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CERTAIN FRACTIONAL CALCULUS OPERATORS ASSOCIATED WITH FOX-WRIGHT GENERALIZED HYPERGEOMETRIC FUNCTION

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ABSTRACT

In this paper Riemann-Lioville fractional integral and derivative formulas for Fox-Wright ${}_p\varphi_q(z)$ generalized hypergeometric functions are obtained. Some special cases of the established formulas are also discussed.

2000 Mathematics Subject Classification : 26A33, 45D05.

Keywords : Fox-Wright generalized hypergeometric Function, Riemann-Liuville fractional Integral and differential.

1. Introduction. The Fox-Wright generalized hypergeometric ${}_p\varphi_q(z)$ for $z \in C$ is defined in series form as [4].

$$ {}_p\varphi_q(z) = {}_p\varphi_q\left[\begin{matrix}(a_i, A_i)_{1,p} \\ (b_i, B_j)_{1,q}\end{matrix}\middle| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma\left(a_i + A_j k\right) z^k}{\prod_{i=1}^{q} \Gamma\left(b_i + B_j k\right) k!} \qquad \text{...(1.1)} $$

where $a_j, b_j \in C, A_i > 0, B_j > 0$

$$ 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_i \geq 0; A_i, B_j \in R\left(A_i, B_j \neq 0\right)(i = 1, ..., p, j = 1, ..., q) $$ for suitably bounded value of $|z|$.

As special case, for $A_1 = ... = A_p = 1$, $B_1 = ... = B_q = 1$, (1.1) reduces to generalized hypergeometric function [3]

$$ {}_p\phi_q\left[\begin{matrix}(a_1,1), ..., (a_p,1) \\ (b_1,1), ..., (b_q,1)\end{matrix}\middle| z\right] = \frac{\Gamma(a_1), ..., \Gamma(a_p)}{\Gamma(b_1), ..., \Gamma(b_q)} {}_pF_q\left[\begin{matrix}a_1, ..., a_p; \\ b_1, ..., b_q;\end{matrix}\, z\right] \qquad \text{...(1.2)} $$

For $p=1$, $q=1$, $a_1=\delta, A_1=1, b_1=\beta$ and $B_1=\alpha$, (1.1) reduces to

$$ {}_1\varphi_1\left[\begin{matrix}(\delta,1)\\(\beta,\alpha)\end{matrix}\middle| z\right]=\sum_{k=0}^{\infty}\frac{\Gamma(\delta+k)}{\Gamma(\beta+\alpha k)}\frac{z^k}{k!}=\Gamma(\delta)E_{\alpha,\beta}^{\delta}(z) \qquad ...(1.3) $$

called generalized Mittag-Leffler function defined by Prabhakar [2].

If $\delta=1$ then (1.3) reduces to generalized Mittag-Leffler function

$$ {}_1\varphi_1\left[\begin{matrix}(1,1)\\(\beta,\alpha)\end{matrix}\middle| z\right]=\sum_{k=0}^{\infty}\frac{\Gamma(1+k)}{\Gamma(\beta+\alpha k)}\frac{z^k}{k!}=E_{\alpha,\beta}(z). $$

If $\delta=1$ and $\beta=1$ then (1.3) reduces to

$$ {}_1\varphi_1\left[\begin{matrix}(1,1)\\(1,\alpha)\end{matrix}\middle| z\right]=\sum_{k=0}^{\infty}\frac{\Gamma(1+k)}{\Gamma(1+\alpha k)}\frac{z^k}{k!}=E_{\alpha}(z) \qquad ...(1.3) $$

called Mittag-Leffler function [1].

The object of this paper is to derive the relations which exist between the Fox-Wright generalized hypergeometric function ${}_p\varphi_q(z)$ and left and right side Riemann-Liouville fractional integral and differential operators.

The fractional integral and differential operators defined by Samko, Kilbas and Marichev [7] for $\alpha>0$ are given by

$$ \left(I_{0+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,dt, \qquad ...(1.4) $$

$$ \left(I_{-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^{\infty} (t-x)^{\alpha-1}f(t)\,dt, \qquad ...(1.5) $$

$$ \left(D_{0+}^{\alpha}f\right)(x)=\frac{1}{\Gamma(1-\{\alpha\})}\left(\frac{d}{dx}\right)^{[\alpha]+1}\int_0^x (x-t)^{-\{\alpha\}}f(t)\,dt, \qquad ...(1.6) $$

$$ \left(D_{-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(1-\{\alpha\})}\left(\frac{d}{dx}\right)^{[\alpha]+1}\int_x^{\infty} (t-x)^{-\{\alpha\}}f(t)\,dt, \qquad ...(1.7) $$

where $[\alpha]$ means the maximal integer not exceeding α and $\{\alpha\}$ is the fractional part of α.

2.Properties of generalized Fox-Wright hypergeometric function. In this section we derive several interesting properties of the generalized Fox-

Wright hypergeometric function ${}_p\varphi_q(z)$ defined by (1.1) with the help of Riemann -Liouville fractional integral and derivative formula [4].

Theorem 1. Let $\alpha>0,\beta>0,\gamma>0$ and $a\in R$, let I_{0+}^{α} be left sided operator of Riemann-Liouville fractional integral (1.4). Then there holds the formula

$$I_{0+}^{\alpha}\left\{t^{\gamma-1}{}_p\varphi_q\left[\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\middle|at^{\beta}\right]\right\}(x)=x^{\gamma+\alpha-1}{}_{p+1}\varphi_{q+1}\left[\begin{matrix}(a_i,A_i)_{1,p},(\gamma,\beta)\\(b_j,B_1)_{1,q},(\alpha+\gamma,\beta)\end{matrix}\middle|ax^{\beta}\right] \quad ...(2.1)$$

Proof. By virtue of (1.1) and (1.4) we have

$$I_{0+}^{\alpha}\left\{t^{\gamma-1}{}_p\varphi_q\left[\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\middle|at^{\beta}\right]\right\}(x)=\frac{1}{\Gamma(\alpha)}\int_0^x\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_j+B_jk)}\frac{(x-t)^{\alpha-1}a^kt^{\beta k+\gamma-1}}{k!}dt\,.$$

Interchanging the order of integration and summation; and evaluating the inner integral with the help of beta functions, by setting $t=xy$, we get

$$L.H.S.=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_i+B_jk)}\frac{a^kx^{\beta k+\gamma+\alpha-1}}{\Gamma(\alpha)k!}\int_0^1(1-y)^{\alpha-1}y^{\beta k+\gamma-1}dy$$

$$=x^{\gamma+\alpha-1}{}_{p+1}\varphi_{q+1}\left[\begin{matrix}(a_i,A_i)_{1,p},(\gamma,\beta)\\(b_i,B_j)_{1,q},(\alpha+\gamma,\beta)\end{matrix}\middle|ax^{\beta}\right].$$

Interchanging the order of integration and summation is permissible under the conditions stated with the theorem, due to convergence of the integral involved in the process. This completes the proof of Theorem 1.

Corollary 1.1 .For $\alpha>0,\beta>0,\gamma,\lambda>0$, there holds the formula

$$I_{0+}^{\alpha}\left\{t^{\gamma-1}{}_1\varphi_1\left[\begin{matrix}(\delta,1)\\(\gamma,\beta)\end{matrix}\middle|at^{\beta}\right]\right\}(x)=\Gamma(\delta)x^{\gamma+\alpha-1}E_{\beta,\alpha+\gamma}^{\delta}\left(ax^{\beta}\right) \quad ...(2.2)$$

Corollary 1.2 . By setting $\delta=1$ in (2.2) there holds the formula

$$I_{0+}^{\alpha}\left\{t^{\gamma-1}E_{\beta,\alpha}\left(at^{\beta}\right)\right\}(x)=x^{\gamma+\alpha-1}E_{\beta,\alpha+\gamma}\left(ax^{\beta}\right) \quad ...(2.3)$$

establised by Saxena and Saigo [6].

Theorem 2. Let $\alpha>0,\beta>0,\gamma>0$ and $a\in R$, let I_{-}^{α} be right operator of Riemann Liouville fractional integral (1.5). Then there holds the formula

$$I_{-}^{\alpha}\left\{t^{-\alpha-\gamma}\ {}_{p}\varphi_{q}\left[\left.\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\right|at^{-\beta}\right]\right\}(x)=x^{-\gamma}\ {}_{p+1}\varphi_{q+1}\left[\left.\begin{matrix}(a_i,A_i)_{1,p},(\gamma,\beta)\\(b_j,B_j)_{1,q},(\alpha+\gamma,\beta)\end{matrix}\right|ax^{-\beta}\right] \quad ...(2.4)$$

Proof. By virtue of (1.1) and (1.5), we have

$$I_{-}^{\alpha}\left\{t^{-\alpha-\gamma}\ {}_{p}\varphi_{q}\left[\left.\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\right|at^{-\beta}\right]\right\}(x)=\frac{1}{\Gamma(\alpha)}\int_x^{\infty}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_j+B_jk)}\frac{(t-x)^{\alpha-1}a^k t^{-\beta k-\alpha-\gamma}}{k!}dt.$$

Interchanging the order of integration and summation and then evaluating the inner integral by beta function formula, we get

$$L.H.S.=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_i+B_ik)}\frac{a^k}{\Gamma(\alpha)k!}\int_x^{1}(t-x)^{\alpha-1}t^{-\beta k-\alpha-\gamma}dt$$

$$=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_j+B_jk)}\frac{a^k}{\Gamma(\alpha)k!}\int_0^{\infty}y^{\alpha-1}(x+y)^{-\beta k-\alpha-\gamma}\,dt \quad \text{(by putting } t\text{–}x\text{=}y\text{)}$$

$$=x^{-\gamma}\ {}_{p+1}\varphi_{q+1}\left[\left.\begin{matrix}(a_i,A_i)_{1,p},(\gamma,\beta)\\(b_j,B_j)_{1,q},(\alpha+\gamma,\beta)\end{matrix}\right|ax^{-\beta}\right] \quad \text{(putting } y\text{=}wx\text{).}$$

Interchanging the order of integration and summation is permissible under the conditions stated with the theorem, due to convergence of the integrals involved in the process. This completes the proof of Theorem 2.

Corollary 2.1. For $\alpha>0,\beta>0,\gamma>0,a\in R$, there holds the formula

$$I_{-}^{\alpha}\left\{t^{-\alpha-\gamma}\ {}_{1}\varphi_{1}\left[\left.\begin{matrix}(\delta,1)\\(\gamma,\beta)\end{matrix}\right|at^{-\beta}\right]\right\}(x)=\Gamma(\delta)x^{-\gamma}E_{\beta,\alpha+\gamma}^{\delta}\left(ax^{-\beta}\right). \quad ...(2.5)$$

Corollary 2.2. By setting $\delta=1$ in (2.5), there holds the formula

$$I_{-}^{\alpha}\left\{t^{-\alpha-\gamma}\ {}_{1}\varphi_{1}\left[\left.\begin{matrix}(1,1)\\(\gamma,\beta)\end{matrix}\right|at^{-\beta}\right]\right\}(x)=x^{-\gamma}E_{\beta,\alpha+\gamma}\left(ax^{-\beta}\right) \quad ...(2.6)$$

estabilished by Saxena and Saigo [6].

Theorem 3. Let $\alpha > 0, \beta > 0, \gamma, \lambda > 0$, $a \in R$ and D_{0+}^{α} be left sided operator of Riemann Liouville fractional integral (1.6). Then there holds the formula

$$D_{0+}^{\alpha} t^{\gamma-1}\, {}_1\varphi_1 \left[t^{\gamma-1}\, {}_p\varphi_q \left[\begin{matrix} (a_i, A_i)_{1,p} \\ (b_j, B_j)_{1,q} \end{matrix} \middle| a t^{\beta} \right] \right](x) = x^{\gamma-\alpha-1}\, {}_{p+1}\varphi_{q+1} \left[\begin{matrix} (a_i, A_i)_{1,p}, (\gamma, \beta) \\ (b_j, B_j)_{1,q}, (\alpha-\gamma, \beta) \end{matrix} \middle| a x^{\beta} \right] \quad ...(2.7)$$

Proof. By virtue of (1.1) and (1.6), we have

$$D_{0+}^{\alpha} t^{\gamma-1}\, {}_p\varphi_q \left[\begin{matrix} (a_i, A_i)_{1,p} \\ (b_j, B_j)_{1,q} \end{matrix} \middle| a t^{\beta} \right](x) = \left(\frac{d}{dx} \right)^{[\alpha]+1} \left(I_{0+}^{1-\{\alpha\}} t^{\gamma-1}\, {}_p\varphi_q \left[\begin{matrix} (a_i, A_i)_{1,p} \\ (b_j, B_j)_{1,q} \end{matrix} \middle| a t^{\beta} \right](x) \right)$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + A_i k)}{\prod_{j=1}^{q} \Gamma(b_j + B_j k)} \frac{a^k}{\Gamma(1-\{\alpha\}) k!} \left(\frac{d}{dx} \right)^{[\alpha]+1} \int_0^x (x-t)^{1-\{\alpha\}-1} t^{\beta k + y - 1} dt.$$

Interchanging the order of integration and summation and evaluating the inner integral by beta function formula (by setting $t=xy$), we obtain

$$L.H.S. = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + A_i k)}{\prod_{j=1}^{q} \Gamma(b_j + B_j k)} \frac{a^k}{k!} \frac{\Gamma(\gamma + \beta k)}{\Gamma(\gamma + \beta k + 1 - \{\alpha\})} \left(\frac{d}{dx} \right)^{[\alpha]+1} x^{\beta k + \gamma - \{\alpha\}}$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + A_i k)}{\prod_{j=1}^{q} \Gamma(b_j + B_j k)} \frac{\Gamma(\gamma + \beta k)}{\Gamma(\gamma + \beta k - \alpha)} \frac{x^{\beta k + \gamma - \alpha - 1} a^k}{k!}$$

$$= x^{\gamma-\alpha-1}\, {}_{p+1}\varphi_{q+1} \left[\begin{matrix} (a_i, A_i)_{1,p}, (\gamma, \beta) \\ (b_j, B_j)_{1,q}, (\alpha-\gamma, \beta) \end{matrix} \middle| a x^{\beta} \right].$$

Interchaning the order of integration and summation is permissible under the condition stated with the theorem, due to convergence of the integral involved in the process. This completes the proof of Theorem 3.

Corollary 3.1. For $\alpha>0,\beta>0,\gamma,\lambda>0,$ there holds the formula

$$D_{0+}^{\alpha}\left\{t^{\gamma-1}\,{}_1\varphi_1\left[\begin{matrix}(\delta,1)\\(\gamma,\beta)\end{matrix}\middle|at^{\beta}\right]\right\}(x)=\Gamma(\delta)x^{\gamma-\alpha-1}E_{\beta,\alpha+\gamma}^{\delta}\left(ax^{\beta}\right) \quad ...(2.8)$$

Corollary 3.2. By setting $\delta=1$ in (2.8) then there holds the formula

$$D_{0+}^{\alpha}\left\{t^{\gamma-1}\,{}_1\varphi_1\left[\begin{matrix}(1,1)\\(\gamma,\beta)\end{matrix}\middle|at^{\beta}\right]\right\}(x)=x^{\gamma-\alpha-1}E_{\beta,\alpha+\gamma}\left(ax^{\beta}\right) \quad ...(2.9)$$

established by Saxena and Saigo [5].

Theorem 4. Let $\alpha>0,\beta>0,\gamma,\lambda>0$, $a\in R$ and D_{-}^{α} be right sided operator of Riemann Liouville fractional integral (1.7). Then there holds the formula

$$D_{-}^{\alpha}\left\{t^{\alpha-\gamma}\,{}_p\varphi_q\left[\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\middle|at^{-\beta}\right]\right\}(x)=x^{-\gamma}\,{}_{p+1}\varphi_{q+1}\left[\begin{matrix}(a_i,A_i)_{1,p},(\gamma,\beta)\\(b_j,B_j)_{1,q},(\alpha+\gamma,\beta)\end{matrix}\middle|ax^{-\beta}\right] \quad ...(2.10)$$

Proof. By virtue of (1.1) and (1.7), we have

$$D_{-}^{\alpha}t^{\alpha-\gamma}\,{}_p\varphi_q\left[\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\middle|at^{-\beta}\right](x)=\left(\frac{d}{dx}\right)^{[\alpha]+1}\left(I_{-}^{1-\{\alpha\}}t^{\alpha-\gamma}\,{}_p\varphi_q\left[\begin{matrix}(a_i,A_i)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\middle|at^{\beta}\right](x)\right)$$

$$=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_j+B_jk)}\frac{a^k}{\Gamma(1-\{\alpha\})k!}\left(\frac{d}{dx}\right)^{[\alpha]+1}\int_x^{\infty}(t-x)^{-\{\alpha\}}\,t^{-\beta k+\alpha-y}dt$$

$$=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_j+B_jk)}\frac{a^k}{k!\Gamma(1-\{\alpha\})}\left(\frac{d}{dx}\right)^{[\alpha]+1}\int_0^{\infty}(x+y)^{-\beta k+\alpha-\gamma}\,y^{1-\{\alpha\}-1}dy$$

(putting *t–x=y)*

$$=\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_ik)}{\prod_{j=1}^{q}\Gamma(b_j+B_jk)}\frac{a^k\Gamma(\beta k+\gamma-[\alpha]-1)}{k!\Gamma(\beta k+\gamma-\alpha)}\left(\frac{d}{dx}\right)^{[\alpha]+1}x^{-\beta k-\gamma+[\alpha]+1} \quad \text{(putting } y=wx\text{)}$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + A_i k)}{\prod_{j=1}^{q} \Gamma(b_j + B_j k)} \frac{a^k \Gamma(\beta k + \gamma)}{k!\,\Gamma(\beta k + \gamma - \alpha)} x^{-\beta k - \gamma}$$

$$= x^{-\gamma}\; {}_{p+1}\varphi_{q+1}\left[\left.\begin{matrix}(a_i, A_i)_{1,p}, (\gamma, \beta)\\ (b_j, B_j)_{1,q}, (\alpha - \gamma, \beta)\end{matrix}\right| a x^{-\beta}\right].$$

Interchanging the order of integration and summation is permissible under the condition stated with the Theorem, due to convergence of the integral involved in the process. This completes the proof of Theorem 4.

Corollary 4.1. For $\alpha > 0, \beta > 0, \gamma, \lambda > 0$, there holds the formula

$$D_{-}^{\alpha}\left\{t^{\alpha-\gamma}\; {}_{1}\varphi_{1}\left[\left.\begin{matrix}(\delta,1)\\ (\gamma,\beta)\end{matrix}\right| a t^{-\beta}\right]\right\}(x) = \Gamma(\delta)\, x^{-\gamma} E^{\delta}_{\beta,\alpha-\gamma}\left(a x^{-\beta}\right) \qquad ...(2.11)$$

Corollary 4.2 . For $\delta = 1$, (2.11) takes the form

$$D_{-}^{\alpha}\left\{t^{\alpha-\gamma}\; {}_{1}\varphi_{1}\left[\left.\begin{matrix}(1,1)\\ (\gamma,\beta)\end{matrix}\right| a t^{-\beta}\right]\right\}(x) = x^{-\gamma} E_{\beta,\alpha-\gamma}\left(a x^{-\beta}\right) \qquad ...(2.12)$$

established by Saxena and Saigo [6].

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J$\widetilde{\text{n}}$āṇābha, Vol. 41, 2011

LIE THEORY AND BASIC GAUSS POLYNOMIALS

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ABSTRACT

In the Present paper, an attempt has been made to bring basic hypergeometric functions within the purview of Lie theory by constructing a dynamical symmetry algebra of basic hypergeometric function ${}_2\phi_1$. Multiplier representation theory is then used to obtain generating function for basic analogue of Gauss polynomial. The results obtained in this paper are extensions of the results derived earlier by Miller [3] and Sarkar-Chatterjea [4].

2000 Mathematics Subject Classification : Primary 60E07, 20G05; Secondary 33C55, 14A17.

Keywords : Lie algebra, Generating function, Basic Gauss polynomials.

1. Introduction. The q-analogue of the Gauss functions or Heine's series [1] may be written as

$$ {}_2\phi_1(a,b;c;q;x) = \sum_{n=0}^{\infty} [a;q,n][b;q,n]/[c;q,n][n;q]! \quad (c \neq 0,-1,-2,...) $$

where $|q|<1$ and $|x|<1$.

Here $[a;q,n]$ and $[n;q]!$ are respectively the basic Pochammer's symbol and basic factorial function defined as $[a;q,n]=[a;q][a+1;q]\ldots[a+n-1;q]$ and $[n;q]!=[1;q][2;q]...[n;q]$.

The basic differential operator $B^{\wedge}_{q,x}$ is defined by [1] through the relation

$$ B^{\wedge}_{q,x}\phi(x) = \{\phi(qx)-\phi(x)\}/x(q-1). \qquad ...(1.1) $$

2. The Dynamical Symmetry Algebra of ${}_2\phi_1$. The dynamical symmetry algebra of the hypergeometric function has been defined by Miller [2]. We use the same technique to define the dynamical symmetry algebra of ${}_2\phi_1$. Let

$$\phi_{\alpha,\beta,\gamma,q} = \Gamma_q(\gamma-\alpha)\Gamma_q(\alpha)/\Gamma_q(\gamma).{}_2\phi_1[\alpha,\beta;\gamma;q;x]s^{\alpha}u^{\beta}t^{\gamma} \qquad ...(2.1)$$

be the basis elements of a subspace of analytical functions of four variables *x,s,u* and *t,* associated with Heine's basic hypergeometric function of Heine's series ${}_2\phi_1$. Introduction of variables *s,u* and *t* renders differential operators independent of parameters α,β and γ and thus facilitates their repeated operation.

The dynamical symmetry algebra of ${}_2\phi_1$ is a 15-dimensional complex Lie algebra isomorphic to *sl* (4), generated by twelve $E^{\wedge}$-operators termed as raising or lowering operators in view of their effect of raising or lowering the corresponding suffix in $\phi_{\alpha\beta\gamma,q}$. The $E^{\wedge}$-operators are

$$\text{(i)}\ E^{\wedge}_{-\alpha,q} = s^{-1}\left(x(1-x)B^{\wedge}_{q,x} + tB^{\wedge}_{q,t} - sB^{\wedge}_{q,s} - xuB^{\wedge}_{q,u}\right),$$

$$\text{(ii)}\ E^{\wedge}_{-\beta,-\gamma,q} = u^{-1}t^{-1}\left(x(1-x)B^{\wedge}_{q,x} - xsB^{\wedge}_{q,x} + tB^{\wedge}_{q,t} - 1\right), \qquad ...(2.2)$$

The action of these operators on $\phi_{\alpha\beta\gamma,q}$ is given by

$$E^{\wedge}_{-\alpha,q}\phi_{\alpha,\beta,\gamma,q} = [\alpha-1;q]\phi^{-1}_{\alpha,q,\beta,\gamma,q}$$

$$E^{\wedge}_{-\beta,-\gamma,q}\phi_{\alpha,\beta,\gamma,q} = [\gamma-\alpha-1;q]\phi^{-1\ \ -1}_{\alpha,\beta q\gamma q,q} \qquad ...(2.3)$$

The upper factor in each bracket is to be associated with plus sign and lower with minus sign. Twelve *E*-operators together with three maintenance operators $J_{\alpha}, J_{\beta}, J_{\gamma}$ and Identity operator *I* form a basic for $gl(4) \cong sl(4)(I)$, where (I) is the 1-dimensional Lie algebra generated by 1.
Here

$$J^{\wedge}_{\alpha,q} = sB^{\wedge}_{q,s} J^{\wedge}_{\beta,q} = uB^{\wedge}_{q,u}, J^{\wedge}_{\gamma,q} = tB^{\wedge}_{q,t} \text{ and } I^{\wedge} = 1 \qquad ...(2.4)$$

with the results

$$J^{\wedge}_{\alpha,q}\phi_{\alpha,\beta,\gamma,q} = [\alpha;q]\phi_{\alpha,\beta,\gamma,q},$$

$$J^{\wedge}_{\beta,q}\phi_{\alpha,\beta,\gamma,q} = [\beta;q]\phi_{\alpha,\beta,\gamma,q},$$

$$J^{\wedge}_{\gamma,q}\phi_{\alpha,\beta,\gamma,q} = [\gamma;q]\phi_{\alpha,\beta,\gamma,q},$$

and

$$I^{\wedge}\phi_{\alpha,\beta,\gamma,q} = \phi_{\alpha,\beta,\gamma,q}. \qquad ...(2.5)$$

3. The Generating Functions for Basic Analogues of Gauss Polynomials. On comparing the results obtained by the action one parameter subgroup $\left(exp_q\, aE^{\wedge}_{-\alpha,-\gamma,q}\right)$ generated by the operator $E^{\wedge}_{-\alpha,q}$ defined in (2.2) on $\phi_{\alpha,\beta,\gamma,q}$ defined in (2.1) and direct expansion, we get the identity

$$\left[st/(ax+st)\right]^{\beta}\left[a/(st+1)\right]^{\gamma-1}\,{}_2\Phi_1\left[\alpha,\beta;\gamma;q;x(a+st)/(ax+st)\right]$$

$$=\sum_{m=0}^{\infty}a^m\left[\gamma-m;q\right]_m/\left[m;q\right]!\,{}_2\Phi_1\left[\alpha q^{-m},\beta;\gamma;q^{-m};q;x\right](st)^{-m} \quad ...(3.1)$$

Taking, $\alpha\to 0,\beta\to\lambda+\mu+m-1,\gamma\to q^{\lambda+m},st\to 1,a\to 1,$ we get

$$(1+x)^{1-\lambda-\mu}\left[2;q\right]^{\lambda-1}=\sum_{m=0}^{\infty}\left[\gamma;q\right]_m/\left[m;q\right]!.\,{}_2\Phi_1\left[-m,\lambda+\mu+m-1;\lambda;q;x\right] \quad ...(3.2)$$

By definition of basic Gauss polynomial [1]

$G_m^{\lambda,\mu}(q;x)={}_2\Phi_1\left[-m,\lambda+\mu+m-1;\lambda;q;x\right],$

where $\lambda\neq 0,-1,-2,3,...$ . ...(3.3)

Using (3.3) in (3.2), we get the generating function

$$(1+x)^{1-\lambda-\mu}\left[2;q\right]^{\lambda-1}=\sum_{m=0}^{\infty}\left[\gamma;q\right]_m/\left[m;q\right]!.G_m^{\lambda,\mu}(q;x)$$

for basic Gauss polynomials.

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