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AN INTEGRAL INVOLVING THE H-FUNCTION OF SEVERAL VARIABLES

by

R. C. BOHARA AND U. C. JAIN
Department of Mathematics, University of Udaipur,
Udaipur-313001, Rajasthan, India

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1. INTRODUCTION

The H-function of n variables, occurring in this paper, is defined and represented as follows by Srivastava and Panda [4, p. 271, Eq. (4.1)]:

$$(1.1)\quad H^{0,\lambda\,:\,(\mu',\nu')\,;\,\cdots\,;\,(\mu^{(n)},\nu^{(n)})}_{A,\,C\,:\,[B',\,D'];\,\cdots\,;\,[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(n)}]:[(b'):\phi'];\cdots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\cdots,\psi^{(n)}]:[(d'):\delta'];\cdots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; x_1,\cdots,x_n\right)$$

$$=\frac{1}{(2\pi w)^n}\int_{L_1}\cdots\int_{L_n}\Psi(\xi_1,\cdots,\xi_n)\prod_{k=1}^{n}\phi_k(\xi_k)\,x_k^{\xi_k}\,d\xi_k,\quad w=\sqrt{-1}$$

where

$$(1.2)\quad \phi_i(\xi_i)=\frac{\prod_{j=1}^{\mu^{(i)}}\Gamma[d_j^{(i)}-\delta_j^{(i)}\xi_i]\prod_{j=1}^{\nu^{(i)}}\Gamma[1-b_j^{(i)}+\phi_j^{(i)}\xi_i]}{\prod_{j=\mu^{(i)}+1}^{D^{(i)}}\Gamma[1-d_j^{(i)}+\delta_j^{(i)}\xi_i]\prod_{j=\nu^{(i)}+1}^{B^{(i)}}\Gamma[b_j^{(i)}-\phi_j^{(i)}\xi_i]},$$

$$(1.3)\quad \Psi(\xi_1;\cdots;\xi_n)=\frac{\prod_{j=1}^{\lambda}\Gamma[1-a_j+\sum_{i=1}^{n}\theta_j^{(i)}\xi_i]}{\prod_{j=\lambda+1}^{A}\Gamma[a_j-\sum_{i=1}^{n}\theta_j^{(i)}\xi_i]\prod_{j=1}^{C}\Gamma[1-c_j+\sum_{i=1}^{n}\psi_j^{(i)}\xi_i]}$$

an empty product is interpreted as unity, the coefficients

$\theta_j^{(i)}, j=1, \cdots, A; \phi_j^{(i)}, j=1, \ldots, B^{(i)}; \Psi_j^{(i)}, j=1, \cdots, G;$

$\delta_j^{(i)}, j=1, \cdots, D^{(i)}$ and $i=1, \ldots, n$ are positive numbers and λ,

$\mu^{(i)}, \nu^{(i)}. A, B^{(i)}, C, D^{(i)}$ are integers such that $0 \leqq \lambda \leqq A$,

$0 \leqq \mu^{(i)} \leqq D^{(i)}, C \geqq 0$ and $0 \leqq \nu^{(i)} \leqq B^{(i)}, i=1, \ldots, n$.

The Contour L_i is in the comolex ξ_i– plane, is of the Mellin – Barnes type, which runs from $-\omega\infty$ to $+\omega\infty$ with indentations, if necessary, in such a manner that all the poles of $\Gamma[d_j^{(i)}-\delta_j^{(i)}\xi_i]$, $(j=1, \ldots, \mu^{(i)})$ are to the right and those of $\Gamma[1-b_j^{(i)}+\phi_j^{(i)}\xi_i]$, $(j=1, \ldots, \nu^{(i)})$ and $\Gamma[1-a_j+\sum_{i=1}^{n}\theta_j^{(i)}\xi_i]$, $(j=1, \ldots, \lambda)$ to the left of L_i, The various parameters are so restricted that none of these poles coincide.

2. MAIN INTEGRALS

If the following set of conditions are satisfied :

(i)
$$\Delta_i = -\sum_{j=1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\nu^{(i)}}\phi_j^{(i)}-\sum_{j=\nu^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}$$

$$-\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0, \quad i=1, \ldots, n$$

(ii) $|\arg x_i| < \frac{1}{2}\pi\Delta_i$

(iii) $0 < Re(a)+\sum_{i=1}^{n} m_i \min_{1\leqslant j\leqslant\mu^{(i)}}\left[Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]$

(iv) $0 < Re(\beta) - \sum_{i=1}^{n} m_i \min_{1 \leqslant j \leqslant \nu^{(i)}} \left[Re \left\{ \frac{b_j^{(i)} - 1}{\phi_j^{(i)}} \right\} \right] + \frac{1}{2}$

and $0 \leqq t < \infty$; $m_1, \ldots, m_n$ and $\eta_1, \cdots, \eta_n$ are positive numbers, then

$$(2.1) \quad \int_0^\infty t^{\alpha-1}(1+t)^{-1/2}\left(t^{1/2}+(1+t)^{1/2}\right)^{2\beta} H^{0,\,0\,:\,(\mu',\nu');\,\ldots;(\mu^{(n)},\nu^{(n)})}_{A,\,C\,:[B',D'];\,\ldots;[B^{(n)},D^{(n)}]}$$

$$\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}]\\ [(c):\Psi',\cdots,\Psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}]\end{matrix}\right|$$

$$\left.\begin{matrix} x_1\, t^{m_1}\left(t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}\right)^{-2\eta_1} \\ x_n\, t^{m_n}\left(t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}\right)^{-2\eta_n}\end{matrix}\right) dt$$

$$= 2^{1-2\alpha} H^{0,\quad 2\,:\,(\mu',\,\nu')\,;\,\ldots\,:\,(\mu^{(n)},\,\nu^{(n)})}_{A+2,C+1;\,[B',D']\,;\,\ldots\,;\,[B^{(n)},\,D^{(n)}]}$$

$$\left(\begin{matrix}[1-2\alpha : 2m_1\cdots,2m_n],\,[\frac{1}{2}+\alpha+\beta : \eta_1-m_1,\cdots,\eta_n-m_n],\,[(a):\theta',\cdots,\theta^{(n)}]:\\ [(c):\Psi',\cdots,\Psi^{(n)}],\,[\frac{1}{2}-\alpha+\beta:\eta_1+m_1,\cdots,\eta_n+m_n]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'];\cdots;[(b^{(n)}):\phi^{(n)}]\\ [(d'):\delta'];\cdots;[(d^{(n)}):\delta^{(n)}]\end{matrix}\right| x_1 \cdot 4^{-m_1},\cdots,x_n . 4^{-m_n}\Bigg)$$

If we use the following analogous property (*cf.* [2,p 27,Eq.(2.2)])

$$H^{0,\,\lambda\,:\,(\mu',\,\nu')\,;\,\ldots\,;\,(\mu^{(n)},\,\nu^{(n)})}_{A,\,C:\,[B',\,D']\,;\,\ldots\,;\,[B^{(n)},D^{(n)}]}$$

$$\left(\begin{matrix}[a:0,\cdots,0],[(a):\theta^{(2)},\cdots,\theta^{(n)}]:[(b'):\phi'];\cdots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\Psi^{(1)},\cdots,\Psi^{(n)}]:[(d'):\delta'];\cdots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; x_1,\cdots,x_n\right)$$

$$=\Gamma(1-a)H^{0,\lambda-1:(\mu',\nu');\cdots;(\mu^{(n)},\nu^{(n)})}_{A-1,C:[B',D'];\cdots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta^{(2)},\cdots,\theta^{(n)}]:(b'):\phi'];\cdots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\Psi^{(1)},\cdots,\Psi^{(n)}]:[(d'):\delta'];\cdots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ x_1,\cdots,x_n\right)$$

then (2.1) reduces to the following interesting result :

$$(2.2)\quad \int_0^\infty t^{a-1}(1+t)^{-\frac{1}{2}}\left(t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}\right)^{2\beta} H^{0,0:(\mu',\nu');\cdots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\cdots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}(a):\theta',\cdots,\theta^{(n)}]:[(b'):\phi'];\cdots;[(b^{(n)}):\phi^{(n)}]\\ [(c):\Psi',\cdots,\Psi^{(n)}]:[(d'):\delta'];\cdots;[(d^{(n)}):\delta^{(n)}]\end{matrix}\left|\begin{matrix}x_1t^{m_1}\left(t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}\right)^{-2m_1}\\ x_nt^{m_n}\left(t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}\right)^{-2m_n}\end{matrix}\right.\right)dt$$

$$=2^{1-2\alpha}\,\Gamma(\tfrac{1}{2}-a-\beta)\,H^{0,1:(\mu',\nu');\cdots;(\mu^{(n)},\nu^{(n)})}_{A+1,C+1:[B',D'];\cdots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[1-2\alpha:2m_1,\cdots,2m_n],[(a):\theta',\cdots,\theta^{(n)}]:\\ [(c):\Psi',\cdots,\Psi^{(n)}],[\frac{1}{2}-\alpha+\beta:2m_1,\cdots,2m_n];\end{matrix}\ \begin{matrix}[(b');\phi'];\cdots;[(b^{(n)});\phi^{(n)}]\\ [(d');\delta'];\cdots;[(d^{(n)});\delta^{(n)}]\end{matrix}\left|\begin{matrix}x_1.4^{-m_1}\\ \vdots\\ x_n.4^{-m_n}\end{matrix}\right.\right)$$

Derivation of (2.1) ;–The integral (2 1) can be easily established if we express the H–function of several variables in the integrand on the left of (2.1) in terms of its Mellin–Barnes type contour integral (1.1), interchange the order of integration (which is justified under the conditions mentioned in (2.1) and evaluate the inner integral with the help of the following known result [1, p. 311, Eq. (28)] ;

$$\int_0^\infty t^{s/2-1}(1+t)^{-\frac{1}{2}}\left(t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}\right)^{\nu}dt$$

$$=2^{-s+1}\,B\left(s,\tfrac{1}{2}-\tfrac{1}{2}s-\tfrac{1}{2}\nu\right),\ 0<Re(s)<1-Re(\nu),$$

and interpret the resulting contour integral with the help of (1.1).

8. PARTICULAR CASES

I. If we take all of the θ's, ϕ's, Ψ's, δ's to be equal to unity in (2.1), then it reduces to the following integrals :

$$(3.1)\quad \int_0^\infty t^{\alpha-1}(1+t)^{-\frac{1}{2}}\left(t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}\right)^{2\beta} G^{\,0,\,0\;:\;(\mu',\nu');\,\dots\,;\,(\mu^{(n)},\nu^{(n)})}_{\,A,C\;:\;[B',D'];\,\dots\,;\,[B^{(n)},D^{(n)}]} \left(\begin{matrix}[(a)]\;:\;b',\dots,b^{(n)}\\ [(c)]\;:\;a',\dots,d^{(n)}\end{matrix}\middle| \begin{matrix} x_1 t^{m_1}\left(t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}\right)^{-2\eta_1}\\ \vdots \\ x_n t^{m_n}\left(t^{\frac{1}{2}}+(1+t)^{\frac{1}{2}}\right)^{-2\eta_n}\end{matrix}\right) dt$$

$$= 2^{1-2\alpha}\, G^{\,0\;\;2\;:\;(\mu',\nu');\,\dots\,;\,(\mu^{(n)},\nu^{(n)})}_{\,A+2,C+1\;:\;[B',D];\,\dots\,;[B^{(n)},D^{(n)}]} \left(\begin{matrix}[1-2\alpha\;:\;2m_1,\dots,2m_n],\,[\tfrac{1}{2}+\alpha+\beta\;:\;\eta_1-m_1,\dots,\eta_n-m_n],\,[(a)]:\\ [\tfrac{1}{2}-\alpha+\beta\;:\;\eta_1+m_1,\dots,\eta_n+m_n],\,[(c)]:\end{matrix}\right.$$

$$\left.\begin{matrix}b',\dots b^{(n)}\\ a',\dots d^{(n)}\end{matrix}\middle| \begin{matrix}x_1.4^{-m_1}\\ \vdots\\ x_n.4^{-m_n}\end{matrix}\right)$$

where the conditions easlly obtainable from (2.1) are assumed to be satisfied.

II. If we take $n=2$ in (2.1) it reduces to the integral obtained by Rakesh [3, p. 65, Eq. (2.1)]. Another integral evaluated by Rakesh [3, p. 65, Eq. (2.2)] is also contained in our integral (2.2).

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