Jñānābha, Vol. 41, 2011

LIE THEORY AND BASIC GAUSS POLYNOMIALS

By

Kishan Sharma

Department of Applied Mathematics

NRI Institute of Technology and Management, Gwalior-474001

Madhya Pradesh, India

E-Mail:drkishan010770@yahoo.com, drkishansharma2006@rediffmail.com

and

Renu Jain

School of Mathematics and Allied Sciences

Jiwaji University, Gwalior-474009, Madhya Pradesh, India

E-mail : renujain3@rediffmail.com

(Received : May 15, 2010, *Revised: October* 15, 2011*)*

ABSTRACT

In the Present paper, an attempt has been made to bring basic hypergeometric functions within the purview of Lie theory by constructing a dynamical symmetry algebra of basic hypergeometric function ${}_2\phi_1$. Multiplier representation theory is then used to obtain generating function for basic analogue of Gauss polynomial. The results obtained in this paper are extensions of the results derived earlier by Miller [3] and Sarkar-Chatterjea [4].

2000 Mathematics Subject Classification : Primary 60E07, 20G05; Secondary 33C55, 14A17.

Keywords : Lie algebra, Generating function, Basic Gauss polynomials.

1. Introduction. The q-analogue of the Gauss functions or Heine's series [1] may be written as

$$ {}_2\phi_1(a,b;c;q;x)=\sum_{n=0}^{\infty}[a;q,n][b;q,n]/[c;q,n][n;q]! \quad (c\neq 0,-1,-2,...)$$

where $|q|<1$ and $|x|<1$.

Here $[a;q,n]$ and $[n;q]!$ are respectively the basic Pochammer's symbol and basic factorial function defined as $[a;q,n]=[a;q][a+1;q]\ldots[a+n-1;q]$ and $[n;q]!=[1;q][2;q]...[n;q]$.

The basic differential operator $B^{\wedge}_{q,x}$ is defined by [1] through the relation

$$B^{\wedge}_{q,x}\phi(x)=\{\phi(qx)-\phi(x)\}/x(q-1). \qquad ...(1.1)$$

2. The Dynamical Symmetry Algebra of ${}_2\phi_1$. The dynamical symmetry algebra of the hypergeometric function has been defined by Miller [2]. We use the same technique to define the dynamical symmetry algebra of ${}_2\phi_1$. Let

$$\phi_{\alpha,\beta,\gamma,q} = \Gamma_q(\gamma-\alpha)\Gamma_q(\alpha)/\Gamma_q(\gamma).{}_2\phi_1[\alpha,\beta;\gamma;q;x]s^{\alpha}u^{\beta}t^{\gamma} \qquad ...(2.1)$$

be the basis elements of a subspace of analytical functions of four variables *x,s,u* and *t,* associated with Heine's basic hypergeometric function of Heine's series ${}_2\phi_1$. Introduction of variables *s,u* and *t* renders differential operators independent of parameters α,β and γ and thus facilitates their repeated operation.

The dynamical symmetry algebra of ${}_2\phi_1$ is a 15-dimensional complex Lie algebra isomorphic to *sl* (4), generated by twelve $E^{\wedge}$-operators termed as raising or lowering operators in view of their effect of raising or lowering the corresponding suffix in $\phi_{\alpha\beta\gamma,q}$. The $E^{\wedge}$-operators are

$$\text{(i)}\ E^{\wedge}_{-\alpha,q} = s^{-1}\left(x(1-x)B^{\wedge}_{q,x} + tB^{\wedge}_{q,t} - sB^{\wedge}_{q,s} - xuB^{\wedge}_{q,u}\right),$$

$$\text{(ii)}\ E^{\wedge}_{-\beta,-\gamma,q} = u^{-1}t^{-1}\left(x(1-x)B^{\wedge}_{q,x} - xsB^{\wedge}_{q,x} + tB^{\wedge}_{q,t} - 1\right), \qquad ...(2.2)$$

The action of these operators on $\phi_{\alpha\beta\gamma,q}$ is given by

$$E^{\wedge}_{-\alpha,q}\phi_{\alpha,\beta,\gamma,q} = [\alpha-1;q]\phi^{-1}_{\alpha,q,\beta,\gamma,q}$$

$$E^{\wedge}_{-\beta,-\gamma,q}\phi_{\alpha,\beta,\gamma,q} = [\gamma-\alpha-1;q]\phi^{-1\ \ -1}_{\alpha,\beta q\gamma q,q} \qquad ...(2.3)$$

The upper factor in each bracket is to be associated with plus sign and lower with minus sign. Twelve *E*-operators together with three maintenance operators $J_\alpha, J_\beta, J_\gamma$ and Identity operator *I* form a basic for $gl(4) \cong sl(4)(I)$, where (I) is the 1-dimensional Lie algebra generated by 1.
Here

$$J^{\wedge}_{\alpha,q} = sB^{\wedge}_{q,s}J^{\wedge}_{\beta,q} = uB^{\wedge}_{q,u}, J^{\wedge}_{\gamma,q} = tB^{\wedge}_{q,t} \text{ and } I^{\wedge} = 1 \qquad ...(2.4)$$

with the results

$$J^{\wedge}_{\alpha,q}\phi_{\alpha,\beta,\gamma,q} = [\alpha;q]\phi_{\alpha,\beta,\gamma,q},$$

$$J^{\wedge}_{\beta,q}\phi_{\alpha,\beta,\gamma,q} = [\beta;q]\phi_{\alpha,\beta,\gamma,q},$$

$$J^{\wedge}_{\gamma,q}\phi_{\alpha,\beta,\gamma,q} = [\gamma;q]\phi_{\alpha,\beta,\gamma,q},$$

and

$$I^{\wedge}\phi_{\alpha,\beta,\gamma,q} = \phi_{\alpha,\beta,\gamma,q}. \qquad ...(2.5)$$

3. The Generating Functions for Basic Analogues of Gauss Polynomials. On comparing the results obtained by the action one parameter subgroup $\left(exp_q\, aE^{\wedge}_{-\alpha,-\gamma,q}\right)$ generated by the operator $E^{\wedge}_{-\alpha,q}$ defined in (2.2) on $\phi_{\alpha,\beta,\gamma,q}$ defined in (2.1) and direct expansion, we get the identity

$$\left[st/(ax+st)\right]^{\beta}\left[a/(st+1)\right]^{\gamma-1}\,{}_2\Phi_1\left[\alpha,\beta;\gamma;q;x(a+st)/(ax+st)\right]$$

$$=\sum_{m=0}^{\infty}a^m\left[\gamma-m;q\right]_m/\left[m;q\right]!\,{}_2\Phi_1\left[\alpha q^{-m},\beta;\gamma;q^{-m};q;x\right](st)^{-m} \quad ...(3.1)$$

Taking, $\alpha\to 0,\beta\to\lambda+\mu+m-1,\gamma\to q^{\lambda+m},st\to 1,a\to 1,$ we get

$$(1+x)^{1-\lambda-\mu}\left[2;q\right]^{\lambda-1}=\sum_{m=0}^{\infty}\left[\gamma;q\right]_m/\left[m;q\right]!.{}_2\Phi_1\left[-m,\lambda+\mu+m-1;\lambda;q;x\right] \quad ...(3.2)$$

By definition of basic Gauss polynomial [1]

$G_m^{\lambda,\mu}(q;x)={}_2\Phi_1\left[-m,\lambda+\mu+m-1;\lambda;q;x\right],$

where $\lambda\neq 0,-1,-2,3,...$ . ...(3.3)

Using (3.3) in (3.2), we get the generating function

$$(1+x)^{1-\lambda-\mu}\left[2;q\right]^{\lambda-1}=\sum_{m=0}^{\infty}\left[\gamma;q\right]_m/\left[m;q\right]!.G_m^{\lambda,\mu}(q;x)$$

for basic Gauss polynomials.

Acknowledgement

The authors are very much thankful to the referee for giving useful suggestions in the improvement of the paper.

REFERENCES

[1] H. Exton, *q-Hypergeometric Functions and Applications*. Ellis Horwood Limited, John Wiley and Sons, New York, Brisbane, Chichester and Toronto (1983).

[2] W.Jr. Miller, *Lie Theory and Special Functions,* Academic Press. New York (1968).

[3] W. Jr. Miller : *Lie Theory and Generalization of Hypergeometric Functions, SIAM. J. Appl. Math.,* **25** No. 2 (1973).

[4] S. Sarkar and S.K. Chatterjea, Extensions of a class of bilateral generating relations for certain special functions by group theoretic method, *Pure Math. Manuscript,* **9** (1990-91), 251-259.