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ON DISCRETE HERMITE TRANSFORM OF GENERALIZED FUNCTIONS

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ABSTRACT

In this paper discrete Hermite transformation of generalized functions belonging to a certain testing function space has been defined and an inversion formula has been established.

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1. Introduction. The aim of the present work is to extend the Hermite transform

$$F(n) = H\{f(x)\} = \int_{-\infty}^{\infty} e^{-x^2} H_n(x) f(x)\,dx \tag{1.1}$$

and its inversion formula

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi}} \frac{F(n)}{2^n n!} H_n(x) (-\infty < x < \infty) \tag{1.2}$$

studied by Debnath [1], to a class of generalized functions. Zemanian [3,4] has also studied (1.1), but here the present work is treated differently.

2. The Notation and Terminology. Throughout this work the interval $(-\infty,\infty)$ is denoted by I. $D(I)$ denotes the space of infinitely differentiable functions on I which have compact support in I. The topology of $D(I)$ is that which makes its dual $D'(I)$ the space of Schwartz distributions. $E(I)$ denotes the space of all infinitely differentiable functions on I. Its dual $E'(I)$ is the space of distributions with compact support. The symobls Ω_x^k denotes the operator

$$\Omega_x^k = \left(D_x e^{-x^2} D_x e^{x^2}\right)^k \equiv \left(D_x^2 + 2xD_x + 2\right)^k, \; D_x \equiv d/dx \tag{2.1}$$

3. Testing Function Space *H(I)* and Its Dual *H'(I)*. We define $H(I)$ as the space of all complex valued infinitely differentiable functions $\phi(x)$ on I such that

$$\rho_k(\phi) = \sup_{-\infty<x<\infty} \left|\Omega_x^k \phi(x)\right| < \infty \;,\; k = 0,1,2,3,\ldots \;. \tag{3.1}$$

It is seen that $H(I)$ is a linear space. The topology of $H(I)$ is that generated by the countable multinorm $\{\rho_k\}, k = 0,1,2,3,....$.

$H'(I)$ is the dual of $H(I)$ and is equipped with the usual weak topology. The members of H'(I) are called generalized functions. It can be seen than $H(I)$ is a complete countably multinormed space.

We now give below some properties of the space $H(I)$ and it dual $H'(I)$.

(i) $D(I) \subset H(I)$ and topology of $D(I)$ is stronger than the topology induced on it by $H(I)$. Hence, the restriction of any member of $H'(I)$ to $D(I)$ is in $D'(I)$.

(ii) $D(I) \subset H(I) \subset E(I)$. As $D(I)$ is dense in $E(I)$. $H(I)$ is also a dense subspace of $E(I)$. Consequently, $E'(I)$ can be identified as a subspace of $H'(I)$.

(iii) For each $f \in H'(I)$, there exists a non-negative integer r and a positive constant C such that for all $\phi \in H(I)$.

$$|< f,\phi >| \leq C \max_{0 \leq k \leq r} \rho_k(\phi) \tag{3.2}$$

The proof of this statement follows from the boundedness property of generalized functions.

(iv) If $f(x)$ be a function of x defined in the interval $(-\infty,\infty)$ such that $\int_{-\infty}^{\infty} |f(x)| dx < \infty$, then $f(x)$ generates a regular generalized function of $H'(I)$ defined by

$$< f,\phi > = \int_{-\infty}^{\infty} f(x)\phi(x)dx, \phi \in H(I). \tag{3.3}$$

This result can be easily established as follows

$$|< f,\phi >| \leq \rho_0(\phi) \int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

(v) For each positive integer n, the function $\left|e^{-x^2} H_n(x)\right|, (-\infty < x < \infty)$ is a member of $H(I)$. This can be proved as follows :

$$\Omega_x\left[e^{-x^2} H_n(x)\right] = -2n\left\{e^{-x^2} H_n(x)\right\}$$

Hence,

$$\rho_x\left|e^{-x^2} H_n(x)\right| = \sup_{-\infty<x<\infty}\left|\Omega_x^k\left\{e^{-x^2} H_n(x)\right\}\right| = \sup_{-\infty<x<\infty}\left|(-2n)^k e^{-x^2} H_n(x)\right| < \infty \forall x, k = 0,1,2,3,.... \tag{3.4}$$

4. The Discrete Hermite Transform of Generalized Functions. Members of $H'(I)$ are called Discrete Hermite transformable generalized functions.

The Discrete Hermite transform of $f \in H'(I)$ is defined as an application of $f \in H'(I)$ to the kernel $e^{-x^2} H_n(x) \in H(I)$,

i.e. $F(n) = H(f) \equiv \left\langle f(x), e^{-x^2} H_n(x) \right\rangle, n = 0,1,2,\dots$. (4.1)

As we require time to write the expression $\sum_{n=0}^{N} \frac{1}{\sqrt{\pi}} \frac{H_n(x) H_n(t)}{2^n n!}$,

therefore again, we denote it by $T_n(t,x); N$ being any positive integer and $-\infty < x < \infty$, $-\infty < t < \infty$.

We now state below several lemmas whose proofs are based upon the similar lemmas proved in Dube [2].

Lemma 4.1. Let $f \in H'(I)$, then for any positive integer N and for any arbitrary $\phi(x) \in D(I)$

$$\int_{-R}^{R} \left\langle f(t), e^{-t^2} T_N(t,x) \right\rangle e^{-x^2} \phi(x)\, dx = \left\langle f(t), \int_{-R}^{R} e^{-t^2} T_N(t,x) \right\rangle e^{-x^2} \phi(x)\, dx \qquad (4.2)$$

Lemma 4.2. $\lim_{N \to \infty} \int_{-R}^{R} T_N(t,x) e^{-x^2} dx = 1$.

Lemma 4.3. Let $\phi(x)$ be an arbitrary member of $D(I)$ with compact support in $(-R,R) \in$ I, then

$$\int_{-R}^{R} T_N(t,x) \left[\phi(x) - \phi(t) \right] e^{-x^2} dx \to 0 \text{ as } N \to \infty .$$

5. Inversion Theorem. We now prove the following inversion theorem for our Discrete Hermite transform:

Theorem 5.1. If $F(n)$ denotes the Discrete Hermite transform of $f(t) H'(I)$ as defined in (4.1), then in the sense of convergence in $D'(I)$,

$$f(t) = \lim_{N \to \infty} \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{H_n(x)}{2^n n!} F(n).$$

Proof. Let $\phi(x) \in D(I)$ be an arbitrary member of $D(I)$. We are to prove that

$$\left\langle \sum_{n=0}^{N} \frac{1}{\sqrt{\pi}} \frac{H_n(x)}{2^n n!} F(n), \phi(x) \right\rangle \to \; < f(t), \phi(t) > as\ N \to \infty . \qquad (5.1)$$

Now $\phi(x) \in D(I) \Leftrightarrow e^{-x^2}, \phi(x) \in D(I)$, hence (5.1) is equivalent prove

$$\left\langle \sum_{n=0}^{N} \frac{1}{\sqrt{\pi}} \frac{H_n(x)}{2^n n!} F(n), e^{-x^2}\phi(x) \right\rangle \rightarrow \quad < f(t), e^{-x^2}\phi(n) > as\ N \rightarrow \infty . \tag{5.2}$$

We have

$$\left\langle \sum_{n=0}^{N} \frac{1}{\sqrt{\pi}} \frac{H_n(x)}{2^n n!} F(n), e^{-x^2}\phi(x) \right\rangle \tag{5.3}$$

$$= \int_{-R}^{R} \left\{ \sum_{n=0}^{N} \frac{1}{\sqrt{\pi}} \frac{H_n(x)}{2^n n!} F(n) \right\} e^{-x^2}\phi(x)\, dx \tag{5.4}$$

[As $\phi(x) \in D(I)$, the support of $\phi(x)$ is contained in $(-R,R)$].

$$= \int_{-R}^{R} \sum_{n=0}^{N} \frac{1}{\sqrt{\pi}} \frac{H_n(x)}{2^n n!} \left\langle f(t), e^{-x^2} H_n(t) \right\rangle e^{-x^2}\phi(x)\, dx \tag{5.5}$$

$$= \int_{-R}^{R} \left\langle f(t), e^{-t^2} \sum_{n=0}^{N} \frac{1}{\sqrt{\pi}} \frac{H_n(x) H_n(t)}{2^n n!} \right\rangle e^{-x^2}\phi(x)\, dx \tag{5.6}$$

$$= \int_{-R}^{R} \left\langle f(t), e^{-t^2} T_N(t,x) \right\rangle e^{-x^2}\phi(x)\, dx \tag{5.7}$$

$$= \left\langle f(t), e^{-t^2} \int_{-R}^{R} T_N(t,x) e^{-x^2}\phi(x)\, dx \right\rangle \tag{5.8}$$

$$\rightarrow \left\langle f(t), e^{-t^2}\phi(t) \right\rangle as N \rightarrow \infty . \tag{5.9}$$

(5.3) equals to (5.4) as the function $\sum_{n=0}^{N} \frac{1}{\sqrt{\pi}} \frac{H_n(x)}{2^n n!} F(n)$ is locally integrable over the interval I and $\phi(x)$ is in $D(I)$ with support in $(-R,R)$.

(5.6) is obvious because of linearity property of functionals. From Lemma 4.1 we obtain (5.8). Finally, Lemma (4.3) helps us to derive (5.9).

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