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ON DECOMPOSITION OF CURVATURE TENSOR FIELDS IN A KAEHLERIAN RECURRENT SPACE OF FIRST ORDER

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ABSTRACT

Takano [13] studied decomposition of curvature tensor in a recurrent space. Sinha and Singh [12] defined and studied defined decomposition of recurrent curvature tensor field in a Finsler space. Negi and Rawat ([2][3],[4]) studied decomposition of recurrent curvature tensor field in a Kaehlerian space. Rawat [5], Rawat and Silswal ([6],[7]) defined and studied decomposition of recurrent curvature tensor fields in a Tachibana space. Further, Rawat and Dobhal ([8],[9]) studied decomposition of recurrent corvature tensor field in a Kaehlerian recurrent space. Rawat and Singh ([10],[11]) studied the decomposition of curvature tensor fields in a Kaehlerian recurent space of first order.

In the present paper, we consider the decomposition of curvature tensor field Rh^i_{jk} in terms of two non-zero vectors and a tensor field. Also several theorems are established and proved therein.

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1. Introduction. When in a $2n$-dimensional real space X_{2n} of class $C^r(r\geq 2)$, there is a mixed tensor field $F_i^h; R_{,i,j} = 1,2,3,...,2n$ satisfying

$$F_i^h F_l^h = -A_j^h, \qquad ...(1.1)$$

we say that the space admits an almost complex structure and we call such a space an almost complex space.

If an almost complex space has a positive definite Riemannian metric $ds^2 = g_{ji} d\xi^j d\xi^i$ which setisfies

$$F_j^l F_i^k g_{lk} = g_{ji}, \qquad ...(1.2)$$

Then the space is called an almost-Hermitian space.

In this case the tensor $F_{ih} \overset{def}{=} F_i^l g_{lh}$ is anti-symmetric(or skew-symmetric) in i and h.

If an almost-Hermitian space satisfies

$$\nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji} = 0, \quad \text{...(1.3)}$$

where ∇_j denotes the operator of covariant differentiation with respect to the metric tensor g_{ji} of the Riemannian space then it is called an almost-Kaehlerian space and if it satisties

$$\nabla_j F_{ih} + \nabla_i F_{jh} = 0, \quad \text{...(1.4)}$$

then it is called a K-space.

In an Almost-Hermitian space, if

$$\nabla_j F_{ih} = 0, \; or \; F_{ih,j} = 0, \quad \text{...(1.5)}$$

then it is called a Kaehlerian space.

The Riemannian curvature tensor field is defined by

$$R^h_{ijk} = \partial_i \left\{ {h \atop jk} \right\} - \partial_j \left\{ {h \atop ik} \right\} + \left\{ {h \atop il} \right\} \left\{ {l \atop jk} \right\} - \left\{ {h \atop jl} \right\} \left\{ {l \atop ik} \right\}, \quad \text{...(1.6)}$$

where $\partial_i = \partial / \partial x^i$ and $\left\{ x^i \right\}$ denotes the real local coordinates.

The Ricci tensor and the scalar curvature are given by

$R_{ij} = R^a_{aij}$ and $R = R_{ij} g^{ij}$ respectively.

It is well known that these tensors satisfy the following identities

$$R^a_{ijk,a} = R_{jk,i} - R_{ik,j} \quad \text{...(1.7)}$$

$$R_{,i} = 2R^a_{i,a} \quad \text{...(1.8)}$$

$$F^a_i R_{aj} = -R_{ia} F^a_j \quad \text{...(1.9)}$$

and $$F^a_i R^j_a = R^a_i F^j_a . \quad \text{...(1.10)}$$

The holomorphically projective curvature tensor is defined by

$$P^h_{ijk} = R^h_{ijk} + \frac{1}{(n+2)} \left(R_{ik} \delta^h_j - R_{jk} \delta^h_i + S_{ik} F^h_j - S_{jk} F^h_i + 2S_{ij} F^h_k \right), \quad \text{...(1.11)}$$

where $S_{ij} = F^a_i R_{aj}$.

The Bianchi identities in K_n are given by

$$R^h_{ijk} + R^h_{jki} + R^h_{kij} = 0, \quad \text{...(1.12)}$$

and $$R^h_{ijk,a} + R^h_{ika,j} + R^h_{iaj,k} = 0 . \quad \text{...(1.13)}$$

The commutative formulae for the curvature tensor field are given as follows

$$T^i_{,jk} - T^i_{,kj} = T^a R^i_{ajk} \quad \text{...(1.14)}$$

and $T^h_{i,ml} - T^h_{i,lm} = T^a_i R^h_{aml} - T^h_a R^a_{iml}$. ...(1.15)

A Kaehlerian space K_n is said to be Kaehlerian recurrent space of first order, if its curvature

$$\nabla_a R^h_{ijk} = \lambda_a R^h_{ijk},$$

i.e. $R^h_{ijk,a} = \lambda_a R^h_{ijk}$, ...(1.16)

where λ_a is a non-zero vector and is known as recurrent vector field. The space is said to be Ricci-recurrent space of first order, if it satisfies the condition

$$R_{ij,a} = \lambda_a R_{ij}, \quad ...(1.17)$$

Multiplying the above equation by g^{ij}, we have

$$R_{,a} = \lambda_a R. \quad ...(1.18)$$

2. Decomposition of Curvature Tensor Field R^h_{ijk}. We consider the decomposition of recurrent curvature tensor field R^h_{ijk} in the following form

$$R^h_{ijk} = v'^h \phi_i \psi_{j,k} \quad ...(2.1)$$

where two vectors u'^h, ϕ_i and tensor field $\psi_{j,k}$ are such that

$$v'^h \lambda_h = 1. \quad ...(2.2)$$

Theorem 2.1. Under the decomposition (2.1), the Bianchi identities for R^h_{ijk} take the forms

$$\phi_i \psi_{j,k} + \phi_j \psi_{k,i} + \phi_k \psi_{i,j} = 0 \quad ...(2.3)$$

and $\lambda_a \psi_{j,k} + \lambda_j \psi_{k,a} + \lambda_k \psi_{a,j} = 0$. ...(2.4)

Proof. From equations (1.12) and (2.1), we have

$$\phi_i \psi_{j,k} + \phi_j \psi_{k,i} + \phi_k \psi_{i,j} = 0. \quad ...(2.5)$$

Since $v'^h \neq 0$.

From equations (1.13), (1.16) and (2.1), we get

$$v'^h \phi_i \left[\lambda_a \psi_{j,k} + \lambda_j \psi_{k,a} + \lambda_k \psi_{a,j}\right] = 0. \quad ...(2.6)$$

Multiplying (2.6) by λ_h and using (2.2), we obtain

$$\phi_i \left[\lambda_a \psi_{j,k} + \lambda_j \psi_{k,a} + \lambda_k \psi_{a,j}\right] = 0. \quad ...(2.7)$$

Since $\phi_i \neq 0$, therefore, we derive

$$\lambda_a \psi_{j,k} + \lambda_j \psi_{k,a} + \lambda_k \psi_{a,j} = 0 .$$

This completes the proof of the theorem.

Theorem 2.2. Under the decomposition (2.1), The tensor field R^h_{ijk}, R_{ij} and $\psi_{j,k}$ satisfy the relations

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} = \phi_i \psi_{j,k} \qquad ...(2.8)$$

Proof. With the help of equations (1.7), (1.16) and (1.17), we have

$$\lambda_a R^a_{ijk} = \lambda_i R_{jk} - \lambda_j R_{ik} \qquad ...(2.9)$$

Multiplying (2.1) by λ_h and using relation (2.2), we get

$$\lambda_h R^h_{ijk} = \phi_i \psi_{j,k} \qquad ...(2.10)$$

From equations (2.9) and (2.10, we derive the required relation (2.8).

Theorem 2.3. Under the decomposition (2.1), the quantities λ_a and v'^h behave the recurrent vectors. The recurrent form of these quantities are given by

$$\lambda_{a,m} = \mu_m \lambda_a \qquad ...(2.11)$$

and

$$v'^h_{,m} = -\mu_m v'^h \qquad ...(2.12)$$

Proof. Differentiating (2.8) covariantly w.r.t. x^m and using (2.1) and (2.8), we obtain

$$\lambda_{a,m} v'^h \phi_i \psi_{j,k} = \lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik} . \qquad ...(2.13)$$

Multiplying (2.13) by λ_a and using (2.1) and (2.9), we have

$$\lambda_{a,m} \left(\lambda_i R_{jk} - \lambda_j R_{ik} \right) = \lambda_a \left(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik} \right), \qquad ...(2.14)$$

Now, multiplying equation (2.14) by λ_h we get

$$\lambda_{a,m} \left(\lambda_i R_{jk} - \lambda_j R_{ik} \right) \lambda_h = \lambda_a \lambda_h \left(\lambda_{i,m} R_{jk} - \lambda_{j,m} R_{ik} \right) . \qquad ...(2.15)$$

Since the expression on the right hand side of above equation is symmetric in a and h, therefore

$$\lambda_{a,m} \lambda_h = \lambda_{h,m} \lambda_a , \qquad ...(2.16)$$

provided that

$$\lambda_i R_{jk} - \lambda_j R_{ik} \neq 0.$$

The vector field λ_a being non-zero, we can have a proportional vector μ_m such that

$$\lambda_{a,m} = \mu_m \lambda_a \qquad ...(2.17)$$

Further, differentiating the equation (2.2) w.r.t. x^m, we have

$$\lambda_h v'^h_{,m} + \lambda_{h,m} v'^h = 0.$$

Making use of equation (2.11), we derive

$$v^{h}_{,m} = -\mu_m v'^{h} \quad (\text{since } \lambda_h \neq 0). \qquad ...(2.18)$$

This proves the theorem.

Theorem 2.4 Under the decomposition (2.1), the vector ϕ_i and the tensor $\psi_{j,k}$ satisfy the equation

$$\phi_i \psi_{j,k}(\lambda_m + \mu_m) = \phi_i \psi_{j,km} + \psi_{j,k}\phi_{i,m}, \qquad ...(2.19)$$

where $\psi_{j,km} = \psi_{j,k,m}$.

Proof. Differentiating (2.1) covariantly w.r.t. x^m and using equations (1.16), (2.1) and (2.12), we get the required result of the Theorem.

Theorem 2.5 Under the decomposition (2.1), the curvature tensor and holomorphically projective curvature tensor are equal if

$$\psi_{k,m}\left\{\left(\phi_i \delta^h_j - \phi_j \delta^h_i\right) + \phi_l\left(F^h_j F^l_i - F^h_i F^l_j\right)\right\} + 2\phi_l \psi_{j,m} F^h_k F^l_i = 0. \qquad ...(2.20)$$

Proof. The equation (1.11) may be written in the form

$$P^h_{ijk} = R^h_{ijk} + D^h_{ijk}, \qquad ...(2.21)$$

where

$$D^h_{ijk} = \frac{1}{(n+2)}\left(R_{ik}\delta^h_j - R_{jk}\delta^h_i + S_{ik}F^h_j - S_{jk}F^h_i + 2S_{ij}F^h_k\right). \qquad ...(2.22)$$

Contracting indices h and k in (2.1), we have

$$R_{ij} = v'^{k}\,\phi_i \psi_{j,k}. \qquad ...(2.23)$$

In view of equation (2.23), we get

$$S_{ij} = F^l_i v'^{m}\,\phi_l \psi_{j,m}. \qquad ...(2.24)$$

Making use of relations (2.23) and (2.24) in equation (2.22), we have

$$D^h_{ijk} = \frac{1}{(n+2)}\left[v'^{m}\,\psi_{k,m}\left\{\left(\phi_i\delta^h_j - \phi_j\delta^h_i\right) + \phi_l\left(F^h_j F^l_i - F^h_i F^l_j\right)\right\} + 2v'^{m}\,\phi_l \psi_{j,m} F^h_k F^l_i\right] = 0 \qquad ...(2.25)$$

From (2.21), it is clear that $P^h_{ijk} = R^h_{ijk}$ if $D^h_{ijk} = 0$,

which, in view of (2.25), becomes

$$v'^{m}\,\psi_{k,m}\left\{\left(\phi_l\delta^h_j - \phi_j\delta^h_l\right) + \phi_l\left(F^h_j F^l_i - F^h_i F^l_j\right)\right\} + 2v'^{m}\,\phi_l \psi_{j,m} F^h_k F^l_i = 0 \qquad ...(2.26)$$

Multiplying the above equation by λ_m and using relation (2.2), we obtain the required result (2.20).

Theorem 2.6 Under the decomposition (2.1), the scalar curvature R, satisfies the relation

$$\lambda_k R = R_{,k} = g^{ij}\phi_i \psi_{j,k}.$$

Proof. Contracting indices h and k in (2.1), we have

$$R_{ij} = v'^k \phi_i \psi_{j,k}. \quad ...(2.27)$$

Multiplying (2.27) by g^{ij} both sides, we have

$$R = g^{ij} v'^k \phi_i \psi_{j,k}. \quad ...(2.28)$$

Multiplying (2.28) by λ_k and using (2.2), we get

$$\lambda_k R = g^{ij}\phi_i \psi_{j,k}$$

or, $R_{,k} = g^{ij}\phi_i \psi_{j,k}$ [by using (1.18)] .

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