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RECURRENCE RELATIONS FOR THE $\bar{H}$-FUNCTION

By

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ABSTRACT

The aim of the present paper is to establish two new recurrence relations for the $\bar{H}$-function. Some results for the generalized Wright hypergeometric function and generalized Riemann Zeta function are special cases of our main findings.

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1. Introduction. The $\bar{H}$-function was introduced by Inayat Hussain [4] and studied by Bushman and Srivastava [1].

The $\bar{H}$-function is defined and represented in the following manner:

$$\bar{H}_{P,Q}^{M,N}[z] = \bar{H}_{P,Q}^{M,N}\left[z \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j,B_j)_{M+1,Q} \end{matrix} \right. \right]$$

$$= \frac{1}{2\pi\omega}\int_L \bar{\phi}(\xi) z^{\xi} d\xi, (z \neq 0) \qquad ...(1.1)$$

where

$$\bar{\phi}(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{N}\left\{\Gamma(1-a_j+\alpha_j\xi)\right\}^{A_j}}{\prod_{j=M+1}^{Q}\left\{\Gamma(1-b_j+\beta_j\xi)\right\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)} \qquad ...(1.2)$$

The behaviour of $\bar{H}$ -function for small valkues of $|z|$ follows easily from a result recently given by Rathie {[6],p.306, eq.(6.9)}.

2. Main Results. In this section, we establish two recurrence relations.

First Recurrence Relation

$$\bar{H}_{P,Q}^{M,N}[z] = \bar{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j,B_j)_{M+1,Q}\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi\omega}\left\{e^{\omega\pi b_{M+1}}\bar{H}_{P,Q}^{M.N}\left[ze^{-\omega\pi\beta_{M+1}}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j,B_j)_{M+1,Q}\end{matrix}\right.\right]\right.$$

$$\left.-e^{\omega\pi b_{M+1}}\bar{H}_{P,Q}^{M,N}\left[ze^{-\omega\pi\beta_{M+1}}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M+1},(b_j,\beta_j,B_j)_{M+2,Q}\end{matrix}\right.\right]\right\} \quad ...(2.1)$$

Second Recurrence Relation

$$\bar{H}_{P,Q}^{M,N}[z] = \bar{H}_{P,Q}^{M,N}\left[z\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j,B_j)_{M+1,Q}\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi\omega}\left\{e^{\omega\pi a_{N+1}}\bar{H}_{P,Q}^{M.N}\left[ze^{-\omega\pi a_{N+1}}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+2,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j,B_j)_{M+1,Q}\end{matrix}\right.\right]\right.$$

$$\left.-e^{\omega\pi a_{N+1}}\bar{H}_{P,Q}^{M,N}\left[ze^{-\omega\pi a_{N+1}}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N+1},(a_j,\alpha_j)_{N+2,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j,B_j)_{M+1,Q}\end{matrix}\right.\right]\right\} \quad ...(2.2)$$

Recurrence relations (2.1) and (2.2) are valid under the conditions of (1.1) and (1.2).

Proof. The recurrence relations (2.1) and (2.2) can be easily established by appealing to the definition [4], [1] and use the well known relation (Rainville [5], p.21, Th.8)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin\,\pi z} = \frac{2\pi\omega}{e^{\omega\pi z}-e^{-\omega\pi z}},$$

involving Γ -function and reinterpret the result in terms of the $\bar{H}$ -function, we arrive at the right hand side of the desire results (2.1) and (2.2).

3. Particulars Cases.

(i) If in the recurrence relation (2.1) we reduce $\bar{H}_{P,Q}^{M,N}$ to generalized Wright hypergeometric function [3, p.271, eq. (7)]

$$ {}_P\psi_Q\left(\begin{matrix}(a_j;\alpha_j;A_j)_{1,P};\\ (b_j;\beta_j;B_j)_{1,Q};\end{matrix}\, z\right)=\frac{1}{2\pi\omega}\left\{e^{\omega\pi b_{M+1}}\sum_{t=0}^{\infty}\frac{\prod_{j=1}^{P}\left\{\Gamma(a_j+\alpha_j t)\right\}^{A_j}}{\prod_{j=M+1}^{P}\left\{\Gamma(b_j+\beta_j t)\right\}^{B_j}}\frac{\left(ze^{-\omega\pi\beta_{M+1}}\right)^t}{t!}\right. $$

$$ \left.-e^{-\omega\pi b_{M+1}}\sum_{t=0}^{\infty}\frac{\prod_{j=1}^{P}\left\{\Gamma(a_j+\alpha_j t)\right\}^{A_j}}{\prod_{j=M+1}^{P}\left\{\Gamma(b_j+\beta_j \xi)\right\}^{B_j}}\frac{\left(ze^{-\omega\pi\beta_{M+1}}\right)^t}{t!}\right\}. $$

(ii) If in the recurrence relation (2.2), we reduce $\bar{H}_{P,Q}^{M,N}$ to generalized Riemann Zeta function [3,p.27, eq.(1); 7, p.314-315, eq. (1.6) and (1.7)]

$$ \phi(z,p,\eta)=\frac{1}{2\pi\omega}\left\{e^{\omega\pi a_{N+1}}\bar{H}_{2,2}^{1,2}\left[-ze^{-\omega\pi a_{N+1}}\left|\begin{matrix}(0,1,1),(1-\eta,1;P)\\(0,1),(-\eta,1;P)\end{matrix}\right.\right]\right. $$

$$ \left.-e^{-\omega\pi a_{N+1}}\bar{H}_{2,2}^{1,2}\left[-ze^{-\omega\pi a_{N+1}}\left|\begin{matrix}(0,1,1),(1-\eta,1;P)\\(0,1),(-\eta,1;P)\end{matrix}\right.\right]\right\}. $$

(iii) If $A_j=B_j=1$ in (2.1) and (2.2) then $\bar{H}$ reduces to Fox H-function and we get known recurrence relations given earlier by Chaurasia [2].

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