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SOME RESULTS ON A τ-GENERALIZED RIEMANN ZETA FUNCTION

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ABSTRACT

In this paper we give a τ-generalization of the new zeta function due to Goyal and Laddha. This function is defined by series and the corresponding integral representations are established. We have also investigated the properties of the new type of generating functions such as integral representations and generating functions. Several interesting results obtained earlier by Goyal and Laddha [3], Katsurada [4] and Bin-Saad [1] follow special cases of our main findings.

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1. Introduction. An interesting definition of the zeta functions, due to Goyal and Laddha [3], is as follows:

$$\phi_\mu^*(y;z,a)=\sum_{n=0}^{\infty}\frac{(\mu)_n\, y^n}{(a+n)^z\, n!},|y|<1, \tag{1.1}$$

$\mathrm{Re}\,a>0, \mu\ge 1,$ where $(\lambda)_n=\dfrac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$.

This is expressed as the integral form

$$\phi_\mu^*(y,z,a)=\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}e^{-at}\left(1-ye^{-t}\right)^{-\mu}dt\,. \tag{1.2}$$

This function is continued to a meromorphic function over the whole z-plane (cf. Erdélyi [2]). Obviously, when μ=1, (1.1) reduces to the zeta function studied by Erdélyi [2]. Katsurada [4] introduced two hypergeometric type generating functions of the Riemann zeta function as follows:

$$e_z(x)=\sum_{m=0}^{\infty}\zeta(z+m)\frac{x^m}{m!}, \qquad |x|<\infty\,, \tag{1.3}$$

and

$$f_z(v;x) = \sum_{m=0}^{\infty} (v)_m \zeta(z+m) \frac{x^m}{m!}, \qquad |x| < 1 \tag{1.4}$$

where v and z are arbitrary fixed complex parameters.
Recently, Bin-Saad [1] has defined two new type of generating functions suggested by (1.3) and (1.4) as :

$$\zeta(x,y;z,a) = \sum_{m=0}^{\infty} \phi(y;z+m,a) \frac{x^m}{m!}, \qquad |y| < 1 \tag{1.5}$$

$$\zeta_v(x,y;z,a) = \sum_{m=0}^{\infty} (v)_m \phi(y;z+m,a) \frac{x^m}{m!}, \qquad |y| < 1, |x| < |a| \tag{1.6}$$

where ϕ is the generalized zeta function.

Its various properties are investigated including the integral representations, generating functions, partial sums and N-fractional calculus. In a recent paper the authors [5] have defined a new generalized zeta function and derived its hypergeometric types of generating functions. In this paper we aim at giving τ-generalizations of the generalized zeta function and at deriving their various properties and formulas including their integral representations, series and genrating functions.

2. Definition and Integral Representations. A τ-generalization of generalized zeta function is defined in terms of series representations as :

$$\phi_\mu^{*\tau}(y;\tau,z,a) = \sum_{n=0}^{\infty} \frac{(\mu)_n y^n}{(a+\tau n)^z n!} \quad , \tag{2.1}$$

where $|y| < 1, \text{Re}\, a > 0, \mu \geq 1, t \in R, t > 0$.

It is interesting to note that for $\tau = 1$, (2.1) reduces to the generalized zeta function studied by Goyal and Laddha [3].

Here we prove the following integral representation

$$\phi_\mu^{*\tau}(y;\tau,z,a) = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-at} \left(1 - ye^{-\tau t}\right)^{-\mu} dt, \tag{2.2}$$

where $|y| < 1, \text{Re}\, a > 0, \mu \geq 1, t \in R, t > 0$.

Proof of (2.2). Let

$$I = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-at} \left(1 - ye^{-\tau t}\right)^{-\mu} dt$$

$$= \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-at} \sum_{n=0}^\infty \frac{(\mu)_n \left(ye^{-\tau t}\right)^n}{n!} dt$$

On interchanging the order of integration and summation, we see that

$$I = \sum_{n=0}^\infty \frac{(\mu)_n\, y^n}{n!(a+\tau n)^z} \frac{1}{\Gamma(z)} \int_0^\infty e^{-p} p^{z-1} dp$$

$$= \sum_{n=0}^\infty \frac{(\mu)_n\, y^n}{(a+\tau n)^z\, n!}$$

$$= \phi_\mu^{*\tau}(y;\tau,z,a).$$

This completes the proof of (2.2).

We now give the τ-generalization of two hypergeometric type generating functions of the Riemann zeta function in the form:

$$e_z^\tau(x;\tau) = \sum_{m=0}^\infty \zeta^\tau(z+m) \frac{x^m}{m!}, \tag{2.3}$$

$|x| < \infty, \tau \in R, \tau > 0,$

and

$$f_z^\tau(v;\tau,x) = \sum_{m=0}^\infty (v)_m\, \zeta^\tau(z+m) \frac{x^m}{m!}, \tag{2.4}$$

$|x| < 1, \tau \in R, t > 0,$

where v and z are arbitrary fixed complex parameters.

3. Integral Involving $e_z^\tau(x;\tau)$ and $f_z^\tau(v;\tau,x)$. In this section we evaluate definite integrals involving the function $e_z^\tau(x;\tau)$ and $f_z^\tau(v;\tau,x)$ in terms of other kind of zeta and hypergeometric functions. First, we recall the Eulerian integral formula of first kind (cf. Srivastava and Manocha [6]);

$$\int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \mathrm{Re}(x) > 0, \mathrm{Re}(y) = 0. \tag{3.1}$$

From the term by term integration, we can derive the following

Theorem 1. Let $\mathrm{Re}(c-b) > 0$ and $\mathrm{Re}(b) > 0$. Then

$$\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\, e_z^{\tau}\left(xt^{\tau};\tau\right)dt = G_z^{\tau}(b,c;\tau,x) \tag{3.2}$$

and

$$\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\, f_z^{\tau}\left(v;\tau,xt^{\tau}\right)dt = G_z^{\tau}(v,b,c;\tau,x)\,, \tag{3.3}$$

where

$$G_z^{\tau}(b,\mu;c;\tau,x) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{m=0}^{\infty}\frac{(\mu)_m\,\Gamma(b+\tau m)}{\Gamma(c+\tau m)}\zeta^{\tau}(z+\tau m)\frac{x^m}{m!}, \tag{3.4}$$

and

$$G_z^{\tau}(b,c;\tau,x) = \frac{\Gamma(c)}{\Gamma(b)}\sum_{m=0}^{\infty}\frac{\Gamma(b+\tau m)}{\Gamma(c+\tau m)}\zeta^{\tau}(z+\tau m)\frac{x^m}{m!}, \tag{3.5}$$

For $\tau = 1$, (3.2) and (3.3) reduce to the known results given earlier by Katsurada [4].

Proof of (3.2). Denote for convenience the left-hand side of relation (3.2) by *I*. Then in view of (2.3), it is easily seen that

$$I = \sum_{m=0}^{\infty}\zeta^{\tau}(z+\tau m)\frac{x^m}{m!}\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{b+\tau m-1}(1-t)^{c-b-1}\,dt\,.$$

Using (3.1), we obtain

$$I = \sum_{m=0}^{\infty}\zeta^{\tau}(z+\tau m)\frac{x^m}{m!}\frac{\Gamma(c)\Gamma(b+\tau m)\Gamma(c-b)}{\Gamma(c-b)\Gamma(b)\Gamma(c+\tau m)}$$

$$= \frac{\Gamma(c)}{\Gamma(b)}\sum_{m=0}^{\infty}\frac{\Gamma(b+\tau m)}{\Gamma(c+\tau m)}\zeta^{\tau}(z+\tau m)\frac{x^m}{m!}.$$

After some simplification, we get the right-hand side of (3.2).

Proof of (3.3). Denote for convenience the left-hand side of relation (3.3) by *J*. Then in view of (2.3), it is easily seen that

$$J = \sum_{m=0}^{\infty}\frac{(v)_m\,x^m}{m!}\zeta^{\tau}(z+\tau m)\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{b+\tau m-1}(1-t)^{c-b-1}\,dt\,.$$

Upon using the Eulerian integral formula (3.1) and the definition (3.4), we are finally led to right-hand side of formula (3.3).

Theorem 2. Let $\operatorname{Re} z > 0$, $\operatorname{Re}\mu > 0$ and $\operatorname{Re}\lambda < 1, \tau \in R, \tau > 0$ Then

$$\frac{1}{\Gamma(\mu)\Gamma(z)}\int_0^\infty\int_0^\infty u^{\mu-1}v^{z-1}e^{-\mu-av}\zeta^\tau\left(xue^{-\tau v},y;\tau,z,a\right)dudv$$

$$=\sum_{n=0}^\infty \phi_\mu^{*\tau}\left(\frac{x}{(a+\tau n)};\tau,z,a\right)\frac{y^n}{(a+\tau n)^z}, \tag{3.6}$$

and

$$\frac{1}{\Gamma(z)}\int_0^\infty t^{z-1}e^{-at}\zeta_v^\tau\left(xe^{\tau t},y;\tau,z,a\right)dt$$

$$=\sum_{n=0}^\infty \phi_v^{*\tau}\left(\frac{x}{(a+\tau n)};\tau,z,a\right)\frac{y^n}{(a+\tau n)^z}, \tag{3.7}$$

where

$$\zeta^\tau(x,y;\tau,z,a)=\sum_{m=0}^\infty \phi^\tau(y;z+\tau m,a)\frac{x^m}{m!}, \tag{3.8}$$

$|y|<1, \tau \in R, \tau > 0, \operatorname{Re}(z) > 1, a \neq 0,-1,-2,\ldots$

and

$$\zeta_v^\tau(x,y;\tau,z,a)=\sum_{m=0}^\infty (v)_m\,\phi^\tau(y;z+\tau m,a)\frac{x^m}{m!}, \tag{3.9}$$

$|y|<1, \tau \in R, \tau > 0, \operatorname{Re}(z) > 1, a \neq 0,-1,-2,\ldots$,

while ϕ^τ is the τ-generalized zeta function.

For $\tau=1$, (3.8) and (3.9) reduce to the hypergeometric type generating function studied by Bin-Saad [1].

Proof of (3.6). Denote for convenience the left-hand side of equation (3.6) by L. Then, in view of (3.8), we have

$$L=\sum_{m,n=0}^\infty \frac{x^m y^n}{m!(a+\tau n)^{z+m}}\frac{1}{\Gamma(\mu)}\int_0^\infty u^{\mu+m-1}e^{-\mu}du\frac{1}{\Gamma(z)}\int_0^\infty e^{-v(a+\tau m)}v^{z-1}dv\,.$$

Applying the Eulerian integral for gamma function, we get

$$L=\sum_{n=0}^\infty \frac{y^n}{(a+\tau n)^z}\sum_{m=0}^\infty \frac{(\mu)_m\left[x/(a+\tau n)\right]^m}{m!(a+\tau m)^z}\,.$$

After some simplification, we obtain the right-hand side of (3.6).

This completes the proof of (3.6). Proof of (3.7) can be developed on the same line.

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