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# $D_{RS}$–TRIANGULATION BASED ON $2^N$–RAY ALGORITHM

By

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**ABSTRACT**

In order to improve the efficiency of the $2^N$-Ray algorithm; we propose a variant of the $D_{RS}$-Triangulation. A nice property of this triangulation is that it subdivides all the subsets, on which the $2^N$-Ray algorithm works, into simplices according to the $D_{RS}$-Triangulation. Numerical tests shows that $2^N$-Ray algorithm based on $D_{1/2}$-Triangulation is much more efficient.

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**1. Introduction.** Various triangulations have been situdied by Dang [1], Dang and Talman [2], Scarf [4], Todd [6,7], Vander and Talman [8], Vertgeim [9]. The $2^n$-ray algorithm was proposed by Wright in [10] to compute solutions of non-linear equations. The $2^n$-ray algorithm partitions $R^n$ into $2n$ cones which have the same vertex. Then a triangulation of $R^n$ subdivides each cone into simplices. The $2^n$-ray algorithm starts at the vertex and leaves it along an edge of some cone. It follows a sequence of adjacent simplices with varying dimension. Under some mild conditions, the $2^n$-ray algorithm terminates at an $n$-dimensional simplex that yields an approximate solutions to the system of non-linear equations. Since these $2n$ cones have $2^n$ edges, each of which is a ray, the $2^n$–ray algorithm has $2^n$ possible ray to leave the vertex.

Motivated by this work, we have tried to develop a new $D_{RS}$–triangulation. The $D_{RS}$-triangulation based on the $2^n$ ray algorithm is much more efficient than $2^n$ ray algorithm proposed by Wright [10] as well as $K_1$ triangulation introduced by Kuhn in [3] and $J_1$ triangulation given by Todd [5].

**2. Notations and Definitions.** The following notations have been used in this paper :

*R*: Set of real numbers,

Z: Set of all integers,

*N:* Set of positive integers (1,2,...,$n$),

$N^0$: Set of all non-negative inegers $N \cup (0)$,

$R^n$ : $n$ dimensionally space, having co-ordinates indexed 1 through $n$,
$R^{n+1}$: $n+1$ dimensional space, with coordinates indexed 0 through $n$,
$\pi$: Group of permutation on $(1,2,...,n)$ and $\pi+1$ group of permutation on $(0,1,2,...,n)$,

$u^i$ : $i^{\text{th}}$ unit vector in $R^n, j\varepsilon N$ and $u = \sum_{i\varepsilon N} u^i$ ,

$R_+^m$ : Non negative orthant of $R^m i.e.\left(x\varepsilon R^m ; x \geq 0\right)$.

Now we consider some standard definitions and explanations which will be used in this paper.

**2.1 Standard Simplex.** The standard $n$ dimensional closed simplex $S^n$ is the convex hull of $v^0, v^1, ..., v^n i.e. S^n = \left\{x\varepsilon R_+^{n+1} : v^T x = 1\right\}$. $s_i^n$ denotes the face of $s^n$ opposite $v^i i.e. S_i^n = \left\{x\varepsilon s^n : x_i = 0\right\}$ and boundary of $s^n$ is denoted by $\partial s = U_{i\varepsilon N} s_i^n$.

Again a $j$-dimensional simplex or [$j$-simplex] is the relative interior of the convex hull of $j+1$ affinely independent points $y^0, y^1, y^2, y^3, ..., y^j$, called its vertices. We write $\sigma = \left\langle y^0, y^1, y^2, ..., y^j \right\rangle$. A simplex $\tau$ is a face of $\sigma$ if its vertices are a subset vertices of the $\sigma$. It is convemient to call the closure of a $(j–1)$ dimensional face of the $j$ simplex $\sigma$ as a facet of $\sigma$. Two $j$ simplices are said to be adjacent if they share a common facet.

**2.2 Triangulation.** A trangulation $G$ of $S^n$ is a collection of $n$ simplices and satisfies the following two conditions :

1. The simplices in $G$ together with all their faces form a partition of $S^n$ and
2. Each point of $S^n$ has a neighbourhood meeting only a finite number of simplices.

**2(a) Pivot Rule.** For a given simplices $G$ and a vertex $y$ of $\sigma$ the rules for obtaining the simplex of $G$ whose vertices include all vertices of $\sigma$ except $y$, are called the pivot rules of $G$.

**2(b) Mesh.** The mesh of a tringulation $G$ is $\sup_{\sigma\varepsilon G} diam\sigma$. We shall use the Euclidian norm through out this paper.

**2.3 Definition.** For each sign vector $s\varepsilon R^n$, let

$E(\text{s}) = \{ x\varepsilon R^n : s_i x_i = \|x\|$ whenever $s_i \neq 0 )$

$= \text{cone} \{ t\varepsilon R^n : t$ is a sign vector, and $s_i \neq 0 \Rightarrow s_i = t_i \}$ .

In case $s$ has $k$ non-zero components for $k>0$ than $E(s)$ is a polyhedral cone of dimension $n-k+1$. Also we have $E(0) = R^n$. Moreover, when $s \neq 0$ each $E(s) \cap B^{n_\infty}$

is a polyhedral of a cubical subdivisions of $B^{n_\infty}$ where $B^{n_\infty}$ donote the unit ball in $1^\infty$ norm.

Wright [10] has also defined another subdivision of $R^n$ into closed convex cone as $n$-dimensional geometric form given as follows :

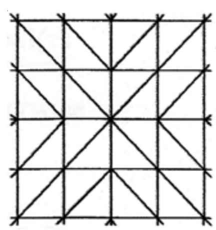
$$Let C(s) = x\varepsilon R^n : \begin{cases} x_i = 0 \text{ if } s_i = 0 \\ s_i x_i \geq 0 \text{ if } s_i \neq 0 \end{cases}$$

=cone $\{s_i u_i : s_i 0$, for each sign vector $s\}$. If $s$ has $k$ non-zero components then $C(s)$ is an orthant of a $k$-dimensional coordinate subspace of $R^n$.

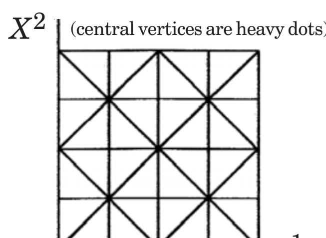
Wright [10] has proposed two type of $T$-triangulations of $R^n$ with the property that $E(t)$ is a subcomplex for every sign vector $t\varepsilon R^n$ for $t \neq 0$. The first triangulation is called a $K^1$ triangulation. This triangulation is obtained by taking the triangulation $K_1$ due to Kuhn [3] in the first orhant and the reflecting through coordinate hyperplane to triangulate the other orhant. A vector $v^1$ of a $n$-simplex $<v^1,...,v^{n+1}>$ of $K^1$ specified by choosing a sign vector $s$ with all the non-zero components, is a member of $C(s)$ when all its components are integrals and $\pi$ is a permutation of $\{1,2,...,n\}$, then $v^1 \varepsilon K^1$ is defined recursively as :

$V^{i+1} = V^i + S_{\pi(i)} U^{\pi(i)}$, for $i=1,2,...,n$.

For $n=2$ triangulation $K^1$ can be illustrated by following diagram.



The second triangulation $J_1$ is defined by Todd [5] and which can be illustrated for $n=2$ by the following figure :




**2.3.1. The $D_{RS}$-*Triangulation.*** Define

$W^n = \{x\varepsilon R_+^n : x_1 = \max x_i, i = 2,3,...,n\}$

taking a vector $Y = (y_1, y_2, ..., y_n)^T$, we have

$$Y_i = \begin{cases} [x_i] \text{ if } [x_i] \text{ is even} \\ [x_i]+1 \text{ otherwise,} \end{cases}$$

where $[\alpha]$ is the greatest integer less than or equal to $\alpha$. Let $D$ is the set of all $Y \varepsilon W^n$ where $Y_i$ is defined above. If $Y \varepsilon D$, we define

$I(y) = \{i \varepsilon N; y_1 = y_i\}$ and $J(y) = \{j \varepsilon N : y_1 \geq y_j\}^T$.

Let $s = (s_1, s_2, ..., s_n)^T$ be a sign vector such that

1. For $i \varepsilon N$, if $y_i = 0$ then $s_i = 1$, and if $y_i \neq 0$ then $s_i = -1$.

Let $K(y,s) = \{i \varepsilon I(y) : s_i = 1\}$.

Let $\ell$ denote the number of element in $I(y)$ and $h$ the number of elements in $K(y,s)$, we take integer $p$ such that

1. when $h=0$, if $\ell = n$ then $p=0$ or 2,
2. when $h>0$, if $h=n$ then $p=0$ and if $h<n$ then $0 \leq p \leq n-1$.

Let $\pi = \{\pi(1), \pi(2), ..., \pi(n)\}$ be permutation of $N$.

When $h=0$, for $j=1,2,...,n$,

1. If $j=1$, define

$$g_i(j) = \begin{cases} -1 \text{ if } i \varepsilon I(y) \\ 0 \text{ otherwise} \end{cases} \quad ...(1)$$

for $i=1,2,...,n$.

2. If $j \neq 1$, we define

$$g_i(j) = \begin{cases} S^i \text{ if } i = j, \\ 0 \text{ otherwise} \end{cases} \quad ...(2)$$

for $i=1,2,...,n$.

When $h>0$, for $j=1,2,...,n$.

1. If $\pi(j) \varepsilon K(y,s)$, define

$$g_i(\pi(j)) = \begin{cases} 1 \text{ if } i \varepsilon K(y,s) \text{ and } j \leq \pi^{-1}(i) \\ 0 \text{ otherwise} \end{cases} \quad ...(3)$$

for $i=1,2,...,n$,

2. If $\pi(j) \notin K(y,s)$, define

$$g_i(\pi(j)) = \begin{cases} s_n(j) \text{ if } i = \pi(j) \\ 0 \text{ otherwise} \end{cases} \quad ...(4)$$

for $i=1,2,...,n$.

If $y$, $\pi$, $s$ and $p$ be as above, then vectors $y^0, y^1, ..., y^n$ are defined as follows :

for $p=0$, we have

$$y^0 = y$$

$$y^k = y + g(\pi(k)), k = 1,2,...,n \quad ...(5)$$

and for $p \geq 1$, we define

$$y^0 = y + s,$$

$$y^k = y^{k-1} - s_{\pi(k)} U^{(\pi(k))}, k = 1,2,..., p-1.$$

$$y^k = y + g(\pi(k)), k = p,...,n. \quad ...(6)$$

The $y^0, y^1, ..., y^n$ vectors obtained from the above definition are affinely independent. Thus their convex hull is a simplex. Let us denote this simplex by $D_{RS}(y,\pi,s,p)$ or $\langle y^0, y^1, ..., y^n \rangle$. Let $D_{RS}$ be the set of all such simplices. Then $D_{RS}$ is a triangulation of $W^n$. Note that simplices of the $D_{RS}$-triangulation can be represented in more than one way. Moreover, the triangulation of a whole cube in $W^n$ is the same as the $D_1$-triangulation.

To be more expatiate let us illustrate $D_{RS}$-triangulation of $W^n$ for $n=2$ and for $x \leq 4$. Obviously, we have that for $y_1 \leq 4$.

$$D = \left\{(0,0,0)^T, (2,2,0)^T, (4,4,0)^T, (4,0,4)^T, (4,4,4)^T\right\}. \quad ...(7)$$

1. Let $y = (0,0,0)^T$. Then, $I(y) = \{i \varepsilon N : y_1 = y_i\} = (1,2,3)$ and $\ell = 3$.

Then $s$ must be $(1,1,1)^T$. Thus

$K(y,s) = \{i \varepsilon I(y) : s_1 = 1\} = (1,2,3)$ and $h = 3$. We have $p = 0$.

(a) Let $\pi = (2,3,1)$. Then $\pi^{-1} = (3,1,2)$ and by applying (3)

$$g(\pi(1)) = g(2) = (g_1(2), g_2(2), g_3(2))$$

$$= (1,1,1)^T,$$

$$g(\pi(2)) = g(3) = (1,0,1)^T,$$

$$g(\pi(3)) = g(3) = (1,0,0)^T.$$

Therefore, $y^0 = y = (0,0,0)^T,$

$$y^1 = y + g(\pi(1)) = (1,1,1)^T,$$

$$y^2 = y + g(\pi(2)) = (1,0,1)^T,$$

$$y^3 = y + g(\pi(3)) = (1,0,0)^T, \text{ Let } \sigma^1 = \langle y^0, y^1, y^2, y^3 \rangle.$$

(b) Let $\pi = (3,2,1)$. Then $\pi^{-1} = (3,2,1)$

$$g(\pi(1)) = g(3) = (1,1,1)^T,$$

$$g(\pi(2)) = g(2) = (1,1,0)^T,$$

$$g(\pi(3)) = g(1) = (1,0,0)^T,$$

Therefore, $y^0 = y = (0,0,0)^T,$

$$y^1 = y + g(\pi(1)) = (1,1,1)^T,$$

$$y^2 = y + g(\pi(2)) = (1,1,0)^T,$$

$$y^3 = y + g(\pi(3)) = (1,0,0)^T, \text{ Let } \sigma^2 = \langle y^0, y^1, y^2, y^3 \rangle.$$

2. Let $y = (2,2,0)^T$. Since $I(y) = \{i \varepsilon N : y_1 = y_i\} = \{1,2\}$ and $\ell = 2$.

So $s$ must be $(-1,-1,1)^T$. Thus $k(y,s) = \{i \varepsilon I(y); s_i = 1\} = \phi$ and $h=0$ while $p$ can be any one of 0,1,2,.

Now by applying (1) and (2), we have

$$g(1) = (-1,-1,0)$$

$$g(2) = (0,-1,0)$$

$$g(3) = (0,0,1).$$

Now considering different values of $p$ and applying (6) and (7), we obtain the following simplices:

(a) For $p$=0. Let $\pi=(1,2,3)$. Therefore,

$y^0=y=(2,2,0)^T$,
$y^1=y+g(\pi(1))=(1,1,0)^T$,
$y^2=y+g(\pi(2))=(2,1,0)^T$,
$y^3=y+g(\pi(3))=(2,2,1)^T$. Let $\sigma^3=\langle y^0,y^1,y^2,y^3\rangle$.

(b) For $p$=1. Let $\pi=(1,2,3)$. Therefore

$y^0=y+s=(1,1,1)^T$,
$y^1=y+g(\pi(1))=(1,1,0)^T$,
$y^2=y+g(\pi(2))=(2,1,0)^T$,
$y^3=y+g(\pi(3))=(2,2,1)^T$. Let $\sigma^4=\langle y^0,y^1,y^2,y^3\rangle$.

(c) For $p$=2. Let $\pi=(1,2,3)$. We have,

$y^0=y+s=(1,1,1)^T$,
$y^1=y^0-s_{\pi(1)}u^{\pi(1)}=(2,1,1)^T$,
$y^2=y+g(\pi(2))=(2,1,0)^T$,
$y^3=y+g(\pi(3))=(2,2,1)^T$. Let $\sigma^5=\langle y^0,y^1,y^2,y^3\rangle$.

**3.** Let $y=(4,4,0)^T$. Therefore, $I(y)=\{i\varepsilon N; y_1=y_i\}=\{1,2)$ and $\ell=2$.

We have that $s$ must be $(-1,-1,1)^T$. Thus $K(y,s)=\{i\varepsilon I(y): s_i=1\}=\phi$ and $h$=0.

We have that $p$ can be any one of 0,1,2. We also have

g(1) = (-1,-1,0),
g(2) = (0,-1,0),
g(3) = (0,0,1).

(a) For $p$=0, and $\pi=(1,2,3)$, we have

$y^0=y=(4,4,0)^T$,
$y^1=y+g(\pi(1))=(3,3,0)^T$,
$y^2=y+g(\pi(2))=(4,3,0)^T$,
$y^3=y+g(\pi(3))=(4,4,1)^T$. Let $\sigma^6=\langle y^0,y^1,y^2,y^3\rangle$.

(b) For $p$=1, let $\pi=(1,2,3)$. Therefore

$y^0 = y+s=(3,3,1)^T$,

$y^1 = y+g(\pi(1))=(3,3,0)^T$,

$y^2 = y+g(\pi(2))=(4,3,0)^T$,

$y^3 = y+g(\pi(3))=(4,4,1)^T$. Let $\sigma^7 = \langle y^0,y^1,y^2,y^3\rangle$.

(c) For $p$=2 and $\pi=(1,2,3)$, we have

$y^0 = y+s=(3,3,1)^T$,

$y^1 = y^0 - s_{\pi(1)}u^{\pi(1)} = (4,4,1)^T$,

$y^2 = y+g(\pi(2))=(4,3,0)^T$,

$y^3 = y+g(\pi(3))=(4,4,1)^T$. Let $\sigma^8 = \langle y^0,y^1,y^2,y^3\rangle$.

**4.** Let $y=(4,0,4)^T$. Since $I(y)=\{i\varepsilon N; y_1 = y_i\}=\{1,3)$ and $\ell=2$. so that $s$ must be (-1,-1,1)$^T$. Thus $K(y,s)=\{i\varepsilon I(y): s_i = 1\}=\phi$ and $h$=0. we have that $p$ can be any one of 0,1,2. We also have

$g(1)$ = (-1,0,-1),

g(2) = (0,1,0),

g(3) = (0,0,-1).

(a) For $p$=0, and $\pi=(1,2,3)$, we have

$y^0 = y=(4,0,4)^T$,

$y^1 = y+g(\pi(1))=(3,0,3)^T$,

$y^2 = y+g(\pi(2))=(4,1,4)^T$,

$y^3 = y+g(\pi(3))=(4,0,3)^T$. Let $\sigma^9 = \langle y^0,y^1,y^2,y^3\rangle$

(b) For $p$=1. Let $\pi=(1,2,3)$. Therefore

$y^0 = y+s=(3,1,3)^T$,

$y^1 = y+g(\pi(1))=(3,0,3)^T$,

$y^2 = y+g(\pi(2))=(4,1,4)^T$,

$y^3 = y+g(\pi(3))=(4,0,3)^T$. Let $\sigma^{10} = \langle y^0,y^1,y^2,y^3\rangle$

(c) For $p$=2 and $\pi=(1,2,3)$, we have

$y^0 = y+s = (3,1,3)^T$,

$y^1 = y^0 - s_{\pi(1)}u^{\pi(1)} = (4,1,3)^T$,

$y^2 = y+g(\pi(2)) = (4,1,4)^T$,

$y^3 = y+g(\pi(3)) = (4,0,3)^T$. Let $\sigma^{11} = \langle y^0, y^1, y^2, y^3 \rangle$

**5.** Let $y = (4,4,4)^T$. Since $I(y) = \{i\varepsilon N; y_1 = y_i\} = \{1,2,3)$ and $\ell = 3$. We have that $a$ must be $(-1,-1,-1)^T$. Thus $K(y,s) = \phi$ and $h=0$. we have that $p$ can be any one of 0,1,2. We also have

$g(1) = (-1,-1,-1)$,
$g(2) = (0,-1,0)$,
$g(3) = (0,0,-1)$.

(a) For $p=0$. Let $\pi = (1,2,3)$. Therefore

$y^0 = y = (4,4,4)^T$,

$y^1 = y+g(\pi(1)) = (3,3,3)^T$,

$y^2 = y+g(\pi(2)) = (4,3,4)^T$,

$y^3 = y+g(\pi(3)) = (4,4,3)^T$. Let $\sigma^{12} = \langle y^0, y^1, y^2, y^3 \rangle$.

(b) For $p=1$. Let $\pi = (1,2,3)$. Therefore

$y^0 = y+s = (3,3,3)^T$,

$y^1 = y+g(\pi(1)) = (3,3,3)^T$,

$y^2 = y+g(\pi(2)) = (4,3,4)^T$,

$y^3 = y+g(\pi(3)) = (4,4,3)^T$. Let $\sigma^{13} = \langle y^0, y^1, y^2, y^3 \rangle$.

(c) For $p=2$. Let $\pi = (1,2,3)$. We have,

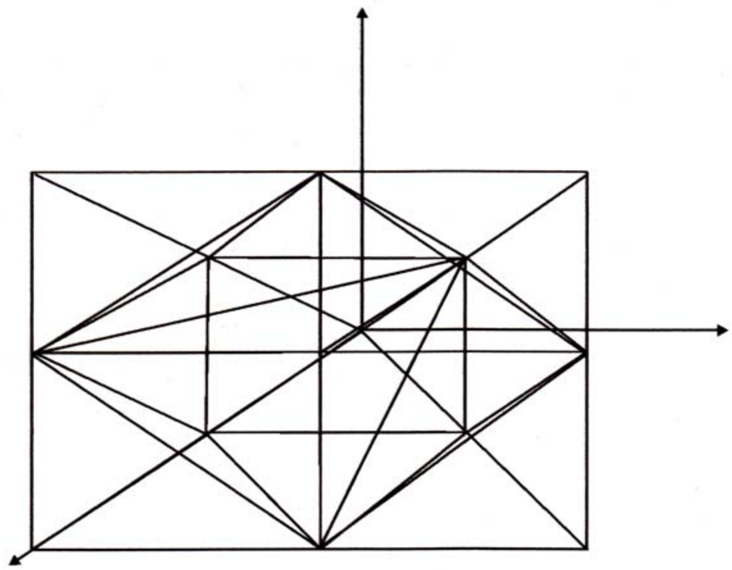
$y^0 = y+s = (3,3,3)^T$,

$y^1 = y^0 - s_{\pi(1)}u^{\pi(1)} = (4,3,3)^T$,

$y^2 = y+g(\pi(2)) = (4,3,4)^T$,

$y^3 = y+g(\pi(3)) = (4,4,3)^T$. Let $\sigma^{14} = \langle y^0, y^1, y^2, y^3 \rangle$

It can be seen that $\{\sigma^i : i = 1,2,\ldots,14\}$ form a triangulation of $W^3$ for $x \le 4$ as shown by the following figure :



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