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STRONG CONVERGENCE THEOREMS FOR UNIFORMLY EQUI-CONTINUOUS AND ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS

By

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ABSTRACT

The purpose of this paper is prove some strong convergence theorems of the modified Ishikawa iterative sequences with errors for unformly equi-continuous and asymptotically quasi-nonexpansive mapping in the setup of unifromly convex Banach spaces by using condition (A) instead of comletely continuous or demicompact condition. Our results improve and generalize the corresponding results of Rhoades [6], Schu [7,8]. Tan and Xu [10,11] Xu and Noor [12] and many others.

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1. Introduction and Preliminaries. Let E be a real normed linear space, K be a nonempty subset of E. Throughout the paper, N denotes the set of positive integers and $F(T)=\{x:Tx=x\}$ the set of fixed points of a mapping T. Let $T:K\to K$ be a given mapping.

(1) T is said to be asymptotically nonexpansive [2] if there exists a sequence $\{k_n\}\subset[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x-y\|, \tag{1.1}$$

for all $x,y\in K$ and $n\in N$.

(2) T is said to be asymptotically quasi-nonexpansive if $F(T)\ne\phi$ and there exists a sequence $\{k_n\}$ in $[1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - p\| \le k_n \|x - p\|, \tag{1.2}$$

for all $x \in K, p \in F(T)$ and $n \in N$.

(3) T is said to be uniformly L-Lipschitzian if there exists a positive constant L such that

$$\|T^n x - T^n y\| \le L\|x - y\|, \tag{1.3}$$

for all $x, y \in K$ and $n \in N$.

(4) T is said to be uniformly Holder continuous [5] if there exist positive constants L and α such that

$$\|T^n x - T^n y\| \le L\|x - y\|^{\alpha}, \tag{1.4}$$

for all $x, y \in K$ and $n \in N$.

(5) T is said to be uniformly equi-continuous [5] if, for any $\in > 0$, there exists $\delta > 0$ such that

$$\|T^n x - T^n y\| \le \in \tag{1.5}$$

whenever $\|x - y\| < \delta$ for all $x, y \in K$ and $n \ge 1$ or, equivalently, T is uniformly equi-continuous if and only if $\|T^n x_n - T^n y_n\| \to 0$ whenever $\|x_n - y_n\| \to 0$ as $n \to \infty$.

Remark 1.1 (i) It is easy to see that, if T is asymptotically nonexpansive, then it is uniformly L-Lipschitzian.
(ii) If T is uniformly L-Lipschitzian, then it is uniformly Holder continuous with constants L > 0 and α = 1.
(iii) If T is uniformly Holder continuous, then it is uniformly equi-continuous.

This class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [2] in 1972. They proved that, if K is a nonempty bounded asymptotically nonexpansive self-mapping of K has a fixed point. Moreover the set *F (T)* of fixed points of T is closed and convex. Since 1972, many authors have studied weak and strong convergence problem of the Mann and Ishikawa iterative sequences (with errors) for asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces (see [2,6,7,8,10,11,12] and references therein).

Recenyly, Liu [3] studied modified Ishikawa iterative sequences with errors for asymptotically quasi-nonexpansive and unifornly Holder continuous mappings in uniformly convex Banach spaces and established some strong convergence theorems which extended some corresponding results of Tan and Xu [11].

The purpose of this paper is to extend and improve some resuits of [3] for uniformly equi-continuous and asymptotically quasi-nonexpansive mappings. Also our results improve and generalize the corresponding results of [6,7,8,11,12,] and many others.

In order to prove the main resuIts in this paper, we need the following lemmas :

Lemma 1.1. (Tan and Xu [10]). *Let* $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of nonnegtive numbers satisfying the inequality

$$\alpha_{n+1} \le (1+\beta_n)\alpha_n + r_n, \forall_n \ge 1.$$

If $\sum_{n=1}^{\infty}\beta_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty}\alpha_n$ exists. In particular, $\{\alpha_n\}_{n=1}^{\infty}$ has a subsequence which converges to zero, then $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 1.2. ([1]) Let X be a uniformly convex Banach space and $B_r(0)$ be a closed ball of X, Then there exists a continuous increasing convex function $g:[0,\infty)\to[0,\infty)$ with $g(0)=0$ such that

$$\|\lambda x + \mu y + \gamma z\|^2 \le \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x-y\|)$$

for all $x, y \in B_r(0)$ and $\lambda, \mu, \gamma \in [0,1]$ with $\lambda+\mu+\gamma = 1$.

2. Main Results. Now, we give the main results of this paper.

Lemma 2.1. Let E be a normed linear space and K be a nonempty convex subset of E. Let $T:K\to K$ be an asymptotically quasi-nonexpansive mapping with $F(T)\neq\phi$. and a seqeunce $\{k_n\}\subset[1,\infty)$ with $\lim_{n\to\infty}k_n = 1$ and $\sum_{n=1}^{\infty}(k_n^2-1)<\infty$. Let $\{x_n\}$ be a sequence in K defined by

$$y_n = a_n' x_n + b_n' T^n x_n + c_n' v_n,$$

$$x_{n+1} = a_n x_n + b_n T^n y_n + c_n u_n, \quad n \ge 1, \tag{2.1}$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in E and $\{a_n\},\{b_n\},\{c_n\},\{a_n'\}, \{b_n'\},\{c_n'\}$ are sequences in [0,1] and $a_n+b_n+c_n = a_n'+b_n'+c_n' = 1$ with the restrictions $\sum_{n=1}^{\infty}c_n<\infty$ and $\sum_{n=1}^{\infty}b_n c_n'<\infty$. Then we have the following:

(i) $\lim_{n\to\infty}\|x_n - p\|$ exists for any $p\in F(T)$,

(ii) $\lim_{n\to\infty} d(x_n, F(T))$ exists, where $d(x,F(T))$ denotes the distance from x to the set $F(T)$.

Proof of (i). Let $p \in F(T)$. Since $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K. So we can set

$$M = \max\left\{\sup_{n\geq 1}\|u_n - p\|, \sup_{n\geq 1}\|v_n - p\|\right\}.$$

Then it follows from (2.1) that

$$\begin{aligned}
\|y_n - p\| &= \|a'_n x_n + b'_n T^n x_n + c'_n v_n - p\| \\
&= \|a'_n (x_n - p) + b'_n (T^n x_n - p) + c'_n (v_n - p)\| \\
&\leq a'_n \|x_n - p\| + b'_n \|T^n x_n - p\| + c'_n \|v_n - p\| \\
&\leq a'_n \|x_n - p\| + b'_n k_n \|x_n - p\| + c'_n \|v_n - p\| \\
&\leq [a'_n + b'_n]_n\, k_n \|x_n - p\| + c'_n \|v_n - p\| \\
&= [1 - c'_n] k_n \|x_n - p\| + c'_n \|v_n - p\| \\
&\leq k_n \|x_n - p\| + c'_n M. \qquad (2.2)
\end{aligned}$$

Again from (2.1) and (2.2), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|a_n x_n + b_n T^n y_n + c_n u_n - p\| \\
&= \|a_n (x_n - p) + b_n (T^n y_n - p) + c_n (u_n - p)\| \\
&\leq a_n \|x_n - p\| + b_n \|T^n y_n - p\| + c_n \|u_n - p\| \\
&\leq a_n \|x_n - p\| + b_n k_n \|y_n - p\| + c_n \|u_n - p\| \\
&\leq a_n \|x_n - p\| + b_n k_n \left[k_n \|x_n - p\| + c'_n M\right] + c_n \|u_n - p\| \\
&\leq [a_n + b_n] k_n^2 \|x_n - p\| + b_n k_n c'_n M + c_n M \\
&= [1 - c_n] k_n^2 \|x_n - p\| + b_n k_n c'_n M + c_n M \\
&\leq k_n^2 \|x_n - p\| + b_n k_n c'_n M + c_n M \\
&\leq \left[1 + (k_n^2 - 1)\right] \|x_n - p\| + (b_n c'_n k_n + c_n) M \qquad (2.3)
\end{aligned}$$

since $\sum_{n=1}^{\infty} (k_n^2 - 1) < \infty, \sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} b_n c'_n < \infty$, it follows from Lemma 1.1, we know that $\lim_{n\to\infty} \|x_n - p\|$ exists. This completes the proof of part (i).

Proof of (ii) From conclusion of part (i), we have

$$\|x_{n+1}-p\| \le \left[1+\left(k_n^2-1\right)\right]\|x_n-p\|+\left(b_n c_n' k_n+c_n\right)M.$$

This gives that

$$d\left(x_{n+1},F(T)\right) \le \left[1+\left(k_n^2-1\right)\right]d\left(x_n,F(T)\right)+\left(b_n c_n' k_n+c_n\right)M.$$

Since $\sum_{n=1}^{\infty}\left(k_n^2-1\right)<\infty, \sum_{n=1}^{\infty}c_n<\infty$ and $\sum_{n=1}^{\infty}b_n c_n'<\infty$, it follows from Lemma 1.1, we know that $\lim_{n\to\infty} d\left(x_n,F(T)\right)$ exists. This completes the proof of part (ii).

Theorem 2.1. Let E be a uniformly convex Banach space, K be a nonempty convex subset of E and $T:K\to K$ be a uniformly equi-continuous and asymptotically quasi-nonexpansive mapping with $F(T)\neq\phi$ and a sequence $\{k_n\}\subset[1,\infty)$ with $\lim_{n\to\infty}k_n=1$ and $\sum_{n=1}^{\infty}\left(k_n^2-1\right)<\infty$. Let $\{x_n\}$be a sequence in K defined by (2.1) with the following restrictions:

(i) $0\le b_n\le b<1$ and $b_{n+1}\le b_n$ for all $n\ge 1$.

(ii) $\sum_{n=1}^{\infty}b_n=\infty$, (iii) $\lim_{n\to\infty}b_n'=0,$ (iv) $\sum_{n=1}^{\infty}c_n=\infty$ and $\sum_{n=1}^{\infty}b_n c_n'<\infty$.

Then $\liminf_{n\to\infty}\|x_n-Tx_n\|=0$.

Proof. Since $T:K\to K$ is asymptotically quasi-nonexpansive, we have

$$\|T^n y_n-p\|\le k_n\|y_n-p\|\le k_n^2\|x_n-p\|$$

for any $p\in F(T)$. By Lemma 2.1(i), we know that $\lim_{n\to\infty}\|x_n-p\|$ exists. Hence $\{x_n-p\}$ and $\{T^n y_n-p\}$ are bounded sequences in E. Set $r_1=\sup\{\|x_n-p\|:n\ge 1\}$, $r_2=\sup\{\|T^n y_n-p\|:n\ge 1\}$, $r_3=\sup\{\|u_n-p\|:n\ge 1\}$, $r_4=\sup\{\|v_n-p\|:n\ge 1\}$ and $r=\max\{r_i:i=1,2,3,4\}$ for any fixed $p\in F(T)$. Then we have $\{x_n-p\},\{T^n y_n-p\}$, $\{u_n-p\},\{v_n-p\}\in B_r(0)$ for all $n\ge 1$. By using Lemma 1.2 and (2.1), we have

$$\begin{aligned}\|y_n-p\|^2 &= \left\|a_n'(x_n-p)+b_n'\left(T^n x_n-p\right)+c_n'(v_n-p)\right\|^2\\ &\le a_n'\|x_n-p\|^2+b_n'\|T^n x_n-p\|^2+c_n'\|v_n-p\|^2-a_n'b_n'g\left(\|x_n-T^n x_n\|\right)\\ &\le a_n'\|x_n-p\|^2+b_n'k_n^2\|x_n-p\|^2+c_n'r^2\end{aligned}$$

$$\leq [a_n' + b_n'] k_n^2 \|x_n - p\|^2 + c_n' r^2$$

$$= [1 - c_n'] k_n^2 \|x_n - p\|^2 + c_n' r^2$$

$$\leq k_n^2 \|x_n - p\|^2 + c_n' r^2. \tag{2.4}$$

Again, using Lemma 1.2, (2.1) and (2.4) we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\|a_n(x_n - p) + b_n(T^n x_n - p) + c_n(u_n - p)\right\|^2 \\
&\leq a_n \|x_n - p\|^2 + b_n \|T^n y_n - p\|^2 + c_n \|u_n - p\|^2 - a_n b_n g\left(\|x_n - T^n y_n\|\right) \\
&\leq a_n \|x_n - p\|^2 + b_n k_n^2 \|y_n - p\|^2 + c_n \|u_n - p\|^2 - a_n b_n g\left(\|x_n - T^n y_n\|\right) \\
&\leq a_n \|x_n - p\|^2 + b_n k_n^2 \left[k_n^2 \|x_n - p\|^2 + c_n' r^2\right] + c_n r^2 - a_n b_n g\left(\|x_n - T^n y_n\|\right) \\
&\leq [a_n + b_n] k_n^4 \|x_n - p\|^2 + b_n c_n' k_n^2 r^2 + c_n r^2 - a_n b_n g\left(\|x_n - T^n y_n\|\right) \\
&= [1 - c_n] k_n^4 \|x_n - p\|^2 + b_n c_n' k_n^2 r^2 + c_n r^2 - a_n b_n g\left(\|x_n - T^n y_n\|\right) \\
&\leq k_n^4 \|x_n - p\|^2 + \left(b_n c_n' k_n^2 + c_n\right) r^2 - a_n b_n g\left(\|x_n - T^n y_n\|\right) \\
&\leq \left[1 + \left(k_n^4 - 1\right)\right] \|x_n - p\|^2 + \left(b_n c_n' k_n^2 + c_n\right) r^2 - a_n b_n g\left(\|x_n - T^n y_n\|\right).
\end{aligned} \tag{2.5}$$

Note that $\sum_{n=1}^{\infty}\left(k_n^2 - 1\right) < \infty$ is equivalent $\sum_{n=1}^{\infty}\left(k_n^4 - 1\right) < \infty$ and so, setting $\rho_n = r^2\left(k_n^4 - 1\right)$, then $\sum_{n=1}^{\infty} \rho_n < \infty$. Furthermore, since $g : [0,\infty) \to [0,\infty)$ is a continuous increasing convex function and $\{x_n - T^n y_n\}$ is a bounded sequence in *E,* we assert that $g\left(\|x_n - T^n y_n\|\right)$ is bounded. Set $\sigma_n = c_n g\left(\|x_n - T^n y_n\|\right)$, we have $\sum_{n=1}^{\infty} \sigma_n < \infty$. Since $\{k_n\}$ is bounded, and by hypothesis $\sum_{n=1}^{\infty} b_n c_n' < \infty$ so $\sum_{n=1}^{\infty} b_n c_n' k_n^2 < \infty$. Now, set

$$\delta_n = \rho_n + \sigma_n + \left(b_n c_n' k_n^2 + c_n\right) r^2 .$$

Then $\sum_{n=1}^{\infty} \delta_n < \infty$. By the assumption (i), we have $(1-b_n) \geq (1-b)$. It follows from (2.5) that

$$\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 - (1 - b_n - c_n) b_n g\left(\|x_n - T^n y_n\|\right) + \rho_n \left(b_n c_n' k_n^2 + c_n\right) r^2$$

$$\leq \|x_n - p\|^2 - (1-b_n) b_n g\left(\|x_n - T^n y_n\|\right) + \rho_n + \sigma_n + \left(b_n c_n' k_n^2 + c_n\right) r^2$$

$$\leq \|x_n - p\|^2 - (1-b_n) b_n g\left(\|x_n - T^n y_n\|\right) + \delta_n \tag{2.6}$$

which leads to

$$(1-b) b_n g\left(\|x_n - T^n y_n\|\right) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \delta_n, \tag{2.7}$$

and

$$(1-b) b_{n+1} g\left(\|x_{n+1} - T^{n+1} y_{n+1}\|\right) \leq \|x_{n+1} - p\|^2 - \|x_{n+2} - p\|^2 + \delta_{n+1}, \text{for all } n \geq 1. \tag{2.8}$$

Adding on both sides of (2.7) and (2.8) and using the condition $b_{n+1} \leq b_n$ for all $n \geq 1$, we have

$$(1-b) \sum_{n=1}^{\infty} b_{n+1} \left[g\left(\|x_{n+1} - T^{n+1} y_{n+1}\|\right) + g\left(\|x_n - T^n y_n\|\right) \right] < \infty. \tag{2.9}$$

Since $\sum_{n=1}^{\infty} b_n = \infty$ by the assumption (ii), we have

$$\liminf_{n\to\infty} \left[g\left(\|x_{n+1} - T^{n+1} y_{n+1}\|\right) + g\left(\|x_n - T^n y_n\|\right) \right] = 0. \tag{2.10}$$

By virtue of the continity and monotonicity of function g, we assert that there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\|x_{n_j} - T^{n_j} y_{n_j}\| \to 0, \|x_{n_{j+1}} - T^{n_{j+1}} y_{n_{j+1}}\| \to 0 \text{ as } j \to \infty. \tag{2.11}$$

By the assumption (iii), we see that

$$\|y_n - x_n\| \leq b_n' \|x_n - T^n x_n\| + c_n' \|v_n - x_n\| \to 0, \text{ as } n \to \infty. \tag{2.12}$$

It follows from the uniform equi-continuity of T that

$$\|T^n y_n - T^n x_n\| \to 0 \text{ as } n \to \infty. \tag{2.13}$$

Now we observe that

$$\|x_n - T^n x_n\| \leq \|x_n - T^n y_n\| + \|T^n y_n - T^n x_n\|. \tag{2.14}$$

It follows from (2.11) and (2.13) that

$$\|x_{n_j} - T^{n_j} x_{n_j}\| \to 0 \text{ and } \|x_{n_{j+1}} - T^{n_{j+1}} x_{n_{j+1}}\| \to 0, \text{ as } j \to \infty.$$

Since $\|T^{n_j} x_{n_j} - x_{n_j}\| \to 0$ as $j \to \infty$, we have

$$\|x_{n_{j+1}} - x_{n_j}\| \leq b_{n_j} \|T^{n_j} y_{n_j} - x_{n_j}\| + c_{n_j} \|u_{n_j} - x_{n_j}\| \to 0 \text{ as } j \to \infty. \tag{2.15}$$

It follows from the uniform equi-continuity of T that

$$\left\|T^{n_j}x_{n_{j+1}} - x_{n_j}\right\| \to 0 \text{ as } j \to \infty. \tag{2.16}$$

Again, from above inequalities, we observe that

$$\left\|T^{n_j}x_{n_{j+1}} - x_{n_{j+1}}\right\| \le \left\|T^{n_j}x_{n_{j+1}} - T^{n_j}x_{n_j}\right\| + \left\|T^{n_j}x_{n_j} - x_{n_{j+1}}\right\|$$

$$\le \left\|T^{n_j}x_{n_{j+1}} - T^{n_j}x_{n_j}\right\| + \left\|T^{n_j}x_{n_j} - x_{n_j}\right\| + \left\|x_{n_j} - x_{n_{j+1}}\right\| \to 0 \text{ as } j \to \infty. \tag{2.17}$$

It follows from the uniform equi-continuity of T that

$$\left\|T^{n_j+1}x_{n_{j+1}} - Tx_{n_{j+1}}\right\| \to 0 \text{ as } j \to \infty, \tag{2.18}$$

and

$$\left\|x_{n_{j+1}} - Tx_{n_{j+1}}\right\| \le \left\|x_{n_{j+1}} - T^{n_{j+1}}x_{n_{j+1}}\right\| + \left\|T^{n_{j+1}}x_{n_{j+1}} - Tx_{n_{j+1}}\right\|. \tag{2.19}$$

Therefore, it follows from (2.17), (2.18) and the above inequality that

$$\left\|x_{n_{j+1}} - Tx_{n_{j+1}}\right\| \to 0 \text{ as } j \to \infty. \tag{2.20}$$

This completes the proof.

Let $\{z_n\}$ be a given sequence in K. Recall that a mapping $T{:}K \to K$ with $F(T) \neq \phi$ is said to satisfy *condition* (A) [9] if there exists a nondecreasing function with $f:[0,\infty) \to [0,\infty)$ with $f(0)=0$, $f(r)>0$ for all $r \in (0,\infty)$ such that

$$\|z_n - Tz_n\| \ge f\left(d\left(z_n, F(T)\right)\right) \text{ for all } n \ge 1,$$

where $d\left(z_n, F(T)\right) = \inf\left\{\|z_n - p\| : p \in F(T)\right\}$.

By using Theorem 2.1, we have the following :

Theorem 2.2. Let E be a uniformly convex Banach space, K be a nonempty convex subset of E and $T{:}K \to K$ be a unifromly equi-continuous and asymptotically quasi-nonexpansive mapping with $F(T) \neq \phi$ and a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and $\sum_{n=1}^{\infty}\left(k_n^2 - 1\right) < \infty$. Let $\{x_n\}$ be a sequence in K defined by (2.1) with the following restrictions:

(i) $0 \le b_n \le b < 1$ and $b_{n+1} \le b_n$ for all $n \ge 1$,

(ii) $\sum_{n=1}^{\infty} b_n = \infty$, (iii) $\lim_{n\to\infty} b'_n = 0$, (iv) $\sum_{n=1}^{\infty} c_n = \infty$ and $\sum_{n=1}^{\infty} b_n c'_n < \infty$.

If T satisfies *condition* (*A*), then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. It follows from Theorem 2.1 that

$$\liminf_{n\to\infty}\|x_n - Tx_n\| = 0.$$

Since T satisfies condition (A), we have

$$\liminf_{n\to\infty} f\left(d\left(x_n, FT\right)\right) = 0.$$

From the property of f, it follows that

$$\liminf_{n\to\infty} d\left(x_n, F\left(T\right)\right) = 0.$$

It follows from Lemma 2.1 that $d\left(x_n, F\left(T\right)\right) \to 0$ as $n \to \infty$. Now, we can take an infinite subsequence $\{x_{n_j}\}$ of $\{x_n\}$ and a sequence $\{p_j\} \subset F(T)$ such that $\|x_{n_j} - p_j\| \le 2^{-j}$. Set $M = \exp\left\{\sum_{n=1}^{\infty}\left(k_n^2 - 1\right)\right\}$ and write $n_j + 1 = n_{j+l}$ for some $l \ge 1$. Then we have

$$\begin{aligned}
\|x_{n_{j+1}} - p_j\| &= \|x_{n_j+l} - p_j\| \\
&\le k_{n_j+l-1}^2\|x_{n_j+l-1} - p_j\| \\
&\le \left[1 + \left(k_{n_j+l-1}^2 - 1\right)\right]\|x_{n_j+l-1} - p_j\| \\
&\le \exp\left\{\left(k_{n_j+l-1}^2 - 1\right)\right\}\|x_{n_j+l-1} - p_j\| \\
&\le \exp\left\{\sum_{m=0}^{l-1}\left(k_{n_j+m}^2 - 1\right)\right\}\|x_{n_j} - p_j\| \\
&\le \frac{M}{2^j}.
\end{aligned} \tag{2.21}$$

It follows from (2.21) that

$$\begin{aligned}
\|p_{j+1} - p_j\| &\le \|p_{j+1} - x_{n_j+1}\| + \|x_{n_j+1} - p_j\| \\
&\le \frac{1}{2^{j+1}} + \frac{M}{2^j} \\
&\le \frac{2M+1}{2^{j+1}}.
\end{aligned} \tag{2.22}$$

Hence $\{p_j\}$ is a Cauchy sequence. Assume that $p_j \to p$ as $j \to \infty$. Then $p \in F\left(T\right)$ since $F(T)$ is closed, which implies that $x_j \to p$ as $j \to \infty$. This completes the proof.

Remark 2.1. We note that, if $T{:}K \to K$ is completely continuous, then it must be demicompact [8], and if T is continuous and demicompact, it must satisfy *condition* (A) [4,9]. In view of this observation, Theorem 2.2 improves the corresponding result of Liu [3] in the following aspects:

(i) K may be not necessarily compact or bounded,

(ii) T may be not uniformly Holder continous.

Remark 2.2. Our results improve and generalize the corresponding results of [6,7,8,10,11,12] and many others from the existing literature.

Remark 2.3. Our results also extend the corresponding results of Cho et al. [1] to the case of more general class of ϕ-continuous asymptotically quasi-nonexpansive mappings.

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