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ON SOME GENERALIZED FRACTIONAL DERIVATIVE FORMULAS

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ABSTRACT

The purpose of the present paper is to derive a number of key formulas for fractional derivatives of generalized multiple hypergeometric functions of several variables, multivariable *H*-function and genralized multivariable polynomials. Each of these formulas can be shown to yield interesting new results for various classes of generalized hypergeometric functions of several variables. Some of the applications of the new formulas provide potentially useful generalizations of known results in the theory of fractional calculus. Our results include all the results of Chandel-Kumar [12] as special cases and all recent results of Ram-Chandk [32] as special cases for the Fox-Wright generalized hypergeometric function of Wright [54].

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1. Introduction. The theory and applications of fractional calculus are based largely upon the familiar differential operator ${}_{\beta}D_x^{\alpha}$ defined by (cf., e.g., [31, p.49]; see also [47, p.356])

$$(1.1)\qquad {}_{\beta}D_x^{\alpha}\{f(x)\}=\begin{cases}\dfrac{1}{\Gamma(-\alpha)}\displaystyle\int_{\beta}^{x}(x-t)^{-\alpha-1}f(t)dt & (Re(\alpha)<0)\\[2ex] \dfrac{d^m}{dx^m}\,{}_{\beta}D_x^{\alpha-m}\{f(x)\} & (0\le Re(\alpha)<m, m\in N_0)\end{cases}$$

where $N_0=N\cup\{0\}, N=\{1,2,3,...\}$.

For $\beta=0$, equation (1.1) defines the classical Riemann Liouville fractional derivative (or integral) of order $\alpha(or-\alpha)$. On the other hand, when $\beta\to\infty$, the equation (1.1) may be identified with the definition of the familiar Weyl fractional derivative (or integral) of order $\alpha(\text{or} -\alpha)$ (see for details, [16, Chap. 13] and [34]). For the sake of simplicity, the special case of the fractional calculus operator ${}_{\beta}D_x^{\alpha}$ when $\beta = 0$ is written as D_x^{α}. Thus we have

(1.2) $$D_x^{\alpha} \equiv {}_0D_x^{\alpha}\,(\alpha \in C).$$

For $0 \le \alpha < 1; \beta, \eta, x \in R; m \in N,$ the generalized modified fractional derivative operator due to Saigo is defined in Samko, kilbas and Marichev [34] as

(1.3) $$D_{0,x,m}^{\alpha,\beta,\eta} f(x) = \frac{d}{dz}\left(\frac{z^{-m(\beta-\eta)}}{\Gamma(1-\alpha)}\int_0^x \left(x^m - t^m\right)^{-\alpha} F(\beta-\alpha), 1-\eta; 1-\alpha; 1-t^m/x^m\right) f(t)\,dt^m.$$

The Multiplicity of $x^m - t^m$ in above equation is removed by requiring $log\left(x^m - t^m\right)$ as real for $x^m - t^m > 0$ and is assumed to be well defined in the unit disk.

It is remarkable that

(1.4) $$D_{0,x,1}^{\alpha,\alpha,\eta} f(x) = D_x^{\alpha} f(x),$$

where D_x^{α} is the familiar Riemann-Liouville fractional derivative operator defined by Miller and Ross [28]

For $0 \le \alpha < 1, m \in N; \beta, \eta, x \in R;\ \ \mu > max(0, \beta - \eta),$ the refined form due to Bhatt and Raina [2] is given by

(1.5) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\right\} = \frac{\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)} x^{(\mu-\beta-1)m}.$$

Making an appeal to the Saigo modified fractional derivative operator $D_{0,x,m}^{\alpha,\beta,\eta}$, Miller and Rasso [28] investigated fractional derivative formulas. Kilbas [21] established the fractional integral formulas for the Wright ([54]; see also Erdélyi [14]) introduced the generalized hypergeometric function ${}_p\Psi_q$ Fox-Wright generalized hypergeometric function, defined by

(1.6) $${}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{matrix}(a_i, A_i)_{1,p} \\ (b_j, B_j)_{1,q}\end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i + A_i k)}{\prod_{j=1}^{q}\Gamma(b_j + B_j k)} \frac{z^k}{k!}$$

where $a_i, b_j \in C; A_i > 0, B_j > 0; 1 + \sum_{j=1}^{q} - \sum_{i=1}^{p} A_i \geq 0;$ $(A_i, B_j \neq 0)(i = 1, \ldots, p; j = 1, .., q)$, for suitable bounded values of $|z|$.

For details of conditions of its existence and the *H*-function due to Mathai and Saxena [26], see Kilbas [21].

Wright [55] also introduced the special case of (1.6) (called Wright function) defined in the form:

$$\phi(\alpha, \beta; z) = {}_0\Psi_1\left[\left.\begin{matrix} \\ (\alpha, \beta) \end{matrix}\right| z\right] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\beta k + \alpha)} \frac{z^k}{k!}, \tag{1.7}$$

where $\alpha, z \in C$ and $\beta \in R$.

Kiryakova [23] introduced a function $J_\nu^\delta(z)$ called Bessel-Maitland function or Wright generalized Bessel function defined by

$$J_\nu^\delta(z) = \phi(v + 1, \delta; -z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\delta k + \nu + 1)} \frac{(-z)^k}{k!}. \tag{1.8}$$

Other special case of (1.4) but generalizing the classical Mittag-Leffler function (Erdélyi [16]) is given by Kilbas et al. [22].

Srivastava and Garg [38] introduced a general class of multivariable polynomials defined by

$$S_L^{h_1, \ldots, h_s}(x_1, \ldots, x_s) = \sum_{k_1, \ldots, k_s = 0}^{k_1 k_1 + \ldots + h_s k_s \leq L} (-L)_{hk_1 + \ldots + h_s k_s} A(L; k_1, \ldots, k_s) \frac{x_1^{k_1}}{k_1!} \cdots \frac{x_s^{k_s}}{k_s!} \tag{1.9}$$

where $h_1, \ldots, h_s$ are arbitrary positive integers and the coefficients $A(L; k_1, \ldots, k_s)$, $(L; k_i \in N_0; i = 1, \ldots, s)$ are arbitrary constants real or complex, $N_0 = N \cup \{0\}$.

It is clear that for $s = 1$, the polynomials (1.9) reduce to the polynomials of Srivsatava [37] defined by

$$S_L^h(x) = \sum_{k=0}^{[l/h]} \frac{(-l)_{hk}}{k!} A_{l,k} x^k \qquad (l \in N_0 = \{0, 1, 2, \ldots\}), \tag{1.10}$$

where h is arbitrary positive integer and the coefficients are arbitrary constants, real or complex.

The computation of fractional derivatives (and fractional integrals) of special functions of one and more variables is important from the point of view of the usefulness of these results in (for example) the evaluation of series and integrals (cf., e.g., [29] and [52]), the derivative of generating functions [46, Chap. 5] and solutions of differential and integral equation (cf. [29] and [39], Chap. 3. see also [27, 30 and 51]. Making an appeal to the operator (1.4), Chandel and Vishwakarma ([6],[7]) have obtained fractional derivatives of confluent forms due to Chandel-Vishwakarma [5]) of Karlsson's multiple hypergeometric function ${}^{(k)}F_{CD}^{(n)}$ [20] and other multiple hypergeometric functions of Lauricella [24], Chandel [3], Chandel and Gupta [4] including their confluent forms. Srivastava and Goyal [43] derived fractional derivatives of the multivariable H-function of Srivastava and Panda ([48]-[50]). Srivastava, Chandel and Vishwakarma [40] obtained several new fractional derivative formulas involving the multivariable H-function defined by Srivastava and Panda (see [50, p.271, eq. [4.1] et Seq.]) and studied systematically by them (see [48] [50] also [44]).

Further for special interest, Chandel and Vishwakarma [8] and Chandel-Sharma [11] derived fractional derivatives involving hypergeometric functions of four variables defined by Exton [18] and Sharma-Parihar [35], while Chandel-Sharma [11] established fractional derivative formulas for their own hypergeometric functions of four variables ([9],[10]). Recently, Chandel and Kumar [12] derived generalizations and unifications of various key formulas of Srivatava Chandel and Vishwakarma [40]. Very recently employing the operator (1.3), Ram and Chandak [32] derived a generalized derivative formula involving the product of Fox-Wright generalized hypergeometric function ${}_p\Psi_q$ defined by (1.6) and a general class of multivariable polynomials defined by (1.9). Some special cases are also discussed.

In the present paper, employing the operator (1.3), with the motivation of Chandel-Kumar [12] and Ram-Chandak [32], we derive generalizations and unifications of various key formulas of Chandel-Kumar [12] and Ram-Chandak [32]. Each of these formulas can be shown to yield interesting new results for various classes of generalized hypergeometric functions of several variables and genearlized hypergeometric functions of one variable. Some of applications of the key formulas provide potentially useful genearlizations of known results in the theory of fractional calculus. Some special cases are also discussed.

2. Key Formulas. In this section, making an appeal to the result (1.5), we derive the following key formulas on generalized fractional derivatives involving multiple hypergeometric function of Srivastava-Daoust ([41],[42]; also see Srivastava-Manocha [46, p.64 (18), (19), (20)]), multivariable H-function of

Srivsatava-Panda ([48]-[50]; see also Srivastava, Gupta-Goyal [44]), generalized multivariable polynomials due to Srivastava-Garg [38]:

$$(2.1)\quad D_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1}\dots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r}\Big\{x_1^{(\mu_1-1)m_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{\lambda_1}\dots x_r^{(\mu_r-1)m_r}\left(x_r^{m_r\nu_r}+\xi_r\right)^{\lambda_r}$$

$$F_{C:D';\dots;D^{(n)}}^{A:B';\dots;B^{(n)}}\left(\begin{matrix}\left[(a):\theta',\dots,\theta^{(n)}\right]:\left[(b'):\phi'\right];\dots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[(c);\psi',\dots,\psi^{(n)}\right]:\left[(d'):\delta'\right];\dots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix}\right.$$

$$\left. z_1x_1^{m_1\rho'_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{-\sigma'_1}\dots x_r^{m_r\rho'_r}\left(x_r^{m_r\nu_r}+\xi_r\right)^{-\sigma'_r},\dots,z_nx_1^{m_1\rho_1^n}\left(x_1^{m_1\nu_1}+\xi_1\right)^{-\sigma_1^n}\dots x_r^{m_r\rho_r^n}\left(x_r^{m_r\nu_r}+\xi_r\right)^{-\sigma_r^n}\right)\Big\}$$

$$=\prod_{j=1}^{r}\frac{\Gamma\left(\mu_j\right)\Gamma\left(\mu_j+\eta_j-\beta_j\right)}{\Gamma\left(\mu_j-\beta_j\right)\Gamma\left(\mu_j+\eta_j-\alpha_j\right)}x_j^{\left(\mu_j-\beta_j-1\right)m_j}\xi_j^{\lambda_j}F_{C+3r:D';\dots;D^{(n)};0;\dots;0}^{A+3r:\beta';\dots;\beta^{(n)};0;\dots;0}\left(\begin{matrix}\left[(a):\theta',\dots,\theta^{(n)},0,\dots,0\right],\\ \left[(c):\psi',\dots,\psi^{(n)},0,\dots,0\right],\end{matrix}\right.$$

$$\begin{matrix}\left[\mu_1:\rho'_1,\dots,\rho_1^n,\nu_1,0,\dots,0\right],\left[\mu_1+\eta_1-\beta_1:\rho'_1,\dots,\rho_1^n,\nu_1,0,\dots,0\right],\dots,\left[\mu_r:\rho'_r,\dots,\rho_r^n,0,\dots,0,\nu_r\right],\\ \left[\mu_1-\beta_1:\rho'_1,\dots,\rho_1^n,\nu_1,0,\dots,0\right],\left[\mu_1+\eta_1-\alpha_1:\rho'_1,\dots,\rho_1^n,\nu_1,0,\dots,0\right],\dots,\left[\mu_r-\beta_r:\rho'_r,\dots,\rho_r^n,0,\dots,0,\nu_r\right],\end{matrix}$$

$$\begin{matrix}\left[\mu_r+\eta_r-\beta_r:\rho'_r,\dots,\rho_r^n,0,\dots,0,\nu_r\right],\left[-\lambda_1:\sigma'_1,\dots,\sigma_1^n,1,0,\dots,0\right],\dots,\left[-\lambda_r:\sigma'_r,\dots,\sigma_r^n,0,\dots,0,1\right]:\\ \left[\mu_r-\eta_r-\alpha_r:\rho'_r,\dots,\rho_r^n,0,\dots,0,\nu_r\right],\left[-\lambda_1:\sigma'_1,\dots,\sigma_1^n,0,\dots,0\right],\dots,\left[-\lambda_r:\sigma'_r,\dots,\sigma_r^n,0,\dots,0\right]:\end{matrix}$$

$$\left.\begin{matrix}\left[(b'):\phi'\right];\dots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];-;\dots;-;\\ \left[(d'):\delta'\right];\dots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];-;\dots;-;\end{matrix}Z_1,\dots,Z_n,\frac{-x_1^{m_1}}{\xi_1},\dots,\frac{-x_r^{m_r}}{\xi_r}\right),$$

where $F_{C:D';\dots;D^{(n)}}^{A:B';\dots;\beta^{(n)}}$ is generalized multiple hypergeometric function of Srivastava and Daoust ([41],[42]), $0\le\alpha_i<1;m_i\in N;\beta_i,\eta_ix_i\in R;\mu_i>max\left(0,\beta_i-\eta_i\right)$,

$$Z_i=\frac{z_ix_1^{\rho_1^im_1}\dots x_r^{\rho_r^im_r}}{\xi_1^{\sigma_1^i}\dots\xi_r^{\sigma_r^i}},\quad i=1,\dots,n;$$

$$1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\prod_{j=1}^{B^{(i)}}\phi_j^{(i)}>0,\ i=1,\dots,n$$

and

$$\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0 \quad i=n+1,\ldots,n+r.$$

(2.2) $$D_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1}\ldots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r}\Big\{x_1^{(\mu_1-1)m_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{\lambda_1}\ldots x_r^{(\mu_r-1)m_r}\left(x_r^{m_r\nu_r}+\xi_r\right)^{\lambda_r}$$

$$H_{A,C:[B',D'];\ldots;\left[B^{(n)},D^{(n)}\right]}^{0,\lambda:(\mu',\nu');\ldots;\left(\mu^{(n)},\nu^{(n)}\right)}\left(\begin{matrix}\left[(a),\theta',\ldots,\theta^{(n)}\right]:\left[(b'):\phi'\right];\ldots;\left[\left(b^{n}\right):\phi^{n}\right];\\ \left[(c):\psi',\ldots,\psi^{(n)}\right]:\left[(d'):\delta'\right];\ldots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix}\right.$$

$$\left.z_1x_1^{m_1\rho'_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{-\sigma'_1}\ldots x_r^{m_r\rho'_r}\left(x_r^{m_r\nu_r}+\xi_n\right)^{-\sigma'_r},\ldots,z_nx_1^{m_1\rho_1^n}\left(x_1^{m_1\nu_1}+\xi_1\right)^{-\sigma_1^n}\ldots x_r^{m_r\sigma_r^n}\left(x_r^{m_r\nu_r}+\xi_r\right)^{-\sigma_r^n}\right)\Big\}$$

$$=\xi_1^{\lambda_1}\ldots\xi_r^{\lambda_r}x_1^{(\mu_1-\beta_1-1)m_1}\ldots x_r^{(\mu_i-\beta_r-1)m_r}\sum_{N_1,\ldots,N_r=0}^{\infty}\frac{\left(-x_1^{\nu_1m_1}/\xi_1\right)^{N_1}}{N_1!}\cdots\frac{\left(-x_r^{\nu_rm_r}/\xi_r\right)^{N_r}}{N_r!}$$

$$H_{A+3r,C+3r:[B',D'];\ldots;\left[B^{(n)},D^{(n)}\right]}^{0,\lambda+3r:(\mu',\nu');\ldots;\left(\mu^{(n)},\nu^{(n)}\right)}\left(\begin{matrix}\left[(a):\theta',\ldots,\theta^{(n)}\right],\left[1-\mu_1-\nu_1N_1:\rho'_1,\ldots,\rho_1^n\right],\\ \left[(c):\psi',\ldots,\psi^{(n)}\right],\left[1-\mu_1-\nu_1N_1+\beta_1:\rho'_1,\ldots,\rho_1^n\right],\end{matrix}\right.$$

$$\begin{matrix}\left[1-\mu_1-\eta_1+\beta_1-\nu_1N_1:\rho'_1,\ldots,\rho_1^n\right],\left[1+\lambda_1-N_1:\sigma'_1,\ldots,\sigma_1^n\right],\ldots,\left[1-\mu_r-\nu_rN_r:\rho'_r,\ldots,\rho_r^n\right],\\ \left[1-\mu_1-\nu_1N_1-\eta_1+\alpha_1:\rho'_1,\ldots,\rho_1^n\right],\left[1+\lambda_1:\sigma'_1,\ldots,\sigma_1^n\right],\ldots,\left[1-\mu_r-\nu_rN_r+\beta_r:\rho'_r,\ldots,\rho_r^n\right],\end{matrix}$$

$$\begin{matrix}\left[1-\mu_r-\eta_r+\beta_r-\nu_rN_r:\rho'_r,\ldots,\rho_r^n\right],\left[1+\lambda_r-N_r:\sigma'_r,\ldots,\sigma_r^n\right]:\left[(b'):\phi'\right];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[1-\mu_r-\eta_r+\alpha_r-\nu_rN_r:\rho'_r,\ldots,\rho_r^n\right],\left[1+\lambda_r:\sigma'_r,\ldots,\sigma_r^n\right]:\quad\left[(d'):\delta'\right];\ldots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix}$$

$$\left.Z_1,\ldots,Z_n\right)$$

provided that $0\le\alpha_i<1, m_i\in N;\beta_i\eta_i,x_i\in R;\mu_i>Max\left(0,\beta_i-\eta_i\right)$,

$max\left\{arg\left(x_1^{\nu_1m_1}/\xi_1\right),\ldots,arg\left(x_r^{\nu_rm_r}/\xi_r\right)\right\}<\pi, min\left(\nu_1,\ldots,\nu_r;\rho_1^i,\ldots,\rho_r^i;\sigma_1^i,\ldots,\sigma_r^i\right)>0,$

$Z_i=\dfrac{z_ix_1^{\rho_1^im_1}\ldots x_r^{\rho_r^im_r}}{\xi_1^{\sigma_1^i}\ldots\xi_r^{\sigma_r^i}}$, $i=1,\ldots,n$ and where $H_{A,C:[B',D'];\ldots;\left[B^{(n)},D^{(n)}\right]}^{0,\lambda:(\mu',\nu');\ldots;\left(\mu^{(n)},\nu^{(n)}\right)}$ is multivariable *H*-

function due to Srivastava and Panda ([48]-[50]).

(2.3) $$D_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1}\dots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r}\Big\{x_1^{(\mu_1-1)m_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{\lambda_1}\dots x_r^{(\mu_r-1)m_r}\left(x_r^{x_r\nu_r}+\xi_r\right)^{\lambda_r}$$

$$F_{C:D';\dots;D^{(n)}}^{A:B';\dots;B^{(n)}}\left(\begin{matrix}\left[(a):\theta',\dots,\theta^{(n)}\right]:\left[(b'):\phi'\right];\dots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[(c):\psi',\dots,\psi^{(n)}\right]:\left[(d'):\delta'\right];\dots;\left[\left(d^{n}\right):\delta^{(n)}\right];\end{matrix}\; z_1x_1^{m_1\rho'_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{-\sigma'_1}\right.$$

$$\left.\dots x_r^{m_r}\rho'_r\left(x_r^{m_r\nu_r}+\xi_r\right)^{-\sigma'_r},\dots,z_nx_1^{m_1\rho_1^n}\left(x_1^{m_1\nu_1}+\xi_1\right)^{-\sigma_1^n}\dots x_r^{m_r}\rho_r^n\left(x_r^{m_r\nu_r}+\xi_r\right)^{-\sigma_r^n}\right)$$

$$F_{G:H';\dots;H^{(s)}}^{E:F';\dots;F^{(s)}}\left(\begin{matrix}\left[(e):\alpha',\dots,\alpha^{(s)}\right]:\left[(f'):\beta'\right];\dots;\left[\left(f^{(s)}\right):\beta^{(s)}\right];\\ \left[(g):\gamma',\dots,\gamma^{(s)}\right]:\left[(h'):\eta'_1\right];\dots;\left[\left(h^{(s)}\right):\eta^{(s)}\right];\end{matrix}\; w_1x_1^{k'_1m_1}\dots x_r^{k'_rm_r},\dots,\right.$$

$$\left.w_sx_1^{k_1^sm_1}\dots x_r^{k_r^sm_r}\right)\Big\}$$

$$=\prod_{i=1}^{r}\frac{\xi_i^{\lambda_i}\Gamma\left(\mu_i\right)\Gamma\left(\mu_i+\eta_i-\beta_i\right)}{\Gamma\left(\mu_i-\beta_i\right)\Gamma\left(\mu_i+\eta_i-\alpha_i\right)}x_i^{(\mu_i-\beta_i-1)m_i}$$

$$F_{C+G+3r:D';\dots;D^{(n)};H';\dots;H^{(s)};0;\dots;0}^{A+E+3r:\beta';\dots;\beta^{(n)};F';\dots;F^{(s)};0;\dots;0}\left(\begin{matrix}\left[(a):\theta',\dots,\theta^{(n)},0,\dots,0\right],\left[(e):0,\dots,0,\alpha',\dots,\alpha^{(s)},0,\dots,0\right],\\ \left[(c):\psi',\dots,\psi^{(n)},0,\dots,0\right],\left[(g):0,\dots,0,\gamma',\dots,\gamma^{(s)},0,\dots,0\right],\end{matrix}\right.$$

$$\begin{matrix}\left[\mu_1;\rho'_1,\dots,\rho_1^n,k'_1,\dots,k_1^s,\nu_1,0,\dots,0\right],\left[\mu_1+\eta_1-\beta_1:\rho'_1,\dots,\rho_1^n,k'_1,\dots,k_1^s,\nu_1,0,\dots,0\right],\dots,\\ \left[\mu_1-\beta_1:\rho'_1,\dots,\rho_1^n,k'_1,\dots,k_1^s,\nu_1,0,\dots,0\right],\left[\mu_1+\eta_1-\alpha_1:\rho'_1,\dots,\rho_1^n,k'_1,\dots,k_1^s,\nu_1,0,\dots,0\right],\dots,\end{matrix}$$

$$\begin{matrix}\left[\mu_r;\rho'_r,\dots,\rho_r^n,k'_r,\dots,k_r^s,0,\dots,0,\nu_r\right],\left[\mu_r+\eta_r-\beta_r:\rho'_r,\dots,\rho_r^n,k'_r,\dots,k_r^s,0,\dots,0,\nu_r\right],\\ \left[\mu_r-\beta_r:\rho'_r,\dots,\rho_r^n,k'_r,\dots,k_r^s,0,\dots,0,\nu_r\right],\left[\mu_r+\eta_r-\alpha_r:\rho'_r,\dots,\rho_r^n,k'_r,\dots,k_r^s,0,\dots,0,\nu_r\right],\end{matrix}$$

$$\begin{matrix}\left[-\lambda_1:\sigma'_1,\dots,\sigma_1^n,0,\dots,0,1,0,\dots,0\right];\dots;\left[-\lambda_r:\sigma'_r,\dots,\sigma_r^n,0,\dots,0,1\right]:\\ \left[-\lambda_1:\sigma'_1,\dots,\sigma_1^n,0,\dots,0\right],\dots,\left[-\lambda_r:\sigma'_r,\dots,\sigma_r^n,0,\dots,0\right]:\end{matrix}$$

$$\left.\begin{matrix}\left[(b'):\phi'\right];\dots;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\left[(f'):\beta'\right];\dots;\left[\left(f^{(s)}\right):\beta^{(s)}\right];-;\dots;-;\\ \left[(d'):\delta'\right];\dots;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\left[(h'):\eta'\right];\dots;\left[\left(h^{(s)}\right):\eta^{(s)}\right];-;\dots;-;\end{matrix}\; Z_1,\dots,Z_n,\right.$$

$$w_1 x_1^{k'_1 m_1} \dots x_r^{k'_r m_r}, \dots, w_r x_1^{k_1^s m_1} \dots x_r^{k_r^s m_r}, -x_1^{\nu_1 m_1}/\xi_1, \dots, -x_r^{\nu_r m_r}/\xi_r \Big),$$

valid if $Z_i = \dfrac{z_i x_1^{\rho_1^i m_1} \dots x_r^{\rho_r^i m_r}}{\xi_1^{\sigma_1^i} \dots \xi_r^{\sigma_r^i}}$, $0 \le \alpha_i < 1, m_i \in N; \beta_i, \eta_i, x_i \in R; \mu_i > Max(0, \beta_i - \eta_i)$, $min(\nu_1, \dots, \nu_r; \rho_1^i, \dots, \rho_r^i; \sigma_1^i; \dots, \sigma_r^i) > 0, i = 1, \dots, n.$

(2.4) $$D_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1} \dots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r} \Big\{ x_1^{(\mu_1-1)m_1} \left(x_1^{m_1\nu_1} + \xi_1\right)^{\lambda_1} \dots x_r^{(\mu_r-1)m_r} \left(x_r^{m_r\nu_r} + \xi_r\right)^{\lambda_r}$$

$$F_{C:D';\dots;D^{(n)}}^{A:B';\dots;B^{(n)}} \left(\begin{matrix} \left[(a):\theta',\dots,\theta^{(n)}\right]:\left[(b'):\phi'\right];\dots;\left[\left(b^{(n)}\right):\phi^{(n)}\right]; \\ \left[(c):\psi',\dots,\psi^{(n)}\right]:\left[(d'):\delta'\right];\dots;\left[\left(d^{n}\right):\delta^{(n)}\right]; \end{matrix} z_1 x_1^{m_1\rho'_1} \left(x_1^{m_1\nu_1} + \xi_1\right)^{-\sigma'_1} \right.$$

$$\left. \dots x_r^{m_r\rho'_r} \left(x_r^{m_r\nu_r} + \xi_r\right)^{-\sigma'_r}, \dots, z_n x_1^{m_1\rho_1^n} \left(x_1^{m_1\nu_1} + \xi_1\right)^{-\sigma_1^n} \dots x_r^{m_r\rho_r^n} \left(x_r^{m_r\nu_r} + \xi_r\right)^{-\sigma_r^n} \right)$$

$$S_L^{h_1,\dots,h_s} \left(w_1 x_1^{k'_1 m_1} \dots x_r^{k'_r m_r}, \dots, w_s x_1^{k_1^s m_1} \dots x_r^{k_r^s m_r} \right) \Big\}$$

$$= \frac{\xi_1^{\lambda_1} \dots \xi_r^{\lambda_r} \Gamma(\mu_1)\Gamma(\mu_1+\eta_1-\beta_1) \dots \Gamma(\mu_r)\Gamma(\mu_r+\eta_r-\beta_r)}{\Gamma(\mu_1-\beta_1)\Gamma(\mu_1+\eta_1-\alpha_1) \dots \Gamma(\mu_r-\beta_r)\Gamma(\mu_r+\eta_r-\alpha_r)} x_1^{(\mu_1-\beta_1-1)m_1} \dots x_r^{(\mu_r-\beta_r-1)m_r}$$

$$\sum_{R_1,\dots,R_s=0}^{h_1R_1+\dots+h_sR_s \le L} (-L, h_1R_1 + \dots + h_sR_s) A(L : R_1, \dots, R_s)$$

$$\frac{(\mu_1, k'_1 R_1 + \dots + k_1^s R_s) \dots (\mu_r, k'_r R_1 + \dots + k_r^s R_s)}{(\mu_1 - \beta_1, k'_1 R_1 + \dots + k_1^s R_s) \dots (\mu_r - \beta_r, k'_r R_1 + \dots + k_r^s R_s)}$$

$$\frac{(\mu_1 + \eta_1 - \beta_1, k'_1 R_1 + \dots + k_1^s R_s) \dots (\mu_r + \eta_r - \beta_r, k'_r R_1 + \dots + k_r^s R_s)}{(\mu_1 + \eta_1 - \alpha_1, k'_1 R_1 + \dots + k_1^s R_s) \dots (\mu_r + \eta_r - \alpha_r, k'_r R_1 + \dots + k_r^s r_s)}$$

$$\frac{\left(w_1 x_1^{k'_1 m_1} \dots x_r^{k'_r m_r}\right)^{R_1}}{R_1!} \dots \frac{\left(w_s x_1^{k_1^s m_1} \dots x_r^{k_r^s m_r}\right)^{R_s}}{R_s!} \; F_{C+3r:D';\dots;D^{(n)};0;\dots;0}^{A+3r:B';\dots;B^{(n)};0;\dots;0} \left(\begin{matrix} \left[(a):\theta',\dots,\theta^{(n)},0,\dots,0\right], \\ \left[(c):\psi',\dots,\psi^{(n)},0,\dots,0\right], \end{matrix} \right.$$

$$\begin{matrix} \left[-\lambda_1 : \sigma'_1, \dots, \sigma_1^n, \nu_1, 0, \dots, 0\right], \dots, \left[-\lambda_r : \sigma'_r, \dots, \sigma_r^n, 0, \dots, 0, \nu_r\right], \\ \left[-\lambda_1 : \sigma'_1, \dots, \sigma_1^n, 0, \dots, 0\right], \dots, \left[-\lambda_r : \sigma'_r, \dots, \sigma_r^n, 0, \dots, 0\right], \end{matrix}$$

$$\left[\mu_1+k'_1R_1+...+k_1^sR_s:\rho'_1,...,\rho_1^n,\nu_1,0,...,0\right],...,\left[\mu_r+k'_rR_1+...+k_r^sR_s:\rho'_r,...,\rho_r^n,0,...,0,\nu_r\right],$$
$$\left[\mu_1-\beta_1+k'_1R_1+...+k_1^sR_s:\rho'_1,...,\rho_1^n,\nu_1,0,...,0\right],...,\left[\mu_r-\beta_r+k'_rR_1+...+k_r^sR_s:\rho'_r,...,\rho_r^n,0,...,0,\nu_r\right],$$

$$\left[\mu_1+\eta_1-\beta_1+k'_1R_1+...+k_1^sR_s:\rho'_1,...,\rho_1^n,\nu_1,0,...,0\right],...,\left[\mu_r+\eta_r-\beta_r+k'_rR_1+...+k_r^sR_s:\rho'_r,...,\rho_r^n,0,...,0,\nu_r\right]:$$
$$\left[\mu_1+\eta_1-\alpha_1+k'_1R_1+...+k_1^sR_s:\rho'_1,...,\rho_1^n,\nu_1,0,...,0\right],...,\left[\mu_r+\eta_r-\alpha_r+k'_rR_1+...+k_r^sR_s:\rho'_r,...,\rho_r^n,0,...,0,\nu_r\right]:$$

$$\left.\begin{matrix}\left[(b'):\phi'\right];...;\left[\left(b^{(n)}\right):\phi^{(n)}\right];-;...;-;\\ \left[(d'):\delta'\right];...;\left[\left(d^{(n)}\right):\delta^{(n)}\right];-;...;-;\end{matrix}\; Z_1,...,Z_n,\frac{-x_1^{\nu_1m_1}}{\xi_1},...,\frac{-x_r^{\nu_rm_r}}{\xi_r}\right),$$

provided that $Z_i=\dfrac{z_ix_1^{\rho_1^im_1}...x_r^{\rho_r^im_r}}{\xi_1^{\sigma_1^i}...\xi_r^{\sigma_r^i}}$, $0\le\alpha_i<1,m_i\in N;\beta_i,\eta_i,x_i\in R;\mu_i>max\left(0,\beta_i-\eta_i\right)$,

$$1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0,i=1,...,n\ ;\ \sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0,$$

$i=n+1,...,n+r$.

$F_{C:D';...;D^{(n)}}^{A:B';...;B^{(n)}}$ is generalized multiple hypergeometric function of Srivastava and Daoust ([41],[42]] while $S_L^{h_1,...,h_s}\left(x_1,...,x_s\right)$ are generalized multivariable polynomials due to Srivastava and Garg [38], defined by (1.9).

(2.5) $$D_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1}...D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r}\left\{x_1^{(\mu_1-1)m_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{\lambda_1}...x_r^{(\mu_r-1)m_r}\left(x_r^{m_r\nu_r}+\xi_r\right)^{\lambda_r}\right.$$

$$H_{A,C:[B',D'];...;\left[B^{(n)},D^{(n)}\right]}^{0,\lambda:\left[\mu^1,\nu^1\right];...;\left[\mu^{(n)},\nu^{(n)}\right]}\left(\begin{matrix}\left[(a):\theta',...,\theta^{(n)}\right]:\left[(b'):\phi'\right];...;\left[\left(b^{(n)}\right):\phi^{(n)}\right];\\ \left[(c):\psi',...,\psi^{(n)}\right]:\left[(d'):\delta'\right];...;\left[\left(d^{(n)}\right):\delta^{(n)}\right];\end{matrix}\right.$$

$$\left.z_1x_1^{m_1\rho'_1}\left(x_1^{m_1\nu_1}+\xi_1\right)^{-\sigma'_1}...x_r^{m_r\rho_r^1}\left(x_r^{m_r\nu_r}+\xi_r\right)^{-\sigma'_r},...,z_nx_1^{m_1\rho_1^n}\left(x_1^{m_1\nu_1}+\xi_1\right)^{-\sigma_1^n}...x_r^{m_r\rho_r^n}\left(x_r^{m_r\nu_r}+\xi_r\right)^{-\sigma_r^n}\right)$$

$$\left.S_L^{h_1,...,h_s}\left(w_1x_1^{k'_1m_1}...x_r^{k'_rm_r},...,w_sx_1^{k_1^sm_1}...x_r^{k_r^sm_r}\right)\right\}$$

$$\sum_{R_1,\ldots,R_s=0}^{h_1R_1+\ldots+h_sR_s\le L}(-L,h_1R_1+\ldots+h_sR_s)A(L;R_1,\ldots,R_s)\frac{w_1^{R_1}}{R_1!}\cdots\frac{w_s^{R_s}}{R_s!}x_1^{(\nu_1N_1+k'_1R_1+\ldots+k_1^sR_s)m_1}\cdots$$

$$x_r^{(\nu_rN_r+k'_rR_1+\ldots+k_r^sR_s)m_r}H^{0,\lambda+3r:(\mu',\nu');\ldots;(\mu^{(n)},\nu^{(n)})}_{A+3r,C+3r:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],\\ [(c):\psi',\ldots,\psi^{(n)}],\end{matrix}\right.$$

$$\begin{matrix}[1+\lambda_1-N_1:\sigma'_1,\ldots,\sigma_1^n],[1-\mu_1-\nu_1N_1-k'_1R_1-\ldots-k_1^sR_s:\rho'_1,\ldots,\rho_1^n],\\ [1+\lambda_1:\sigma'_1,\ldots,\sigma_1^n],[1-\mu_1-\nu_1N_1+\beta_1-k'_1R_1-\ldots-k_1^sR_s:\rho'_1,\ldots,\rho_1^n],\end{matrix}$$

$$\begin{matrix}[1-\mu_1-\eta_1+\beta_1-\nu_1N_1-k'_1R_1-\ldots-k_1^sR_s:\rho'_1,\ldots,\rho_1^n],\ldots,\\ [1-\mu_1-\eta_1+\alpha_1-\nu_1N_1-k'_1R_1-\ldots-k_1^sR_s:\rho'_1,\ldots,\rho_1^n],\ldots,\end{matrix}$$

$$\begin{matrix}[1+\lambda_r-N_r:\sigma'_r,\ldots,\sigma_r^n],[1-\mu_r-\nu_rN_r-k'_rR_1-\ldots-k_r^sR_s:\rho'_r,\ldots,\rho_r^n],\\ [1+\lambda_r:\sigma'_r,\ldots,\sigma_r^n],[1-\mu_r-\nu_rN_r+\beta_r-k'_rR_1-\ldots-k_r^sR_s:\rho'_r,\ldots,\rho_r^n],\end{matrix}$$

$$\begin{matrix}[1-\mu_r-\eta_r+\beta_r-\nu_rN_r-k'_rR_1-\ldots-k_r^sR_s:\rho'_r,\ldots,\rho_r^n]:\\ [1-\mu_r-\eta_r+\alpha_r-\nu_rN_r-k'_rR_1-\ldots-k_r^sR_s:\rho'_r,\ldots,\rho_r^n]:\end{matrix}$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}Z_1,\ldots,Z_n\right),$$

where $Z_i=\dfrac{z_ix_1^{\rho_1^im_1}\ldots x_r^{\rho_r^im_r}}{\xi_1^{\sigma_1^i}\ldots\xi_r^{\sigma_r^i}}$, $min(\nu_1,\ldots,\nu_r;\rho_1^i,\ldots,\rho_r^i;\sigma_1^i,\ldots,\sigma_r^i)>0,\quad 0\le\alpha_j<1,m_j\in N;$ $\beta_j,\eta_j,x_j\in R;\mu_j>max(0,\beta_j-\eta_j),j=1,\ldots,r$ and $S_L^{h_1,\ldots,h_s}(w_1,\ldots,w_s)$ are generalized multivariable polynomials due to Srivastava and Garg [38] defined by (1.9). Our result (2.5) among others also includes (2.4) as special case.

3. Proofs of the Key Formulas. In this section, we prove the results of Section 2.

Proof of (2.1). For brevity, we denote

$$S\equiv\sum_{M_1,\ldots,M_n=0}^{\infty}\frac{1}{M!\ldots M_n!}\frac{\prod_{j=1}^{A}(a_j,M_1\theta'_j+\ldots+M_n\theta_j^{(n)})\prod_{j=1}^{B'}(b'_j,M_1\phi'_j)\ldots\prod_{j=1}^{B^{(n)}}(b_j^{(n)},M_n\phi_j^{(n)})}{\prod_{j=1}^{C}(c_j,M_1\phi'_j+\ldots+M_n\phi_j^{(n)})\prod_{j=1}^{D'}(d'_j,M_1\delta'_j)\ldots\prod_{j=1}^{D^{(n)}}(d_j^{(n)},M_n\delta_j^{(n)})}.$$

Therefore, **left hand side of (2.1)**

$$= S z_1^{M_1} \ldots z_n^{M_n} D_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1} \ldots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r} \left\{ x_1^{(\mu_1 - 1 + \rho'_1 M_1 + \ldots + \rho_1^n M_n) m_1} \left(x_1^{\nu_1 m_1} + \xi_1 \right)^{\lambda_1 - (\sigma'_1 M_1 + \ldots + \sigma_1^n M_n)} \ldots \right.$$

$$\left. x_r^{(\mu_{r-1} + \rho'_r M_1 + \ldots + \rho_r^n M_n) m_r} \left(x_r^{\nu_r m_r} + \xi_r \right)^{\lambda_r - (\sigma'_r M_1 + \ldots + \sigma_r^n M_n)} \right\}$$

$$= S \frac{z_1^{M_1} \ldots z_n^{M_n} \xi_1^{\lambda_1} \ldots \xi_r^{\lambda_2}}{\xi_1^{\sigma'_1 M_1 + \ldots + \sigma_1^n M_n} \ldots \xi_r^{\sigma'_r M_1 + \ldots + \sigma_r^n M_n}} \sum_{N_1, \ldots, N_r = 0}^{\infty} \frac{(-1/\xi_1)^{N_1}}{N_1!} \ldots \frac{(-1/\xi_r)^{N_r}}{N_r!}$$

$$\left(\sigma'_1 M_1 + \ldots + \sigma_1^n M_n - \lambda_1, N_1 \right) \ldots \left(\sigma'_r M_1 + \ldots + \sigma_r^n M_n - \lambda_r, N_r \right)$$

$$D_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1} \ldots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r} \left\{ x_1^{(\mu_1 - 1 + \rho'_1 M_1 + \ldots + \rho_1^n M_n + \nu_1 N_1) m_1} \ldots x_r^{(\mu_r - 1 + \rho'_r M_1 + \ldots + \rho_r^n M_r + \nu_r N_r) m_r} \right\}$$

$$= \prod_{j=1}^{r} \frac{\xi_j^{\lambda_j} \Gamma(\mu_j) \Gamma(\mu_j + \eta_j - \beta_j)}{\Gamma(\mu_j - \beta_j) \Gamma(\mu_j + \eta_j - \alpha_j)} x_j^{(\mu_j - \beta_j - 1) m_j} S \sum_{N_1, \ldots, N_r = 0}^{\infty} \prod_{j=1}^{r} \frac{(\mu_j, \rho'_j M_1 + \ldots + \rho_j^n M_n + \nu_j N_j)}{(\mu_j - \beta_j, \rho'_j M_1 + \ldots + \rho_j^n M_n + \nu_j N_j)}$$

$$\frac{(\mu_j + \eta_j - \beta_j, \rho'_j M_1 + \ldots + \rho_j^n M_n + \nu_j N_j)}{(\mu_j + \eta_j - \alpha_j, \rho'_j M_1 + \ldots + \rho_j^n M_n + \nu_j N_j)} \frac{(-\lambda_j, \sigma_j^1 M_1 + \ldots + \sigma_j^n M_n + 1)}{(-\lambda_j, \sigma'_j M_1 + \ldots + \sigma_j^n M_n)}$$

$$\frac{\left(\frac{z_1 x_1^{\rho'_1 m_1} \ldots x_r^{\rho'_r m_r}}{\xi_1^{\sigma'_1} \ldots \xi_r^{\sigma'_r}} \right)^{M_1}}{M_1!} \ldots \frac{\left(\frac{z_n x_1^{\rho_1^n m_1} \ldots x_r^{\rho_r^n m_r}}{\xi_1^{\sigma_1^n} \ldots \xi_r^{\sigma_r^n}} \right)^{M_n}}{M_n!} \frac{\left(\frac{-x_1^{m_1}}{\xi_1} \right)^{N_1}}{N_1!} \ldots \frac{\left(\frac{-x_r^{m_r}}{\xi_r} \right)^{N_r}}{N_r!}$$

[By making an appeal to (1.5)],

which can be written in the form of right hand side of (2.1).

Proof of (2.2). For brevity, we denote

$$I \equiv \frac{1}{(2\pi w)^n} \int_{L_1} \ldots \int_{L_n} \frac{\prod_{j=1}^{\lambda} \Gamma\left(1 - a_j + \sum_{i=1}^{n} \theta_j^{(i)} \zeta_i\right)}{\prod_{j=\lambda+1}^{A} \Gamma\left(a_j - \sum_{i=1}^{r} \theta_j^{(i)} \zeta_i\right) \prod_{j=1}^{C} \Gamma\left(1 - c_j + \sum_{i=1}^{r} \psi_j^{(i)} \zeta_i\right)}$$

$$\prod_{i=1}^{n} \frac{\prod_{j=1}^{\mu^{(i)}} \Gamma\left(d_j^{(i)} - \delta_j^{(i)} \zeta_i\right) \prod_{j=1}^{\nu^{(i)}} \Gamma\left(1 - b_j^{(i)} + \phi_j^{(i)} \zeta_i\right)}{\prod_{j=\mu^{(i)}+1}^{D^{(i)}} \Gamma\left(1 - d_j^{(i)} + \delta_j^{(i)} \zeta_i\right) \prod_{j=1}^{B^{(i)}} \Gamma\left(b_j^{(i)} - \phi_j^{(i)} \zeta_i\right)}.$$

Therefore, **left hand side of (2.2)**

$$= I\frac{z_1^{\zeta_1}\ldots z_n^{\zeta_n}\xi_1^{\lambda_1}\ldots\xi_r^{\lambda_r}}{\xi_1^{\sigma'_1\zeta_1+\ldots+\sigma_1^n\zeta_n}\ldots\xi_r^{\sigma'_r\zeta_1+\ldots+\sigma_r^n\zeta_n}}$$

$$\sum_{N_1,\ldots,N_r=0}^{\infty}\frac{(-1)^{N_1+\ldots+N_r}\left(\sigma'_1\zeta_1+\ldots+\sigma_1^n\zeta_n-\lambda_1,N_1\right)\ldots\left(\sigma'_r\zeta_1+\ldots+\sigma_r^n\zeta_n-\lambda_r,N_r\right)}{\xi_1^{N_1}\ldots\xi_r^{N_r}N_1!\ldots N_r!}$$

$$D_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1}\ldots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r}\left\{x_1^{\left(\mu_1-1+\rho'_1\zeta_1+\ldots+\rho_1^n\zeta_n+\nu_1N_1\right)m_1}\ldots x_r^{\left(\mu_r-1+\rho'_r\zeta_1+\ldots+\rho_r^n\zeta_n+\nu_nN_r\right)m_r}\right\}$$

$$= I\frac{z_1^{\zeta_1}\ldots z_n^{\zeta_n}\xi_1^{\lambda_1}\ldots\xi_r^{\lambda_r}}{\xi_1^{\sigma'_1\zeta_1+\ldots+\sigma_1^n\zeta_n}\ldots\xi_r^{\sigma'_r\zeta_1+\ldots+\sigma_r^n\zeta_n}}\sum_{N_1,\ldots,N_r=0}^{\infty}\frac{(-1/\xi_1)^{N_1}}{N_1!}\ldots\frac{(-1/\xi_r)^{N_r}}{N_r!}$$

$$\frac{\Gamma\left(-\lambda_1+N_1+\sigma'_1\zeta_1+\ldots+\sigma_1^n\zeta_n\right)}{\Gamma\left(-\lambda_1+\sigma'_1\zeta_1+\ldots+\sigma_1^n\zeta_n\right)}\ldots\frac{\Gamma\left(-\lambda_r+N_r+\sigma'_r\zeta_1+\ldots+\sigma_r^n\zeta_n\right)}{\Gamma\left(-\lambda_r+\sigma'_r\zeta_1+\ldots+\sigma_r^n\zeta_n\right)}$$

$$\frac{\Gamma\left(\mu_1+\rho'_1\zeta_1+\ldots+\rho_1^n\zeta_n+\nu_1\mathrm{N}_1\right)\Gamma\left(\mu_1+\rho'_1\zeta_1+\ldots+\rho_1^n\zeta_n+\nu_1N_1+\eta_1-\beta_1\right)}{\Gamma\left(\mu_1+\rho'_1\zeta_1+\ldots+\rho_1^n\zeta_n+\nu_1N_1-\beta_1\right)\ldots\Gamma\left(\mu_1+\rho'_1\zeta_1+\ldots+\rho_1^n\zeta_n+\nu_1N_1+\eta_1-\alpha_1\right)}\ldots$$

$$\frac{\Gamma\left(\mu_r+\rho'_r\zeta_1+\ldots+\rho_r^n\zeta_n+\nu_r\mathrm{N}_r\right)\Gamma\left(\mu_r+\rho'_r\zeta_1+\ldots+\rho_r^n\zeta_n+\nu_rN_r+\eta_r-\beta_r\right)}{\Gamma\left(\mu_r+\rho'_r\zeta_1+\ldots+\rho_r^n\zeta_n+\nu_rN_r-\beta_r\right)\ldots\Gamma\left(\mu_r+\rho'_r\zeta_1+\ldots+\rho_r^n\zeta_n+\nu_rN_r+\eta_r-\alpha_r\right)}$$

$$x_1^{\left(\mu_1-\beta_1-1+\rho'_1\zeta_1+\ldots+\rho_1^n\zeta_n+\nu_1N_1\right)m_1}\ldots x_r^{\left(\mu_r-\beta_r-1+\rho'_r\zeta_1+\ldots+\rho_r^n\zeta_n+\nu_rN_r+\eta_r-\alpha_r\right)m_r}\quad \text{(By an use of (1.5)]},$$

which can be reformed in the form of right hand side of (2.2).

Proof. of (2.3). For brevity, if we denote

$$K\equiv S\frac{z_1^{M_1}\ldots z_n^{M_n}\xi_1^{\lambda_1}\ldots\xi_r^{\lambda_r}}{\xi_1^{\sigma'_1M_1+\ldots+\sigma_1^nM_n}\ldots\xi_r^{\sigma'_rM_1+\ldots+\sigma_r^nM_n}}\sum_{N_1,\ldots,N_r=0}^{\infty}\frac{(-1/\xi_1)^{N_1}}{N_1!}\ldots\frac{(-1/\xi_r)^{N_r}}{N_r!}$$

$$\left(\sigma'_1M_1+\ldots+\sigma_1^nM_n-\lambda_1,N_1\right)\ldots\left(\sigma'_rM_1+\ldots+\sigma_r^nM_n-\lambda_r,N_r\right)$$

$$\sum_{R_1,\ldots,R_s=0}^{\infty}\frac{\prod_{j=1}^{E}\left(e_j,R_1\alpha'_j+\ldots+R_s\alpha_j^{(s)}\right)}{\prod_{j=1}^{G}\left(g_j,R_1\gamma'_j+\ldots+R_s\gamma_j^{(s)}\right)}\frac{\prod_{j=1}^{F'}\left(f'_j,R_1\beta'_j\right)\ldots\prod_{j=1}^{F^{(s)}}\left(f_j^{(s)},R_s\beta_j^{(s)}\right)}{\prod_{j=1}^{H'}\left(h'_j,R_1\eta'_j\right)\ldots\prod_{j=1}^{H^{(s)}}\left(h_j^{(s)},R_s\psi_j^{(s)}\right)}\frac{w_1^{R_1}}{R_1!}\ldots\frac{w_s^{R_s}}{R_s!}.$$

The left hand side of (2.3)

$$= KD_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1}\dots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r}\left\{x_1^{\left(\mu_1-1+\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+k'_1 R_1+\dots+k_1^s R_s\right)m_1}\dots x_r^{\left(\mu_r-1+\rho'_r M_1+\dots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\dots+k_r^s R_s\right)m_r}\right\}$$

$$= K\frac{\left(\mu_1,\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+k'_1 R_1+\dots+k_1^s R_s\right)}{\Gamma\left(\mu_1-\beta_1,\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+k'_1 R_1+\dots+k_1^s R_s\right)}$$

$$\frac{\left(\mu_1+\eta_1-\beta_1,\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+\dots+k'_1 R_1+\dots+k_1^s R_s\right)}{\left(\mu_1+\eta_1-\alpha_1,\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+\dots+k'_1 R_1+\dots+k_1^s R_s\right)}\dots$$

$$\frac{\left(\mu_r,\rho'_r M_1+\dots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\dots+k_r^s R_s\right)}{\left(\mu_r-\beta_r,\rho'_r M_1+\dots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\dots+k_r^s R_s\right)}$$

$$\frac{\left(\mu_r+\eta_r-\beta_r,\rho'_r M_1+\dots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\dots+k_r^s R_s\right)}{\left(\mu_r+\eta_r-\alpha_r,\rho'_r M_1+\dots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\dots+k_r^s R_s\right)}$$

$$x_1^{\left(\mu_1-\beta_1-1+\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+k'_1 R_1+\dots+k_1^s R_s\right)m_1}\dots x_r^{\left(\mu_r-\beta_r-1+\rho'_r M_1+\dots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\dots+k_r^s R_s\right)m_r}$$

[By making an appeal to (1.5)],

which can be adjusted in the form of right hand side of (2.3).

Proof of (2.4). For brevity considering

$$M \equiv S\frac{z_1^{M_1}\dots z_n^{M_n}\xi_1^{\lambda_1}\dots\xi_r^{\lambda_r}}{\xi_1^{\sigma'_1 M_1+\dots+\sigma_1^n M_n}\dots\xi_r^{\sigma'_r M_1+\dots+\sigma_r^n M_n}}\sum_{N_1,\dots,N_r=0}^{\infty}\left(\sigma'_1 M_1+\dots+\sigma_1^n M_n-\lambda_1,N_1\right)\dots$$

$$\left(\sigma'_r M_1+\dots+\sigma_r^n M_n-\lambda_r,N_r\right)\frac{\left(-1/\xi_1\right)^{N_1}}{N_1!}\dots\frac{\left(-1/\xi_r\right)^{N_r}}{N_r!}$$

$$\sum_{R_1,\dots,R_s=0}^{h_1R_1+\dots+h_sR_s\le L}(-L)_{h_1R_1+\dots+h_sR_s}\,A\left(L;R_1,\dots,R_s\right)\frac{w_1^{R_1}}{R_1!}\dots\frac{w_s^{R_s}}{R_s!}.$$

Left hand side of (2.4)

$$= MD_{0,x_1,m_1}^{\alpha_1,\beta_1,\eta_1}\dots D_{0,x_r,m_r}^{\alpha_r,\beta_r,\eta_r}\left\{x_1^{\left(\mu_1-1+\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+k'_1 R_1+\dots+k_1^s R_s\right)m_1}\dots x_r^{\left(\mu_r-1+\rho'_r M_1+\dots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\dots+k_r^s R_s\right)m_r}\right\}$$

$$= M\frac{\left(\mu_1,\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+k'_1 R_1+\dots+k_1^s R_s\right)}{\Gamma\left(\mu_1-\beta_1,\rho'_1 M_1+\dots+\rho_1^n M_n+\nu_1 N_1+k'_1 R_1+\dots+k_1^s R_s\right)}$$

$$\frac{\left(\mu_1+\eta_1-\beta_1,\rho'_1 M_1+\ldots+\rho_1^n M_n+\nu_1 N_1+\ldots+k'_1 R_1+\ldots+k_1^s R_s\right)}{\left(\mu_1+\eta_1-\alpha_1,\rho'_1 M_1+\ldots+\rho_1^n M_n+\nu_1 N_1+\ldots+k'_1 R_1+\ldots+k_1^s R_s\right)}\ldots$$

$$\frac{\left(\mu_r,\rho'_r M_1+\ldots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\ldots+k_r^s R_s\right)}{\left(\mu_r-\beta_r,\rho'_r M_1+\ldots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\ldots+k_r^s R_s\right)}$$

$$\frac{\left(\mu_r+\eta_r-\beta_r,\rho'_r M_1+\ldots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\ldots+k_r^s R_s\right)}{\left(\mu_r+\eta_r-\alpha_r,\rho'_r M_1+\ldots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\ldots+k_r^s R_s\right)}$$

$$x_1^{\left(\mu_1-\beta_1-1+\rho'_1 M_1+\ldots+\rho_1^n M_n+\nu_1 N_1+k'_1 R_1+\ldots+k_1^s R_s\right)m_1}\ldots x_r^{\left(\mu_r-\beta_r-1+\rho'_r M_1+\ldots+\rho_r^n M_n+\nu_r N_r+k'_r R_1+\ldots+k_r^s R_s\right)m_r}$$

[By making an appeal to (1.5)],

which can be reformed as right hand side of (2.4).

Proof of (2.5). We can write

Left hand side of (2.5)

$$= I\frac{z_1^{\zeta_1}\ldots z_n^{\zeta_n}\xi_1^{\lambda_1}\ldots\xi_r^{\lambda_r}}{\xi_1^{\sigma'_1\zeta_1+\ldots+\sigma_1^n\zeta_n}\ldots\xi_r^{\sigma'_r\zeta_1+\ldots+\sigma_r^n\zeta_n}}\sum_{N_1,\ldots,N_r=0}^{\infty}\frac{\left(-1/\xi_1\right)^{N_1}}{N_1!}\ldots\frac{\left(-1/\xi_r\right)^{N_r}}{N_r!}$$

$$\left(-\lambda_1+\sigma'_1\zeta_1+\ldots+\sigma_1^n\zeta_n,N_1\right)\ldots\left(-\lambda_r+\sigma'_r\zeta_1+\ldots+\sigma_r^n\zeta_n,N_r\right)$$

$$\sum_{R_1,\ldots,R_s=0}^{h_1R_1+\ldots+h_sR_s\le L}\left(-L,h_1R_1+\ldots+h_sR_s\right)A\left(L;R_1,\ldots,R_s\right)\frac{w_1^{R_1}}{R_1!}\ldots\frac{w_s^{R_s}}{R_s!}$$

$$\left\{x_1^{\left(\mu_1-1+\rho'_1\zeta_1+\ldots+\rho_1^n\zeta_n+\nu_1 N_1+k'_1 R_1+\ldots+k_1^s R_s\right)m_1}\ldots x_r^{\left(\mu_r-1+\rho'_r\zeta_1+\ldots+\rho_r^n\zeta_n+\nu_r N_r+k'_r R_1+\ldots+k_r^s R_s\right)m_r}\right\}$$

$$= I\frac{z_1^{\zeta_1}\ldots z_n^{\zeta_n}\xi_1^{\lambda_1}\ldots\xi_r^{\lambda_r}}{\xi_1^{\sigma'_1\zeta_1+\ldots+\sigma_1^n\zeta_n}\ldots\xi_r^{\sigma'_r\zeta_1+\ldots+\sigma_r^n\zeta_n}}\sum_{N_1,\ldots,N_r=0}^{\infty}\sum_{R_1,\ldots,R_0=0}^{h_1R_1+\ldots+h_sR_s\le L}\left(-L,h_1R_1+\ldots+h_sR_s\right)A\left(L;R_1,\ldots,R_s\right)$$

$$\frac{w_1^{R_1}}{R_1!}\ldots\frac{w_s^{R_s}}{R_s!}\frac{\left(-1/\xi_1\right)^{N_1}}{N_1!}\ldots\frac{\left(-1/\xi_r\right)^{N_r}}{N_r!}$$

$$\frac{\Gamma\left(-\lambda_1+\sigma'_1\zeta_1+\ldots+\sigma_1^n\zeta_n+N_1\right)\ldots\Gamma\left(-\lambda_r\sigma'_r\zeta_1+\ldots+\sigma_r^n\zeta_n+N_r\right)}{\left(-\lambda_1+\sigma'_1\zeta_1+\ldots+\sigma_1^n\zeta_n\right)\ldots\Gamma\left(-\lambda_r+\sigma'_r\zeta_1+\ldots+\sigma_r^n\zeta_n\right)}$$

$$\frac{\Gamma\left(\mu_1+\rho'_1\zeta_1+\ldots+\rho_1^n\zeta_n+\nu_1 N_1+k'_1 R_1+\ldots+k_1^s R_s\right)}{\Gamma\left(\mu_1-\beta_1+\rho'_1\zeta_1+\ldots+\rho_1^n\zeta_n+\nu_1 N_1+k'_1 R_1+\ldots+k_1^s R_s\right)}$$

$$\frac{\Gamma\left(\mu_1+\eta_1-\beta_1+\rho'_1\,\zeta_1+\ldots+\rho_1^n\zeta_n+\nu_1 N_1+k'_1\,R_1+\ldots+k_1^s R_s\right)}{\Gamma\left(\mu_1+\eta_1-\alpha_1+\rho'_1\,\zeta_1+\ldots+\rho_1^n\zeta_n+\nu_1 N_1+k'_1\,R_1+\ldots+k_1^s R_s\right)}\cdots$$

$$\frac{\Gamma\left(\mu_r+\rho'_r\,\zeta_1+\ldots+\rho_r^n\zeta_n+\nu_r N_r+k'_r\,R_1+\ldots+k_r^s R_s\right)}{\Gamma\left(\mu_r-\beta_r+\rho'_r\,\zeta_1+\ldots+\rho_r^n\zeta_n+\nu_r N_r+k'_r\,R_1+\ldots+k_r^s R_s\right)}$$

$$\frac{\Gamma\left(\mu_r+\eta_r-\beta_r+\rho'_r\,\zeta_1+\ldots+\rho_r^n\zeta_n+\nu_r N_r+k'_r\,R_1+\ldots+k_r^s R_s\right)}{\Gamma\left(\mu_r+\eta_r-\alpha_r+\rho'_r\,\zeta_1+\ldots+\rho_r^n\zeta_n+\nu_r N_r+k'_r\,R_1+\ldots+k_r^s R_s\right)}$$

$$x_1^{\left(\mu_1-\beta_1-1+\rho'_1\,\zeta_1+\ldots+\rho_1^n\zeta_n+\nu_r N_r+k'_r\,R_1+\ldots+k_1^s R_s\right)m_1}\ldots x_r^{\left(\mu_r-\beta_r-1+\rho'_r\,\zeta_1+\ldots+\rho_r^n\zeta_n+\nu_r N_r+k'_r\,R_1+\ldots+k_r^s R_s\right)m_r}$$

[By making an application of (1.5)],

which can be expressed in the form of right hand side of (2.5).

3. Special Cases. In this section, we mention the special cases of our results.

Special Cases of (2.1)

(a) For $m_i=1,\beta_i=\alpha_i, i=1,\ldots,r;$ (2.1) reduces to Chandel and Kumar [12,p.107, (2.1)].

(b) For $r=1, m_1=1,\beta_1=\alpha_1$, (2.1) gives Srivastava, Chandel and Vishwakarma [40, p. 567 (3.1)].

(c) For $r=2, m_1=m_2=1,\beta_1=\alpha_1,\beta_2=\alpha_2$, (2.1) reduces to Srivastava, Chandel and Vishwakarma [40, p. 568 (3.3)].

Special Cases of (2.2)

(a) For $m_i=1,\beta_i=\alpha_i; i=1,\ldots,r;$ (2.2) reduces to Chandel and Kumar [12, p.108 (2.3)], which also generalizes the earlier results due to Srivastava, Chandel and Vishwakarma [40, p. 563 (2.1), p. 564 (2.3)].

Special Cases of (2.3).

(a) For $m_i=1,\beta_i=\alpha_i, i=1,\ldots,r;$ (2.3) reduces to the result due to Chandel and Kumar [12, p.107, (2.2)].

(b) For $r=1, m_1=1,\beta_1=\alpha_1$, (2.3) reduces to the improved version of the result due to Srivastava, Chandel and Vishwakrama [40, p. 567 (3.2)]

(c) Due to general nature of the functions involved in (2.3), specializing the number of variables and other parameters in (2.3), we can get several interesting known and unknown results.

Special Cases of (2.4).

(a) For $r=1,\lambda_1=0, n=1,\sigma'_1=0$, the result (2.4) will include the result due to Ram and Chandak [32, p.53 (10)] involving Fox-Wright generalized hypergeometric function ${}_p\Psi_q$ and generalized multi-variable polynomials $S_L^{h_1,\ldots,h_s}$ of Srivastava and Garg [38].

(b) For $r=1,\lambda_1=0,n=1,\sigma'_1=0$ and $\beta_1=\alpha_1$, (2.4) includes an interesting result due to Ram-Chandak [32, p. 53 (11)] for the Riemann-Liouville derivative operator-defined by Miller and Ross [28].

(c) For $s=1$, the polynomials $S_L^{h_1,\ldots,h_s}(x_1,\ldots,x_s)$ reduce to the Srivastava polynomials S_l^h defined by (1.10), therefore, for $r=1,\lambda_1=0,n=1,\sigma'_1=0,\beta_1=\alpha_1$ and $s_1=1$, (2.4) gives an interesing result due to Ram-Chandak [32, p. 54(12)] involving Srivastava polynomials.

(d) Similarly all the following results due to Ram and Chandak [32, p.54 (13), (14), (15)] are included in our result (2.4). Since for $l=0,A_{0,0}=1,S_l^k(x)=1,$ therefore our result (2.4), also gives the result due to Kilbas [21] as special case of Ram and Chandak [32 (15)]

(e) Since Wright function $\phi(\alpha,\beta;z)$ defined by (1.7) and Wright generalized Bessel function $J_\nu^\delta(z)$ defined by (1.8) are special cases of Fox-Wright generalized hypergeometric function ${}_p\psi_q(z)$ defined by (1.6), therefore, all the results due to Ram and Chandak [32, pp. 55-57 (16), (17), (18),(19)(20),(21)] are included in our result (2.4) as special cases.

Special Cases of (2.5). Our result (2.5) includes (2.4) along with all results of Ram and Chandak [32] as special cases.

4. Other Special Cases as Application of one Fractional Derivative. In this section, making an appeal to (1.5) and one fractional derivative we derive

$$(4.1)\, D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m} S_{C:D';\ldots;D^{(n)}}^{A:B';\ldots;B^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}]\end{matrix}\, y_1x^{\lambda_1 m},\ldots,y_nx^{\lambda_n m}\right)\right\}$$

$$= x^{(\mu-1-\beta)m} S_{C+2:D';\ldots;D^{(n)}}^{A+2:\beta';\ldots;\beta^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],[\mu:\lambda_1,\ldots,\lambda_n],[\mu+\eta-\beta:\lambda_1,\ldots,\lambda_n]:\\ [(c):\psi',\ldots,\psi^{(n)}],[\mu-\beta:\lambda_1,\ldots,\lambda_n],[\mu+\eta-\alpha:\lambda_1,\ldots,\lambda_n]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\ldots;\left[\left(b^{(n)}\right):\phi^{(n)}\right]\\ [(d'):\delta'];\ldots;\left[\left(d^{(n)}\right):\delta^{(n)}\right]\end{matrix}\, y_1x^{\lambda_1 m},\ldots,y_nx^{\lambda_n m}\right),$$

provided that $0 \le \alpha < 1, m \in N; \beta, \eta, \lambda_i > 0, x \in R, \mu > max(1, \beta - \eta)$ and

$$1+\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}>0 \quad i=1,\ldots,n.$$

(4.2) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}F_D^{(n)}\left(\mu-\beta,b_1,\ldots,b_n;\mu;y_1x^m,\ldots,y_nx^m\right)\right\}$$

$$=\frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}F_D^{(n)}\left(\mu+\eta-\beta,b_1,\ldots,b_n;\mu+\eta-\alpha;y_1x^m,\ldots,y_nx^m\right)$$

valid if $0 \le \alpha < 1, m \in N; \beta, \eta, x \in R, k > max(0, \beta - \eta - 1), |y_1x| < 1, \ldots, |y_nx| < 1,$

where $F_D^{(n)}$ is well known Lauricella function [24].

(4.3) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}F_D^{(n)}\left(\mu+\eta-\alpha,b_1,\ldots,b_n,\mu+\eta-\beta;y_1x^m,\ldots,y_nx^m\right)\right\}$$

$$=\frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}F_D^{(n)}\left(\mu,b_1,\ldots,b_n;\mu-\beta;y_1x^m,\ldots,y_nx^m\right),$$

provided that all conditions of (4.2) are satisfied.

(4.4) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}F_D^{(n)}\left(\mu+\eta-\alpha,b_1,\ldots,b_n,\mu;y_1x^m,\ldots,y_nx^m\right)\right\}$$

$$=\frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}F_D^{(n)}\left(\mu+\eta-\beta,b_1,\ldots,b_n;\mu-\beta;y_1x^m,\ldots,y_nx^m\right),$$

where all conditions of (4.2) hold true.

(4.5) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}F_D^{(n)}\left(\mu-\beta,b_1,\ldots,b_n;\mu+\eta-\beta;y_1x^m,\ldots,y_nx^m\right)\right\}$$

$$=\frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}F_D^{(n)}\left(\mu,b_1,\ldots,b_n;\mu+\eta-\beta;y_1x^m,\ldots,y_nx^m\right),$$

valid if all conditions of (4.2) are satisfied.

(4.6) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\,exp\left(x^m\right)\right\}$$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\, {}_2F_2\left(\mu,\mu+\eta-\beta;\mu-\beta,\mu+\eta-\alpha;x^m\right),$$

provided that $0 \le \alpha < 1, m \in N, \beta, \eta, x \in R, \mu > max(1, \beta - \eta)$.

(4.7) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\, {}_2F_2\left(\mu-\beta,\mu+\eta-\alpha;\mu,\mu+\eta-\beta;x^m\right)\right\}$$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)} exp\left(x^m\right),$$

where all conditions of (4.6) are satisfied.

(4.8) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\, {}_1F_1\left(\mu-\beta;\mu;x^m\right)\right\}$$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\, {}_1F_1\left(\mu+\eta-\beta;\mu+\eta-\alpha;x^m\right),$$

which holds true if all conditions of (4.6) are satisfied.

(4.9) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\, {}_1F_1\left(\mu+\eta-\alpha;\mu;x^m\right)\right\}$$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\, {}_1F_1\left(\mu+\eta-\beta;\mu-\beta;x^m\right),$$

where all conditions of (4.6) hold true.

(4.10) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\, {}_1F_1\left(\mu-\beta;\mu+\eta-\alpha;x^m\right)\right\}$$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\, {}_1F_1\left(\mu;\mu+\eta-\alpha;x^m\right),$$

provided that all conditions of (4.6) hold time.

(4.11) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\, {}_1F_1\left(\mu+\eta-\alpha;\mu+\eta-\beta;x^m\right)\right\}$$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\, {}_1F_1\left(\mu;\mu-\beta;x^m\right),$$

which is true if all conditions of (4.6) are setisfied.

(4.12) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\, {}_2F_1\left(\mu-\beta;\mu+\eta-\alpha;x^m\right)\right\}$$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\left(1-x^m\right)^{-(\mu+\eta-\beta)}$$

valid if all conditions of (4.6) are satisfied.

(4.13) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\,{}_2F_1\left(\mu-\beta;\mu+\eta-\alpha,\mu+\eta-\beta;x^m\right)\right\}$$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\left(1-x^m\right)^{-\mu},$$

where all conditions of (4.6) are satisfied.

(4.14) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\,{}_2F_1\left(\mu-\beta;\mu-\eta-\alpha,\mu;x^m\right)\right\}$$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\left(1-x^m\right)^{-(\mu+\eta-\beta)},$$

provided that all conditions of (4.6) hold true.

(4.15) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\,{}_1F_2\left(\mu-\beta;\mu,\mu+\eta-\beta;x^m\right)\right\}$$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\left(1-x^m\right)^{-(\mu+\eta-\beta)},$$

valid if all conditions of (4.6) are true.

(4.16) $$D_{0,x,m}^{\alpha,\beta,\eta}\left\{x^{(\mu-1)m}\,{}_1F_2\left(\mu+\eta-\alpha;\mu,\mu+\eta-\beta;x^m\right)\right\}$$

$$= \frac{x^{(\mu-\beta-1)m}\Gamma(\mu)\Gamma(\mu+\eta-\beta)}{\Gamma(\mu-\beta)\Gamma(\mu+\eta-\alpha)}\,{}_0F_1\left(-;\mu-\beta;x^m\right),$$

which holds true if all conditions of (4.6) are satisfied.

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