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q-OPERATIONAL FORMULAE FOR A CLASS OF *q*-POLYNOMIALS UNIFYING THE GENERALIZED *q*-HERMITE AND *q*-LAGUERRE POLYNOMIALS

By

R.K. Kumbhat, R.K. Gupta and Madhu Chauhan
Department of Mathematics and Statistics
J.N.V. University, Jodhpur-342005, India
E-Mail : drrkkumbhat@yahoo.com, drrkgupta@yahoo.co.in

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ABSTRACT

In the present paper, certain q-operational formulae for the generalized q-polynomials $J_n^{(\alpha)}(x;r,p,l)$ are developed and make an attempt to unify the various results. The results given earlier by Gould and Hopper [6], Singh and Srivastava [9], Al-Salam [1], Das [4], Carlitz [3], Joshi and Singhal [10] follow as special cases.

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1. Introduction. Burchnall [2] made use of the opertional formula

$$(D-2x)^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_{n-k}(x) D^k, \quad D = d/dx, \qquad \text{...(1.1)}$$

to prove the well-known relation

$$H_{m+n}(x) = \sum_{k=0}^{\min[m,n]} (-2)^k \binom{m}{k}\binom{n}{k} K! H_{m-k}(x) H_{n-k}(x). \qquad \text{...(1.2)}$$

Gould and Hopper [6] studied operational formulae associated with classical polynomials and established that

$$\mathscr{D}^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_{n-k}^r(x,r,p) D^k, \qquad \text{...(1.3)}$$

where the symbol $\mathscr{D}$ is defined by

$$\mathscr{D} = D - prx^{r-1} + \alpha/x$$

and satisfies the relation

$$x^n D^n = \prod_{j=0}^{n-1} \left(xD - prx^r + \alpha - j\right),$$

and

$$H_n^r(x,\alpha,p)=(-1)^n x^{-\alpha} exp(px^r)D^n\left\{x^\alpha \exp(-px^r)\right\} \quad ...(1.4)$$

defines the elegant generalization of the Hermite polynomials to which it reduces when $\alpha=0, p=1, r=2$.

The relation (1.3) provides a generalization of the formula of Burchnall [2] quoted above as well as of Carlitz's formula [3]

$$\prod_{j=1}^{n}(xD-x+\alpha+j)=n!\sum_{k=0}^{n}\frac{x^k}{k!}L_{n-k}^{(\alpha+k)}(x)D^k, \quad ...(1.5)$$

for the Laguerre polynomials.

Joshi and Singhal [10] also introduced certain operational formulae associated with a class of polynomials unifying the generalized Hermite and Laguerre polynomials.

The Rodrigues formula (c.f. Joshi and Singhal [10]) is

$$J_n^{(\alpha)}(x;r,p,l)=c(l,n)x^{-\alpha}\exp(px^r)D^n\left\{x^{\alpha+\ln}\exp(-px^r)\right\}, \quad ...(1.6)$$

where

$$c(l,n)=\frac{(-1)^{n(l-1)(l-2)/2}}{2^{nl(l-1)/2}(1)_{nl(2-l)}},$$

l being a non-negative integer.

In a recent paper, the authors [8] established certain operational formulae for the generalized basic Laguerre polynomials with the help of known results. The object of the present paper is to define q-analogue of the generalized polynomials $J_n^{(\alpha)}(x;r,p,l)$ and develop certain q-operational formulae for the generalized polynomials $J_n^{(\alpha)}(x;r,p,l,q)$ and make an attempt to unify the various results.

For real or complex $\alpha, 0<|q|<1$, the q-shifted factorial is defined as

$$(\alpha;q)_n=(q^\alpha;q)_n=\begin{cases}1 & ,n=0\\ (1-q^\alpha)(1-q^{\alpha+1})...(1-q^{\alpha+n-1}), & n\in N\end{cases}. \quad ...(1.7)$$

In terms of the q-gamma function, (1.7) can be expressed as

$$(\alpha;q)_n=\frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, n>0, \quad ...(1.8)$$

where $\Gamma_q(.)$ is the q-gamma function (c.f. Gasper and Rahman [5]) given by

$$\Gamma_q(\alpha)=\frac{(q;q)_\infty}{(q^\alpha;q)_\infty(1-q)^{\alpha-1}}.$$

Indeed, it is easy to verify that

$$\lim_{q\to1^-}\Gamma_q(\alpha)=\Gamma(\alpha) \text{ and } \lim_{q\to1^-}\frac{(q^\alpha;q)_n}{(1-q)^n}=(\alpha)_n, \quad ...(1.10)$$

where

$$(\alpha)_n=\alpha(\alpha+1)...(\alpha+n-1), n\geq1. \quad ...(1.11)$$

The fractional q-derivative of arbitrary order $\lambda>0$ for a function $f(x)=x^{\mu-1}$, is given by

$$D_{q,x}^{\lambda}\left(x^{\mu-1}\right)=\frac{\Gamma_q(\mu)x^{\mu-\lambda-1}}{\Gamma_q(\mu-\lambda)}, \quad ...(1.12)$$

where $\mu\neq0,-1,-2,...$.

For $\lambda=1$, the equation (1.12) reduces to

$$D_{q,x}\left(x^{\mu-1}\right)=\frac{\Gamma_q(\mu)x^{\mu-2}}{L_q(\mu-1)}=\frac{\left(1-q^{\mu-1}\right)x^{\mu-2}}{(1-q)}. \quad ...(1.13)$$

Further, we shall denote the infinite product

$$\prod_{j=0}^{\infty}\frac{(1-a_1q^j)...(1-a_rq^j)}{(1-b_1q^j)...(a-b_sq^j)}=\prod\left[\begin{matrix}a_1,a_2,...,a_r;\\ b_1,b_2,...,b_s;\end{matrix}\,q\right] \quad ...(1.14)$$

In what follows the other notations and symbols employed in this paper have their usual meanings.

2. q-Extension of $J_n^{(\alpha)}(x;r,p,l)$. In this section, we define a q-extension of the generalized polynomials $J_n^{(\alpha)}(x;r,p,l)$ due to Joshi and Singhal [10], by means of the following relation

$$J_n^{(\alpha)}(x;r,p,l,q)=\frac{(-1)^{n(l-1)(l-2)/2}x^{-\alpha}e_q\left(px^r\right)}{2^{n(l)(l-1)/2}(l;q)_{nl(2-l)}}D_{q,x}^{n}\left(x^{\alpha+\ln}e_q\left(px^r\right)\right), \quad ...(2.1)$$

where *r,p,l* and constants assume integral values, and the q-Leibnitz rule is

$$D_{q,x}^{n}\left(UV\right)=\sum_{r=0}^{n}\begin{bmatrix}n\\r\end{bmatrix}D_{q,x}^{n-r}\left(U\right)D_{q,x}^{r}\left(V\right), \quad \text{...(2.2)}$$

where

$$\begin{bmatrix}n\\r\end{bmatrix}=\frac{(q;q)_n}{(q;q)_r\,(q;q)_{n-r}} . \quad \text{...(2.3)}$$

By virtue of (1.12) and (1.10), we observe

$$\lim_{q\to1^-}(1-q)^{n(l-2)}\,J_n^{(\alpha)}(x;r,p,l,q) = J_n^{(\alpha)}(x;r,p,l) . \quad \text{...(2.4)}$$

3. The *q*-Operational Formulae. In this section, we prove the various q-operational formulae by working use of the q-differential operator $\delta = xD_q$, which possesses the following interesting properties:

(i) $$F(\delta)\left[x^{\alpha} f(x;q)\right]=x^{\alpha}F(\delta+\alpha)\,f(x;q), \quad \text{...(3.1)}$$

(ii) $$F(\delta)\left[e_q(g(x))\,f(x;q)\right]=e_q(g(x))\,F(\delta+xg')\,f(x;q), \quad \text{...(3.2)}$$

(iii) $$x^{n\alpha}F(\delta)F(\delta+\alpha)...F(\delta+(n-1)\alpha) = \left[x^{\alpha}F(\delta)\right]^{n}. \quad \text{...(3.3)}$$

We now consider the expression

$$e_q(px^r)\,x^{-\alpha-ln}D_q^n\left[x^{\alpha+ln}e_q(-px^r)Y\right]$$

$$= e_q(px^r)\,x^{-\alpha-ln-n}x^nD_q^n\left[x^{\alpha+ln}e_q(-px^r)Y\right]$$

$$= e_q(px^r)\,x^{-\alpha-ln-n}\delta(\delta-1)(\delta-2)...(\delta-n+1).\left[x^{\alpha+ln}e_q(-px^r)Y\right] \text{ [By (3.3)]}$$

$$= x^{-n}\prod_{j=1}^{n}\left(\delta+\alpha+(l-1)n-prx^r+j\right) \text{ [By making an appeal to (3.1) and (3.2)].}$$

We thus have

$$D_q^n\left[x^{\alpha+ln}e_q(-px^r)Y\right] = x^{\alpha+ln-n}e_q(-px^r)\prod_{j=1}^{n}\left(\delta+\alpha+(l-1)n-prx^r+j\right)Y .$$

On the other hand, replacing n by $n\text{-}k$, α by $\alpha+lk$ in (2.1), we find that

$$D_q^n\left[x^{\alpha+ln}e_q\left(-px^r\right)Y\right]=2^{nl(l-1)/2}x^{\alpha}e_q\left(-px^r\right)\sum_{k=0}^{n}(-1)^{(n-k)(l-1)(l-2)/2}\begin{bmatrix}n\\k\end{bmatrix}(1;q)_{(n-k)l(2-l)}$$

$$\left(\frac{x^l}{2^{l(l-1)/2}}\right)^k j_{n-k}^{(\alpha+lk)}(x;r,p,l,q). \qquad ...(3.5)$$

Comparision of (3.4) and (3.5), gives the q-operational formula

$$\prod_{j=1}^{n}\left(\delta+\alpha+(l-1)n-prx^r+j\right)=x^{(1-l)n}2^{nl(l-1)/2}\sum_{k=0}^{n}(-1)^{(n-k)(l-1)(l-2)/2}\begin{bmatrix}n\\k\end{bmatrix}$$

$$(1;q)_{(n-k)l(2-l)}\left(\frac{x^l}{2^{l(l-2)/2}}\right)^k J_{n-k}^{(\alpha+lk)}(x;r,p,l,q) \quad . \qquad ...(3.6)$$

Also

$$x^{-\alpha}e_q\left(px^r\right)D_q^n\left[x^{\alpha+ln}e_q\left(-px^r\right)Y\right]$$

$$=x^{-\alpha-n}e_q\left(px^r\right)\delta(\delta-1)(\delta-2)...(\delta-n+1)\left[x^{n-k}x^{\alpha+(l-1)n+k}e_q\left(-px^r\right)Y\right].$$

Therefore, making an appeal to (3.1), (3.2), (3.3) and (3.5), we obtain the another q-operational formula

$$x(\delta-k+1)^n\left[x^{\alpha+(l-1)n+k}e_q\left(-px^r\right)Y\right]$$

$$=x^{k+n+\alpha}2^{nl(l-1)/2}e_q\left(-px^r\right)\sum_{k=0}^{n}(-1)^{(n-k)(l-1)(l-2)/2}\begin{bmatrix}n\\k\end{bmatrix}(1;q)_{(n-k)l(2-l)}\left(\frac{x^l}{2^{l(l-1)/2}}\right)^k$$

$$J_{n-k}^{(\alpha+lk)}(x;r,p,l,q).D_q^kY \qquad ...(3.7)$$

Next we observe that the q-analogue of the recurrence relation of Joshi and Singhal [10]:

$$\left(x^lD_q+\alpha x^{l-1}-prx^{l+r-1}\right)J_n^{(\alpha)}(x;r,p,l,q)=\frac{c(l,n)}{c(l,n+1)}J_{n+1}^{(\alpha-1)}(x;r,p,l,q)$$

suggests the q-operational formula

$$D_{q,l}^mJ_n^{(\alpha)}(x;r,p,l,q)=\frac{c(l,n)}{c(l,m+n)}J_{m+n}^{(\alpha-nl)}(x;r,p,l,q) \qquad ...(3.8)$$

where

$$\mathscr{D}_{q,L}\equiv x^lD_q+\alpha x^{l-1}-prx^{l+r-1},$$

which corresponds to the q-analogue of the formula of Gould and Hopper [6] to which it reduces when $l=0$.

4. Particular Cases. In this section we shall deduce some interesting particular cases of the q-operational formulae.

(i) If we let $q \to 1^-$ and make use of the limit formula (1.10), in (3.8), we obtain known results due to Gould and Hopper [6].

(ii) If we put $l=0$ and make use of (1.10), our formula (3.6) reduces to (1.3) due to Gould and Hopper [6].

(iii) Again, if we set $p=l=r=1$ in (3.6) and make use of the limit formula (1.10), we arrive at the known results due to Carlitz [3].

(iv) On the other hand, when $l=1$, $q \to 1^-$ and use of (1.10), (3.7) yields the known formula due to Joshi [7].

(v) If we put $l=1$, $p=r=1$ $k=0$, $Y=1$ in (3.7), we get the q-operational formula of Das [4]:

$$\left\{\left(xD_q+1\right)\right\}^n\left\{x^n e_q(-x)\right\} = x^{\alpha+n} e_q(-x)(q;q)_n L_n^{(\alpha)}(x;q).$$

(vi) If we take $l=k=0$, $Y=1$ in (3.7), we obtain

$$\left\{x\left(xDq+1\right)\right\}^n\left\{x^{\alpha-n} e_q\left(-px^r\right)\right\} = (-1)^n x^{-\alpha+n} H_n^r(x;r,p,q).$$

5. Applications. Setting $Y=1$ in (3.6), we have

(i) $$\prod_{j=1}^{n}\left(\delta+\alpha+(l-1)n-prx^r+j\right)1 = \frac{x^{(1-l)n}}{c(l,n)} J_n^{(\alpha)}(x;r,p,l,q), \quad \text{...(5.1)}$$

so that

$$\frac{x^{(1-l)(m+n)}}{c(l,m+n)} J_{m+n}^{(\alpha)}(x;r,p,l,q)$$

$$= \prod_{j=1}^{m+n}\left(\delta+\alpha+(l-1)(m+n)-prx^r+j\right)1$$

$$= \prod_{j=1}^{m}\left(\delta+\alpha+(l-1)(m+n)-prx^r+j+n\right)\prod_{j=1}^{n}\left(\delta+\alpha+(l-1)(m+n)-prx^r+j\right)1$$

$$= \frac{x^{(1-l)n}}{c(l,n)}\prod_{j=1}^{m}\left(\delta+\alpha+n+(l-1)m-prx^r+j\right).J_n^{\alpha+(l-1)n}(x;r,p,l,q).$$

Therefore in views of (3.6) we finally derive

(ii) $$\frac{c(l,m)c(l,n)}{c(l,m+n)}J_{m+n}^{(\alpha)}(x;r,p,l,q)$$

$$=\frac{1}{(1;q)_{ml(2-l)}}\sum_{k=0}^{m}(-1)^{k(l-1)(l-2)/2}\begin{bmatrix} m \\ k \end{bmatrix}(1;q)_{(m-k)l(2-l)}\left(\frac{x^l}{2^{l(l-1)/2}}\right)^k$$

$$J_{m-k}^{\alpha+lk+n}(x;r,p,l,q)D_q^kJ_n^{\alpha+(l-1)m}(x;r,p,l,q). \qquad ...(5.2)$$

If we reverse the order of the operator on the left hand side in (3.6) and proceed as above, we obtain an alternative q-operational formula

(iii) $$\frac{c(l,m)c(l,n)}{c(l,m+n)}J_{m+n}^{(\alpha)}(x;r,p,l,q)$$

$$=\frac{1}{(1;q)_{ml(2-l)}}\sum_{k=0}^{m}(-1)^{k(l-1)(l-2)/2}\begin{bmatrix} m \\ k \end{bmatrix}(1;q)_{(m-k)l(2-l)}\left(\frac{x^l}{2^{l(l-1)/2}}\right)^k$$

$$J_{m-k}^{(\alpha+lk)}(x;r,p,l,q)D_q^kJ_n^{(\alpha+lm)}(x;r,p,l,q). \qquad ...(5.3)$$

Comparision of (5.2) and (5.3) gives the identity

(iv) $$\sum_{k=0}^{m}(-1)^{k(l-1)(l-2)/2}\begin{bmatrix} m \\ k \end{bmatrix}(1;q)_{(m-k)l(2-l)}\left(\frac{x^l}{2^{l(l-1)/2}}\right)^k$$

$$J_{m-k}^{(\alpha+n+lk)}(x;r,p,l,q)D_q^kJ_n^{\alpha+(l-1)m}(x;r,p,l,q)$$

$$=\sum_{k=0}^{m}(-1)^{k(l-1)(l-2)/2}\begin{bmatrix} m \\ k \end{bmatrix}(1;q)_{(m-k)l(2-l)}\left(\frac{x^l}{2^{l(l-1)/2}}\right)^k$$

$$J_{m-k}^{(\alpha+lk)}(x;r,p,l,q)D_q^kJ_n^{\alpha+lm}(x;r,p,l,q), \qquad ...(5.4)$$

which in the special case $l=0$, making use of (4.6), gives

(v) $$\sum_{k=0}^{m}(-1)^k\begin{bmatrix} m \\ k \end{bmatrix}H_{m-k}^r(x;\alpha,p,q)D_q^kH_n^r(x;r,p,q)$$

$$=\sum_{k=0}^{m}(-1)^k\begin{bmatrix} m \\ k \end{bmatrix}H_{m-k}^r(x;\alpha+n,p,q)D_q^kH_n^r(x;\alpha q^{-m},p,q) \qquad ...(5.5)$$

for the generalized q-Hermite polynomials.

Next in (5.3) if we replace α by $\alpha-lm$, multiplying on both the sides by

$\frac{t^m}{(q;q)_m}$ and summing from $m=0$ to $m=\infty$, we arrive at

(vi) $$\sum_{m=0}^{\infty} \frac{(l;q)_{(m+n)l(2-l)}}{(q;q)_m} t^m J_{m+n}^{\alpha-lm}(x;r,p,l,q)$$

$$= (1;q)_{nl(2-l)} J_n^{(\alpha)}\left(x + A_l t x^r, r, p, l, q\right) \sum_{m=0}^{\infty} \frac{(l;q)_{ml(2-l)}}{(q;q)_m} t^m J_m^{(\alpha-lm)}(x;r,p,l,q)$$

where

$$A_l = (-1)^{k(l-1)(l-2)/2} (2)^{-l(l-1)/2} \qquad ...(5.6)$$

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REFERENCES

[1] W.A. Al-Salam, Operational representations for the Laguerre and other polynomials, *Duke Math. J.,* **31** (1964), 127-142.

[2] J.l. Burchnall, A note on the polynomials of Hermite, *Quart. J. Math Oxford,* **12** (1941), 9-11.

[3] L. Carlitz, A note on the Laguerre polynomails, *Mich. Maths J.,* **7** (1960), 219-223.

[4] M.K. Das, Operational representation for the Laguerre polynomials, *Acta Math. Acad. Sci. Hungar.* **18** (1967), 335-338.

[5] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, New York (1990).

[6] H.W. Gould and A.T. Hopper, Operational formulae connected with two generalizations of Hermite polynomails, *Duke Math. J.,* **29** (1962), 51-63.

[7] C.M. Joshi, Operational formulae associated with the generalized Laguerre polynomials, *Mathematica,* **11** (34), 2, (1969), 269-276.

[8] R.K. Kumbhat, R.K. Gupta and Madhu Chouhan, Certain operational formulae associated with generalized basic Laguerre polynomials, *Vijanana Priadhad Anushandan Patrika* **52**, (2), (April 2009), 163-172.

[9] R.P. Singh and K.N. Srivastava, A note on generalization of Laguerre and Humbert Polymials, *La Ricerca,* (2), **14** (1963), 11-21. Errata, ibid **(2)15** (1964), 63.

[10] J.P. Singhal and C.M. Joshi, On the Unification of generalized Hermite and Laguerre polynomials, *Revista Matemática Hispanoamericana,* ISSN 0373-0999, **42**, 1-3 (1982) 82-89.