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APPLICATIONS OF SAIGO FRACTIONAL CALCULUS OPERATORS FOR CERTAIN SUBCLASS OF MULTIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT

In the present paper, we introduce the subclass $S_j(n,p,\lambda,q,\alpha)$ of functions with negative coefficients, which are analytic and multivalent in the unit disc $U=\{z:|z|<1\}$. The fractional calculus of functions associated with ingegral operator $J_{c,p}$ in the class $S_j(n,p,\lambda,q,\alpha)$ as applications of the Saigo fractional calculus operator $I_{0,z}^{\beta,\gamma,\eta}$ are established here.

Corresponding to our main theorems some known and unkown results for the multivalent functions are also shown to be deduced as the special cases.

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1. Introduction. Let $T(j,p)$ be the class of function of the form

$$f(z)=z^p-\sum_{k=j+p}^{\infty}a_k z^k \qquad \left(a_k\geq 0;\, p,j\in N=\{1,2,...\}\right), \tag{1}$$

which are analytic and p-valent in the open disc $U=\{z:|z|<1\}$. A function $f(z)\in T(j,p)$ is said to be p-valently close to convex of order α in U if it satisfies the inequality

$$Re\left\{z^{1-p}f(z)\right\} > \alpha \qquad \left(z \in U; 0 \le \alpha < p; p \in N\right). \tag{2}$$

A function $f(z) \in T(j,p)$ is said to be p-valently starlike of order α in U if it satisfies the inequality

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad \left(z \in U; 0 \le \alpha < p; p \in N\right). \tag{3}$$

Further more a function $f(z) \in T(j,p)$ is said to p-valently convex of order α in U if it satisfies the inequality

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad \left(z \in U; 0 \le \alpha < p; p \in N\right). \tag{4}$$

For each $f(z) \in T(j,p)$ we have [2]

$$f^{(q)}(z) = \frac{p!}{(p-q)!}z^{p-q}\sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}a_k z^{k-q} \quad \left(q \in N_0 = N \cup \{0\}; p > q\right). \tag{5}$$

For a function $f(z)$ in class $T(j,p)$, we define the following differential operator:

$$D^0_{p,\lambda}f^{(q)}(z) = f^{(q)}(z),$$

$$D^1_{p,\lambda}f^{(q)}(z) = Df^{(q)}(z) = \frac{z^{1-\lambda}}{(p+\lambda-q)}\left[z^{\lambda}f^{(q)}(z)\right]$$

$$= \frac{p!}{(p-q)!}z^{p-q} - \sum_{k=j+p}^{\infty}\frac{(k+\lambda-q)}{(p+\lambda-q)}\frac{k!}{(k-q)!}a_k z^{k-q},$$

$$D^2_{p,\lambda}f^{(q)}(z) = D\left[D^1_{p,\lambda}f^{(q)}(z)\right] = \frac{z^{1-\lambda}}{(p+\lambda-q)}\left[z^{\lambda}D^1_{p,\lambda}f^{(q)}(z)\right]$$

$$= \frac{p!}{(p-q)!}z^{p-q} - \sum_{k=j+p}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^2 a_k z^{k-q},$$

and so on

$$D^n_{p,\lambda}f^{(q)}(z) = D\left[D^{n-1}_{p,\lambda}f^{(q)}(z)\right] = \frac{z^{1-\lambda}}{(p+\lambda-q)}\left[z^{\lambda}D^{n-1}_{p,\lambda}f^{(q)}(z)\right]$$

$$= \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=j+p}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k+\lambda-q}{p+\lambda-q} \right)^{n} a_k z^{k-q}$$

$$(p, j \in N; q \in N_0; p > q; \lambda \geq 0) \tag{6}$$

A function $f^{(q)}(z) \in T(j,p)$ is said to be in $S_j(n,p,\lambda,q,\alpha)$ if and only if

$$Re\left(\frac{z\left[D^n_{p,\lambda} f^{(q)}(z)\right]'}{D^n_{p,\lambda} f^{(q)}(z)} \right) > \alpha; \tag{7}$$

where $z \in U, \lambda \geq 0, p \in N, q, n \in N_0, 0 \leq \alpha < p-q, p > q$ and $D^n_{p,\lambda}$ is defined in (6).

We note that for $q = 0$ the operator $D^n_{p,\lambda} f(z)$ in view of multiplier transformation is defined and studied recently by Agharaly et al. [1] and Singh et al. [13] for positive coefficients in the following form

$$I_p(n,\lambda) f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda} \right)^{n} a_k z^k \qquad (\lambda \geq 0, n \in Z).$$

Earlier the operator $I_1(n,\lambda)$ was investigated by Cho and Srivastava [6] and Cho and Kim [7]. Whereas the operator $I_1(n,l)$ was studied by Uralegaddi and Somanatha [16]. $I_1(n,0)$ is the well known *Sălăgean* derivative operator [11] defined as

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, n \in N_0 = NU\{0\}.$$

2. Coefficient Estimate.

Theorem 1. Let the function *f*(*z*) defined by (1) be in the class $T(j,p)$. Then the function *f*(*z*) belongs to the class $S_j(n,p,\lambda,q,\alpha)$ if and only if

$$\sum_{k=j+p}^{\infty} \left(\frac{k+\lambda-q}{p+\lambda-q} \right)^{n} (k-q-\alpha)\delta(k,q) a_k \leq (p-q-\alpha)\delta(p,q) \tag{8}$$

$$(0 \leq \alpha < p-q; p, j \in N; q, n \in N_0; p > q; \lambda \geq 0;)$$

where

$$\delta(p,q)=\frac{p!}{(p-q)!}=\begin{cases} p(p-1)\ldots(p-q+1), & q\neq 0 \\ 1 & ,q=0 \end{cases} \tag{9}$$

Proof. Let us suppose the inequality (8) holds true. Then in view of (7), we have

$$\left|\frac{z\left[D^n_{p,\lambda}f^{(q)}(z)\right]'}{D^n_{p,\lambda}f^{(q)}(z)}-(p-q)\right|\leq\frac{\sum\limits_{k=j+p}^{\infty}\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^n(k-p)\delta(k,q)a_k|z|^{k-p}}{\delta(p,q)-\sum\limits_{k=j+p}^{\infty}\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^n\delta(k,q)a_k|z|^{k-p}}$$

$$\leq\frac{\sum\limits_{k=j+p}^{\infty}\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^n(k-p)\delta(k,q)a_k}{\delta(p,q)-\sum\limits_{k=j+p}^{\infty}\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^n\delta(k,q)a_k}\leq p-q-\alpha$$

Therefore the values of function

$$\phi(z)=\frac{z\left[D^n_{p,\lambda}f^{(q)}(z)\right]}{D^n_{p,\lambda}f^{(q)}(z)} \tag{10}$$

lie in a circle which is centered at $w=(p-q)$ and whose radius is $(p-q-\alpha)$. Hence the function *f*(*z*) satisfies the condition given in (7)

Now conversely, assume that the function *f*(*z*) is in the class $S_j(n,p,\lambda,q,\alpha)$. Then we have

$$Re\left\{\frac{z\left[D^n_{p,\lambda}f^{(q)}(z)\right]}{D^n_{p,\lambda}f^{(q)}(z)}\right\}=Re\left\{\frac{(p-q)\delta(p,q)-\sum\limits_{k=j+p}^{\infty}(k-q)\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^n\delta(k,q)a_kz^{k-q}}{\delta(p,q)-\sum\limits_{k=j+p}^{\infty}\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^n\delta(k,q)a_kz^{k-q}}\right\}>\alpha \tag{11}$$

for some $\alpha(0\leq\alpha<p-q;\,p,j\in N;\,q,n\in N_0;\,p>q;\,\lambda\geq 0)$ and $z\in U$. Choose value of z on the real axis so that $\phi(z)$ given by (10) is real. Upon clearing the denominator in (11) and letting $z\to 1^-$ through the real values we can see that

$$(p-q)\delta(p,q)-\sum\limits_{k=j+p}^{\infty}(k-q)\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^n\delta(k,q)$$

$$a_k \geq \alpha\left\{\delta(p,q)-\sum_{k=j+p}^{\infty}\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^n \delta(k,q)a_k\right\}, \tag{12}$$

which leads to inequality (8). It completes the proof of Theorem 1.

Corollary 1. Let the function *f*(*z*) defined by (1) be in the class $S_j(n,p,\lambda,q,\alpha)$ then the following inequality hold true

$$a_k \leq \frac{(p-q-\alpha)\delta(p,q)}{\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^n (k-q-\alpha)\delta(k,q)} \tag{13}$$

$$(k \geq j+p;\, p,j \in N;\, q,n \in N_0;\, \lambda \geq 0;\, p > q).$$

The result is sharp for the function *f*(*z*) given by

$$f(z) = z^p - \frac{(p-q-\alpha)\delta(p,q)}{\left(\frac{k+\lambda-q}{p+\lambda-q}\right)^n (k-q-\alpha)\delta(k,q)} z^k \tag{14}$$

$$(k \geq j+p;\, p,j \in N;\, q,n \in N_0;\, \lambda \geq 0;\, p > q).$$

3. Applicatrions of Fractional Calculus. In our present investigation, we shall make use of the familiar integral operator $J_{c,p}$ defined by [5,p.676, eq. (1.8)]

$$(J_{c,p}f)(z) = \frac{c+p}{z^c}\int_0^z t^{c-1}f(t)\,dt \qquad (f \in T(j,p);\, c > -p;\, p \in N). \tag{15}$$

Definition 1. The Reimann-Liouville fractional integral of order λ is defined, for a function $f(z)$, by [15,p.224,eq.(3.1)]

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}}\,d\zeta, \tag{16}$$

where $\lambda > 0, f(z)$ is an analytic function in a simply-connected region of the *z*-plane containing the origin, and hte multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring log $(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 2. The Reimann-Liouville fractional derivative of order λ is defined, for a function $f(z)$, by [15,p.224, eq. (3.2)]

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta, \tag{17}$$

where $0 \le \lambda < 1, f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed as in Definition 1 above.

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\lambda)$ is defined by [15,p.225, eq. (3.3)]

$$D_z^{(n+\lambda)} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z), \tag{18}$$

where and $n \in N_0 = NU\{0\}$.

Srivastava, Saigo and Owa defined the following fractional integral operator involving Gauss's hypergeometric function:

Definition-4. For real number $\alpha > 0, \beta$ and η, the Saigo fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}$ is defined by [12,p.112, Eq. (8)] (See also [15]):

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = \frac{z^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^z (z-\zeta)^{\alpha-1} F\left(\alpha+\beta, -\eta; \alpha; 1-\frac{\zeta}{z}\right) f(\zeta) d\zeta, \tag{19}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin with the order

$f(z) = O\left(|z|^{\epsilon}\right)$, as $z \to 0$,

where $\epsilon > max\{0, \beta-\eta\} - 1$ and the many-valuedness of $(z-\xi)^{\alpha-1}$ is removed by requiring $log(z-\xi)$ to be real when $(z-\xi) > 0$.

From Definition (1) and Definition (4), it is easy to see that

$D_z^{-\alpha} f(z) = I_{0,z}^{\alpha,\alpha,\eta} f(z)$.

Lemma. If $\alpha > 0$ and $k > \beta - \eta - 1$, then [12]

$$I_{0,z}^{\alpha,\beta,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+1)\Gamma(k+\alpha+\eta+1)} z^{k-\beta}. \tag{20}$$

For a function $f(z) \in T(j,p)$ defined by (1), we obtain easily the following results in view of (15) and (20) respectively :

$$\left(J_{c,p}f\right)(z) = z^p - \sum_{k=j+p}^{\infty} \left(\frac{c+p}{c+k}\right) a_k z^k \qquad \left(c > -p; p, j \in N\right) \tag{21}$$

and

$$I_{0,z}^{\beta,\gamma,\eta}\left\{f(z)\right\} = \frac{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)}{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)} z^{p-\gamma} - \sum_{k=j+p}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\gamma+\eta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\beta+\eta+1)} a_k z^{k-\gamma} . \tag{22}$$

Now for the function $\left(J_{c,p}f\right)(z)$ defined in (21) we obtain the following fractional integral in view of (22)

$$I_{0,z}^{\beta,\gamma,\eta}\left\{\left(J_{c,p}f\right)(z)\right\} = \frac{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)}{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)} z^{p-\gamma} -$$

$$\sum_{k=j+p}^{\infty} \left(\frac{c+p}{c+k}\right) \frac{\Gamma(k+1)\Gamma(k-\gamma+\eta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\beta+\eta+1)} a_k z^{k-\gamma} \tag{23}$$

$$\left(\beta > 0, c > -p; p, j \in N\right) .$$

Theorem 2. Let β,γ and η satisfy inequalities $\beta > 0, \gamma < p+1, \gamma - \eta < p+1,$ $\beta + \eta > -(p+1)$. Choose a positive integer such that $n \geq \gamma(\beta+\eta)/\beta - p - 1$.

If $f(z) \in S_j(n,p,\lambda,q,\alpha)$ then

$$\left|I_{0,z}^{\beta,\gamma,\eta}\left\{\left(J_{c,p}f\right)(z)\right\}\right| \geq \left\{\frac{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)}{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)} - \frac{(c+p)}{c+p+j}\right.$$

$$\left.\frac{\Gamma(p+j+1)\Gamma(p+j-\gamma+\eta+1)(p-q-\alpha)\delta(p,q)}{\Gamma(p+j-\gamma+1)\Gamma(p+j+\beta+\eta+1)\left(\frac{j+p+\lambda-q}{p+\lambda-q}\right)^n (j+p-q-\alpha)\delta(j+p,q)} |z|^j\right\} |z|^{p-\gamma}$$

$$\left(z \in U_0; 0 \leq \alpha < p-q; c > -p; p, j \in N; q, n \in N_0; p > q; \lambda \geq 0\right) \tag{24}$$

and

$$\left|I_{0,z}^{\beta,\gamma,\eta}\left\{\left(J_{c,p}f\right)(z)\right\}\right| \leq \left\{\frac{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)}{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)} + \frac{(c+p)}{c+p+j}\right.$$

$$\left.\frac{\Gamma(p+j+1)\Gamma(p+j-\gamma+\eta+1)(p-q-\alpha)\delta(p,q)}{\Gamma(p+j-\gamma+1)\Gamma(p+j+\beta+\eta+1)\left(\frac{j+p+\lambda-q}{p+\lambda-q}\right)^n(j+p-q-\alpha)\delta(j+p,q)}|z|^j\right\}|z|^{p-\gamma}$$

$$\left(z\in U_0;0\leq\alpha<p-q;c>-p;p,j\in N;q,n\in N_0;p>q;\lambda\geq 0\right), \tag{25}$$

where

$$U_0=\begin{cases}U, & \gamma\leq p\\ U-\{0\}, & \gamma>p\end{cases}.$$

These results are sharp for function $f(z)$ given by

$$\left(J_{c,p}f\right)(z)=z^p-\frac{(c+p)(p-q-\alpha)\delta(p,q)}{(c+p+j)\left(\frac{j+p+\lambda-q}{p+\lambda-q}\right)^n(j+p-q-\alpha)\delta(j+p,q)}z^{j+p}. \tag{26}$$

Proof. To prove the Theorem 2, we start from eq. (23), i.e.

$$\left|I_{0,z}^{\beta,\gamma,\eta}\left\{\left(J_{c,p}f\right)(z)\right\}\right|=\frac{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)}{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)}z^{p-\gamma}-$$

$$\sum_{k=j+p}^{\infty}\left(\frac{c+p}{c+k}\right)\frac{\Gamma(k+1)\Gamma(k-\gamma+\eta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\beta+\eta+1)}a_kz^{k-\gamma}\ . \tag{27}$$

Now on setting

$$H(z)=\frac{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)}{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)}z^{\gamma}\left|I_{0,z}^{\beta,\gamma,\eta}\left\{\left(J_{c,p}f\right)(z)\right\}\right| \tag{28}$$

the above result given in (27) takes the following form

$$H(z)=z^p-\sum_{k=j+p}^{\infty}a_k\psi(k)z^k \tag{29}$$

where

$$\psi(k)=\frac{\Gamma(k+1)\Gamma(k-\gamma+\eta+1)}{\Gamma(k-\gamma+1)\Gamma(k+\beta+\eta+1)}\frac{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)}{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)}\frac{(c+p)}{(c+k)}\ .$$

Since $\psi(k)$ is a decreasing function of *k*, therefore

$$\psi(k) \le \psi(p+j)$$

$$= \frac{\Gamma(j+p+1)\Gamma(j+p-\gamma+\eta+1)}{\Gamma(j+p-\gamma+1)\Gamma(j+p+\beta+\eta+1)} \frac{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)}{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)} \frac{(c+p)}{(c+p+j)} \quad (30)$$

Now in view of Theorem 1, we have

$$\sum_{k=j+p}^{\infty} a_k \le \frac{(p-q-\alpha)\delta(p,q)}{\delta(j+p,q)\left(\frac{j+p+\lambda-q}{p+\lambda-q}\right)^n (j+p-q-\alpha)}. \quad (31)$$

Hence for the function $H(z)$ obtained in (29), we have

$$|H(z)| \ge |z|^p - |z|^{j+p} \psi(j+p) \sum_{k=j+p}^{\infty} a_k .$$

Therefore, making an appeal to (30) and (31), we obtain

$$|H(z)| \ge |z|^p - \frac{(c+p)}{(c+p+j)} \frac{\Gamma(j+p+1)\Gamma(j+p-\gamma+\eta+1)}{\Gamma(j+p-\gamma+1)\Gamma(j+p+\beta+\eta+1)} \frac{\Gamma(p-\gamma+1)\Gamma(p+\beta+\eta+1)}{\Gamma(p+1)\Gamma(p-\gamma+\eta+1)}$$

$$\frac{(p-q-\alpha)\delta(p,q)}{\left(\frac{j+p+\lambda-q}{p+\lambda-q}\right)^n (j+p-q-\alpha)\delta(j+p,q)} |z|^{j+p} .$$

Now in view of (28), we atonce arrive at the desired result in (24).
The second result of Theorem 2 is proved similarly in view of (30) and (31) for inequality

$$|H(z)| \le |z|^p + |z|^{j+p} \psi(j+p) \sum_{k=j+p}^{\infty} a_k .$$

4. Special Cases. (I) If in differential operator $D^n_{p,\lambda} f^{(q)}(z)$ defined in (6) we consider $\lambda = 0$ then it reduces to the operator studied by Aouf [2]. Therefore, at $\lambda = 0$ all the results established in the Sections-2, 3 will provide the known results of Aouf [2].

(II) The results obtained in class $S_j(n,p,\lambda,q,\alpha)$ here in turn provide many known results studied in various subclasses by specializing the parameters j,n,p,q,α,λ. To illustrate we give few classes as follows:

(i) $S_j(n,p,0,q,\alpha)=S_j(n,p,q,\alpha)$ (Aouf [2]) ,

(ii) $S_j(0,p,0,q,\alpha)=S_j(p,q,\alpha)$ and $S_j(1,p,0,q,\alpha)=C_j(p,q,\alpha)$ (Chen et al.[4]),

(iii) $S_j(n,1,0,0,\alpha)=P(j,\alpha,n)(j\in N;n\in N_0;0\le\alpha<1)$ (Aouf and Srivastava [3]),

(iv) $S_1(n,1,0,0,\alpha)=T(n,\alpha)(n\in N_0;0\le\alpha<1)$ (Hur and Oh [8]),

(v) $S_j(0,p,0,0,\alpha)=\begin{cases}T_j^*(p,\alpha) & (Owa[10])\\ T_\alpha(p,j) & (Yamakawa[17])\end{cases},$

(vi) $S_j(1,p,0,0,\alpha)=\begin{cases}C_j^*(p,\alpha) & (Owa[10])\\ CT_\alpha(p,j) & (Yamakawa[17])\end{cases},$

(vii) $S_1(0,p,0,0,\alpha)=T^*(p,\alpha)$ and $S_1(1,p,0,0,\alpha)=C(p,\alpha)(p\in N;0\le\alpha<p)$ (Owa [9] and Salagean et al. [11]),

(viii) $S_j(0,1,0,0,\alpha)=T_\alpha(j)$ and $S_j(1,1,0,0,\alpha)=C_\alpha(j)(n\in N_0;0\le\alpha<1)$ (Srivastava et al. [14]),

(ix) $S_j(n,p,0,0,\alpha)=S_j(n,p,\alpha)(p,j\in N,n\in N_0;0\le\alpha<p)$ (Aouf [2]),

(III) If in (24) and (25) we take $\gamma=-\beta$ then these inequalities reduce to the following inequalities involving *R-L* fractional integral operator defined in (16).

Corollary 2. Let the function $f(z)$ defined by (1) be in the class $S_j(n,p,\lambda,q,\alpha)$. Then

$$D_z^{-\beta}\left\{(J_{c,p}f)(z)\right\}\ge\left\{\frac{\Gamma(p+1)}{\Gamma(p+\beta+1)}-\frac{(c+p)\Gamma(p+j+1)(p-q-\alpha)(\delta(p,q))}{(c+p+j)\Gamma(p+j+\beta+1)\left(\frac{j+p+\lambda-q}{p+\lambda-q}\right)^n(j+p-q-\alpha)\delta(j+p,q)}|z|^j\right\}|z|^{p+\beta} \quad (32)$$

$(z\in U;0\le\alpha<p-q;\beta>0;c>-p;p,j\in N;n\in N_0;p>q;\lambda>0)$,

and

$$D_z^{-\beta}\left\{\left(J_{c,p}f\right)(z)\right\} \leq \left\{\frac{\Gamma(p+1)}{\Gamma(p+\beta+1)}+\right.$$

$$\left.\frac{(c+p)\Gamma(p+j+1)(p-q-\alpha)(\delta(p,q))}{(c+p+j)\Gamma(p+j+\beta+1)\left(\frac{j+p+\lambda-q}{p+\lambda-q}\right)^n(j+p-q-\alpha)\delta(j+p,q)}|z|^j\right\}|z|^{p+\beta} \quad (33)$$

$\left(z\in U; 0\leq\alpha<p-q; \beta>0; c>-p; p,j\in N; n\in N_0; p>q; \lambda>0\right)$.

Each of the assertion (32) and (33) is sharp for the function $f(z)$ givey by (26). The results in (32) and (33) in turn at $\lambda=0$ give the known results due to Aouf [2, p.32, eq. (6.6) and eq. (6.7)].

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