m}{p}+r\right)}\left(\beta_2\right)_{\alpha_2\left(\frac{n}{p}-r\right)}\left(\beta_3\right)_{\alpha_3 r}}. \tag{2.7}$$

For $\alpha_1=\alpha_2=\alpha_3=\beta_1=\beta_2=\beta_3=1$ equation (2.7) reduces to (2.2) and also to (1.5) for $p=1$.

(iii) If $\gamma_1=\gamma_2=\gamma_3=1$ and $p=q=l=1$ in equation (2.5), we get known generating function of Kamarujjama and Khursheed [4]

$$E_{\alpha_1,\beta_1}\left(s\right)E_{\alpha_2,\beta_2}\left(t\right)E_{\alpha_3,\beta_3}\left(\frac{-xt}{s}\right)$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{\Gamma\beta_1\Gamma\beta_2\Gamma\beta_3}\sum_{r=0}^{n}\frac{\left(-x\right)^{r}}{\left(\beta_1\right)_{\alpha_1(m+r)}\left(\beta_2\right)_{\alpha_2(n-r)}\left(\beta_3\right)_{\alpha_3 r}}. \tag{2.8}$$

(iv) When $p=2$ in equation (2.7) or $p=q=l=2$ in equation (2.6), we get

$$E_{\alpha_1,\beta_1}\left(s^2\right)E_{\alpha_2,\beta_2}\left(t^2\right)E_{\alpha_3,\beta_3}\left(\frac{-xt}{s}\right)^2$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{\Gamma\beta_1\Gamma\beta_2\Gamma\beta_3}\sum_{r=0}^{[n/2]}\frac{\left(-x\right)^{2r}}{\left(\beta_1\right)_{\alpha_1\left(\frac{m}{2}+r\right)}\left(\beta_2\right)_{\alpha_2\left(\frac{n}{2}-r\right)}\left(\beta_3\right)_{\alpha_e r}}. \tag{2.9}$$

For $\alpha_1=\alpha_2=\alpha_3=\beta_1=\beta_2=\beta_3=1$ equation (2.9) reduces to (2.3).

Now putting $\alpha_1=\alpha_2=\alpha_3=2, \beta_1=\beta_2=\beta_3=1$ in equation (2.9) and using a relation (1.4), we obtain a generating function of hypergeometric function ${}_2F_3$ in terms of hyperbolic cosine functions.

$$\cosh s\ \cosh t\ \cosh(xt/s)=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^{2m}t^{2n}}{(2m)!(2n)!}\,{}_2F_3\left[\begin{matrix}-n,-n+1/2 & ; \\ m+1,m+1/2,1/2 & ;\end{matrix}\ x^2/4\right]. \quad (2.10)$$

(v) For $p=q=1$, equation (2.6) reduces to

$$E_{\alpha_1,\beta_1}(s)E_{\alpha_2,\beta_2}(t)E_{\alpha_3,\beta_3}\left(\frac{-xt}{s}\right)^l$$

$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{\Gamma\beta_1\Gamma\beta_2\Gamma\beta_3}\sum_{r=0}^{[n/l]}\frac{(-x)^{lr}}{(\beta_1)_{\alpha_1(m+lr)}(\beta_2)_{\alpha_2(n-lr)}(\beta_3)_{\alpha_3 r}}. \quad (2.11)$$

For $\alpha_1=\alpha_2=\alpha_3=\beta_1=\beta_2=\beta_3=1$, equation (2.11) reuces to (2.4) and also to (1.5) for $l=1$.

(vi) For $\alpha_1=\alpha_2=\alpha_3=1$ and $l=1$, equation (2.11) reduces to a following generating relation

$${}_1F_1[1,\beta_1;s].{}_1F_1[1,\beta_2;t].{}_1F_1[1,\beta_3;-xt/s]=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{s^m t^n}{(\beta_1)_m(\beta_2)_n}.{}_2F_2\left[\begin{matrix}1,1-\beta_2-n & ; \\ \beta_1+m,\beta_3 & ;\end{matrix}\ x\right] \quad (2.12)$$

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